Journal of Classical Analysis
Volume 16, Number 1 (2020), 9–15 doi:10.7153/jca-2020-16-02

ON BERNSTEIN–TYPE INEQUALITIES FOR POLYNOMIALS INVOLVING THE POLAR DERIVATIVE

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Abstract. In this paper, we establish some upper bound estimates for the polar derivative of a polynomial not vanishing in a disk $|z| < k$, $k \geqslant 1$ with a zero of multiplicity s, $0 \leqslant s \leqslant n-1$ at the origin. The obtained results enable us to derive polar derivative analogues of some well known Bernstein-type inequalities as special cases.

1. Introduction

By $\mathbb{P}_n$ we denote the space of all complex polynomials $P(z) := \sum_{v=0}^{n} a_v z^n$ of degree n. If $P \in \mathbb{P}_n$, then by the famous Bernstein inequality [3], we have

$$\max_{|z|=1} |P'(z)| \leqslant n \max_{|z|=1} |P(z)|. \tag{1}$$

Equality holds in (1) if and only if $P(z)$ has all its zeros at origin. If we restrict ourselves to the class of polynomials $P(z)$ having no zero in $|z| < 1$, then (1) can be sharpened. In fact, Erdös conjectured and later Lax [6] proved that if $P(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{2}$$

It was shown by Frappier et al. [4] that if $P \in \mathbb{P}_n$, then

$$\max_{|z|=1} |P'(z)| \leqslant n \max_{1 \leqslant l \leqslant 2n} |P(e^{\frac{il\pi}{n}})|. \tag{3}$$

Clearly (3) is a refinement of (1), since the maximum of $|P(z)|$ on $|z| = 1$ may be larger than maximum of $|P(z)|$ taken over $2n^{th}$ roots of unity as one can show by taking a simple example $P(z) = z^n + ia$, $a > 0$.
The inequality (3) was improved by Aziz [1] by showing that

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{2} (M_\alpha + M_{\alpha+\pi}), \tag{4}$$

Mathematics subject classification (2010): 30A10, 30C10, 30D15.
Keywords and phrases: Polar derivative, Bernstein inequality, zeros.
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Paper JCA-16-02

where

$$M_\alpha = \max_{1\leqslant l\leqslant n} \left| P\left(e^{i\frac{(\alpha+2l\pi)}{n}}\right)\right| \tag{5}$$

for all real α.
In the same paper, Aziz [1] also improved inequality (4) for a restricted class of polynomials not vanishing in the unit disk $|z| < 1$. In fact, he proved that if $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{2}\left(M_\alpha^2 + M_{\alpha+\pi}^2\right)^{\frac{1}{2}}, \tag{6}$$

where M_α is as defined in (5).
As a refinement of (6), Rather and Shah [7] proved that if $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{2}\left(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2\right)^{\frac{1}{2}}. \tag{7}$$

where $m = \min_{|z|=1} |P(z)|$ and M_α is as defined in (5).
As a generalization of (7), Rather and Shah [7] in the same paper proved that if $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < k$, $k \geqslant 1$, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{\sqrt{2(1+k^2)}}\left(M_\alpha^2 + M_{\alpha+\pi}^2 - 2{m_k}^2\right)^{\frac{1}{2}}, \tag{8}$$

where $m_k = \min_{|z|=k} |P(z)|$ and M_α is as defined in (5).
Recently Sunil et al. [8], extended inequality (8) to the class of polynomials $P(z) = z^s\left(a_0 + \sum\limits_{\nu=\mu}^{n-s} a_\nu z^\nu\right)$, $1 \leqslant \mu \leqslant n-s$ having s-fold zero at $z = 0$ and the remaining $n-s$ zeros in $|z| \geqslant k$, $k \geqslant 1$ and obtained the following result.

THEOREM A. *If $P \in \mathbb{P}_n$ and $P(z) = z^s\left(a_0 + \sum\limits_{\nu=\mu}^{n-s} a_\nu z^\nu\right)$, $1 \leqslant \mu \leqslant n-s$, $0 \leqslant s \leqslant n-1$ such that $P(z)$ has s-fold zero at $z = 0$ and the remaining $n-s$ zeros in $|z| \geqslant k$, $k \geqslant 1$, then*

$$\max_{|z|=1} |P'(z)| \leqslant s\max_{|z|=1} |P(z)| + \frac{n-s}{\sqrt{2(1+k^{2\mu})}}\left({M_\alpha^*}^2 + {M_{\alpha+\pi}^*}^2 - \frac{2m_k^2}{k^{2s}}\right)^{\frac{1}{2}}, \tag{9}$$

where $m_k = \min_{|z|=k} |P(z)|$ and

$$M_\alpha^* = \max_{1\leqslant l\leqslant n-s} \left| P\left(e^{\frac{i(\alpha+2l\pi)}{n-s}}\right)\right|. \tag{10}$$

