Journal of Classical Analysis
Volume 16, Number 1 (2020), 23–35
doi:10.7153/jca-2020-16-04

UNIQUENESS OF HOMOGENEOUS DIFFERENTIAL POLYNOMIALS OF MEROMORPHIC FUNCTIONS CONCERNING WEAKLY WEIGHTED SHARING

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Abstract. In 2006 S. Lin and W. Lin [3] first defined the concept of weakly-weighted sharing of functions and proved some results on uniqueness of a meromorphic function f and its n-th derivative $f^{(n)}$. Using this notion of weakly-weighted sharing of functions, in this paper we prove uniqueness of homogeneous differential polynomials $P[f]$ and $P[g]$ generated by meromorphic functions f and g respectively.

1. Introduction and main result

Let $\mathbb{C}$ denote the complex plane and let f be a non-constant meromorphic function defined on $\mathbb{C}$. We assume that the reader is familiar with the standard definitions and notations used in the Nevanlinna value distribution theory, such as $T(r,f)$, $m(r,f)$, $N(r,f)$ (see [1, 10, 11]). By $S(r,f)$ we denote any quantity satisfying the condition $S(r,f) = \circ(T(r,f))$ as $r \to \infty$ possibly outside an exceptional set E of finite linear measure. A meromorphic function a is called a small function with respect to f if either $a \equiv \infty$ or $T(r,a) = S(r,f)$. We denote by $S(f)$ the collection of all small functions with respect to f. Clearly $\mathbb{C} \cup \{\infty\} \subset S(f)$ and $S(f)$ is a field over the set of complex numbers. For $a \in \mathbb{C} \cup \{\infty\}$ the quantities

$$\delta(a,f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)}$$

and

$$\Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a;f)}{T(r,f)}$$

are respectively called the deficiency and ramification index of a for the function f. We use $\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}$ is the order of f.

For any two non-constant meromorphic functions f and g, and $a \in S(f) \cap S(g)$ we say that f and g share a IM (CM) provided that $f-a$ and $g-a$ have the same zeros ignoring (counting) multiplicities. If $\frac{1}{f}$ and $\frac{1}{g}$ share 0 IM (CM), we say that f

Mathematics subject classification (2010): 30D30, 30D35.
Keywords and phrases: Meromorphic functions, Nevanlinna theory, weakly-weighted sharing, differential polynomial, uniqueness.
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and g share ∞ IM (CM). Let f and g share 1 IM and let z_0 be a zero of $f-1$ of multiplicity p and a zero of $g-1$ of multiplicity q. By $\overline{N}_L(r,1;f)$ we denote the reduced counting function of those 1-points of f and g where $p>q\geqslant 1$. $\overline{N}_L(r,1;g)$ is defined similarly.

The subject on sharing values between two non-constant meromorphic functions f and g, and their relationship studied by Yi [12] and proved the following results.

THEOREM 1. [12] *Let f and g be two non-constant meromorphic functions such that $f^{(n)}$, $g^{(n)}$ share the value 1 CM. If*

$$2\delta(0;f)+(n+4)\Theta(\infty;f)>n+5 \text{ and}$$

$$2\delta(0;g)+(n+4)\Theta(\infty;g)>n+5,$$

then either $f\equiv g$ or $f^{(n)}.g^{(n)