Over the last few decades different authors produced a large number of different versions and generalizations of the above inequalities by introducing restrictions on

the multiplicity of zero at $z=0$, the modulus of largest root of $P(z)$, restrictions on coefficients, using higher order derivatives etc. Before proceeding to our main results, let us introduce the concept of polar derivative involved.
For $P\in\mathbb{P}_n$ the polar derivative $D_\beta P(z)$ of $P(z)$ with respect to point β is defined as,

$$D_\beta P(z) := nP(z)+(\beta-z)P'(z).$$

Note that $D_\beta P(z)$ is a polynomial of degree at most $n-1$. This is so called polar derivative of $P(z)$ with respect to β. It generalises the ordinary derivative in the sense that

$$\lim_{\beta\to\infty}\left\{\frac{D_\beta P(z)}{\beta}\right\}=P'(z), \tag{11}$$

uniformly with respect to z for $|z|\leqslant R,\ R>0$.
In 1998, Aziz and Shah [2] established the polar derivative analogue of (1) by proving that if $P(z)$ is a polynomial of degree n, then for every complex number β with $|\beta|\geqslant 1$,

$$\max_{|z|=1}|D_\beta P(z)|\leqslant n|\beta|\max_{|z|=1}|P(z)|. \tag{12}$$

Clearly the above inequality generalizes (1) and to obtain (1) from it, simply divide both sides of (12) by $|\beta|$ and let $|\beta|\to\infty$.
The main aim of the present paper is to obtain some upper bound estimates for the maximum modulus of polar derivative of a polynomial on a unit disk under the assumption that the polynomial has no zeros in the disk $|z|<k,\ k\geqslant 1$, but having s-fold zero at the origin. The obtained results generalizes some already known estimates for the ordinary derivative of polynomial as special cases.

THEOREM 1. (Main) *If* $P(z)=z^s\left(a_0+\sum_{v=\mu}^{n-s}a_v z^v\right)$, $1\leqslant\mu\leqslant n-s,\ 0\leqslant s\leqslant n-1$ *is a polynomial of degree* n *having* s*-fold zero at* $z=0$ *and remaining* $n-s$ *zeros in* $|z|\geqslant k,\ k\geqslant 1$, *then for any complex number* β *with* $|\beta|\geqslant 1$, *we have*

$$\max_{|z|=1}|D_\beta P(z)|\leqslant\Big[n+s(|\beta|-1)\Big]\max_{|z|=1}|P(z)|+\frac{(n-s)(|\beta|-1)}{\sqrt{2(1+k^{2\mu})}}\left({M_\alpha^*}^2+{M_{\alpha+\pi}^*}^2-\frac{2m_k^2}{k^{2s}}\right)^{\frac{1}{2}}, \tag{13}$$

$m_k=\min_{|z|=k}|P(z)|$ *and* M_α^* *is as defined in* (10).

REMARK 1. If we divide both sides of inequality (13) by $|\beta|$ and let $|\beta|\to\infty$ and noting (11), we get inequality (9).
If we take $s=0$ and $\mu=1$ in (13), we get the following polar derivative analogue of (8).

COROLLARY 1. *If* $P(z) = \sum_{v=0}^{n} a_v z^v$, *is a polynomial of degree* n *and having no zeros in* $|z| < k$, $k \geqslant 1$, *then for every complex number* β *with* $|\beta| \geqslant 1$, *we have*

$$\max_{|z|=1} |D_\beta P(z)| \leqslant n \max_{|z|=1} |P(z)| + \frac{n(|\beta|-1)}{\sqrt{2(1+k^2)}} \left({M_\alpha}^2 + {M_{\alpha+\pi}}^2 - 2m_k^2\right)^{\frac{1}{2}}, \tag{14}$$

$m_k = \min_{|z|=k} |P(z)|$ *and* M_α *is as defined in* (5).

REMARK 2. Dividing both sides of inequality (14) by $|\beta|$ and let $|\beta| \to \infty$ and noting (11), we get inequality (8).
By taking $s = 0$, $\mu = 1$ and $k = 1$ in (13), we get the following polar derivative analogue of (7).

COROLLARY 2. *If* $P(z) = \sum_{v=0}^{n} a_v z^v$, *is a polynomial of degree* n *and having no zeros in* $|z| < 1$, *then for every complex number* β *with* $|\beta| \geqslant 1$, *we have*

$$\max_{|z|=1} |D_\beta P(z)| \leqslant n \max_{|z|=1} |P(z)| + \frac{n(|\beta|-1)}{2} \left({M_\alpha}^2 + {M_{\alpha+\pi}}^2 - 2m^2\right)^{\frac{1}{2}}, \tag{15}$$

$m = \min_{|z|=1} |P(z)|$ *and* M_α *is as defined in* (5).

REMARK 3. Dividing both sides of inequality (15) by $|\beta|$ and let $|\beta| \to \infty$ and noting (11), we get inequality (7).
From Corollary 2, we easily get the following result.

COROLLARY 3. *If* $P(z) = \sum_{v=0}^{n} a_v z^v$, *is a polynomial of degree* n *and having no zeros in* $|z| < 1$, *then for every complex number* β *with* $|\beta| \geqslant 1$, *we have*

$$\max_{|z|=1} |D_\beta P(z)| \leqslant n \max_{|z|=1} |P(z)| + \frac{n(|\beta|-1)}{2} \left({M_\alpha}^2 + {M_{\alpha+\pi}}^2\right)^{\frac{1}{2}}, \tag{16}$$

M_α *is as defined in* (5).

REMARK 4. For the class of polynomials having no zeros in the unit disk $\{z \in \mathbb{C}; |z| < 1\}$, the inequality (16) represents a refinement of (12) because, since $|\beta| \geqslant 1$ and the maximum of $|P(z)|$ on $|z| = 1$ may be larger than the quantity $\frac{1}{2}\left({M_\alpha}^2 + {M_{\alpha+\pi}}^2\right)^{\frac{1}{2}}$ for every real α.

REMARK 5. Dividing both sides of inequality (16) by $|\beta|$ and let $|\beta| \to \infty$ and noting (11), we get inequality (6).

2. Lemmas

We need the following lemmas to prove the theorem.
The following lemma is due to Sunil et al. [8].

LEMMA 1. *If* $P(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v,\ 1 \leqslant \mu \leqslant n,\ P(z) \neq 0$ *in* $|z| < k,\ k \geqslant 1$ *and* $m_k = \min_{|z|=k} |P(z)|$, *then for every real* α,

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{\sqrt{2(1+k^{2\mu})}} \left(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m_k^2\right)^{\frac{1}{2}}, \tag{17}$$

where M_α *is as defined in* (5).

The following lemma is a special case of a result due to Govil and Rahman [5].

LEMMA 2. *If* $P(z)$ *is a polynomial of degree n, then for* $|z| = 1$,

$$|P'(z)| + |Q'(z)| \leqslant n \max_{|z|=1} |P(z)|, \tag{18}$$

where $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$.

Next we prove a lemma in which we generalize lemma 1 to the polar derivative of a polynomial. More precisely, we prove the following.

LEMMA 3. *If* $P \in \mathbb{P}_n$ *and* $P(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v,\ 1 \leqslant \mu \leqslant n,\ P(z) \neq 0$ *in* $|z| < k$, $k \geqslant 1$ *and* $m_k = \min_{|z|=k} |P(z)|$, *then for every real* α *and for every complex number* β *with* $|\beta| \geqslant 1$, *we have*

$$\max_{|z|=1} |D_\beta P(z)| \leqslant \frac{n}{\sqrt{2}} \left[\sqrt{2}|P(z)| + \frac{(|\beta|-1)}{\sqrt{1+k^{2\mu}}} \left({M_\alpha}^2 + {M_{\alpha+\pi}}^2 - 2m_k^2\right)^{\frac{1}{2}}\right], \tag{19}$$

where M_α *is as defined in* (5).

Proof. Since $P(z)$ is a polynomial of degree n which does not vanish in $|z| < k$, $k \geqslant 1$. Applying the inequality (17), we have

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{\sqrt{2(1+k^{2\mu})}} \left(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m_k^2\right)^{\frac{1}{2}}. \tag{20}$$

Now for $\beta \in \mathbb{C}$, with $|\beta| \geqslant 1$

$$\begin{aligned} |D_\beta P(z)| &= |nP(z) + (\beta - z)P'(z)| = |nP(z) + \beta P'(z) - zP'(z)| \\ &\leqslant |nP(z) - zP'(z)| + |\beta||P'(z)|. \end{aligned}$$

It can be easily seen that

$$|nP(z) - zP'(z)| = |Q^{'}(z)| \quad \text{for } |z| = 1.$$

Therefore we have for $|z| = 1$

$$|D_\beta P(z)| \leqslant |Q'(z)| + |\beta||P'(z)| = |Q'(z)| + |P'(z)| - |P'(z)| + |\beta||P'(z)|.$$

Using lemma 2, we get for $|z| = 1$,

$$|D_\beta P(z)| \leqslant n|P(z)| + (|\beta| - 1)|P'(z)|,$$

which on using (20), we get

$$|D_\beta P(z)| \leqslant n|P(z)| + \frac{n(|\beta| - 1)}{\sqrt{2(1 + k^{2\mu})}} \left(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m_k^2\right)^{\frac{1}{2}}.$$

Equivalently,

$$\max_{|z|=1} |D_\beta P(z)| \leqslant \frac{n}{\sqrt{2}} \left\{ \sqrt{2}|P(z)| + \frac{(|\beta| - 1)}{\sqrt{(1 + k^{2\mu})}} \left(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m_k^2\right)^{\frac{1}{2}} \right\},$$

which proves lemma 3.

3. Proof of the theorem

Proof. Let $P(z) = z^s \phi(z)$ where $\phi(z) = a_0 + \sum_{v=\mu}^{n-s} a_v z^v$, $1 \leqslant \mu \leqslant n-s$ and $0 \leqslant s \leqslant n-1$ is the polynomial of degree $n-s$ having no zeros in $|z| < k$, $k \geqslant 1$. Applying lemma 3 to polynomial $\phi(z)$ of degree $n-s$, we get for every $\beta \in \mathbb{C}$, with $|\beta| \geqslant 1$,

$$\max_{|z|=1} |D_\beta \phi(z)| \leqslant \frac{(n-s)}{\sqrt{2}} \left\{ \sqrt{2} \max_{|z|=1} |\phi(z)| + \frac{(\beta - 1)}{\sqrt{(1 + k^{2\mu})}} \left({M_\alpha^*}^2 + {M_{\alpha+\pi}^*}^2 - 2m_k'^2 \right)^{\frac{1}{2}} \right\}, \tag{21}$$

where $m_k' = \min_{|z|=k} |\phi(z)|$.
Now for $\beta \in \mathbb{C}$ with $|\beta| \geqslant 1$, we have

$$\begin{aligned} D_\beta P(z) &= nP(z) + (\beta - z)P'(z) = nz^s \phi(z) + (\beta - z)\left[z^s \phi'(z) + \phi(z).sz^{s-1}\right] \\ &= z^s D_\beta \phi(z) + s\beta z^{s-1} \phi(z), \end{aligned}$$

which implies

$$zD_\beta P(z) = z^{s+1} D_\beta \phi(z) + s\beta P(z).$$

Hence for $|z| = 1$, we get from the above inequality that

$$|D_\beta P(z)| \leqslant |D_\beta \phi(z)| + s|\beta||P(z)|,$$

which in particular implies,

$$\max_{|z|=1}|D_\beta P(z)| \leqslant \max_{|z|=1}|D_\beta \phi(z)| + s|\beta| \max_{|z|=1}|P(z)|.$$

This gives by using inequality (21),

$$\max_{|z|=1}|D_\beta P(z)| \leqslant \left[\frac{(n-s)}{\sqrt{2}}\left\{\sqrt{2}\max_{|z|=1}|\phi(z)| + \frac{(\beta-1)}{\sqrt{(1+k^{2\mu})}}\left(M_\alpha^{*\,2} + M_{\alpha+\pi}^{*\,2} - 2m_k'^2\right)^{\frac{1}{2}}\right\}\right] + s|\beta|\max_{|z|=1}|P(z)|. \tag{22}$$

Now the relation between $P(z)$ and $\phi(z)$ is $P(z) = z^s\phi(z)$.
This implies $|P(z)| = |\phi(z)|$ for $|z| = 1$ and $m_k' = \min_{|z|=k}|\phi(z)| = \frac{1}{k^s}\min_{|z|=k}|P(z)| = \frac{1}{k^s}m_k$, where $m_k = \min_{|z|=k}|P(z)|$.
Using these observations in (22), we get (13).
This completes the proof of Theorem 1.

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(Received December 18, 2019)

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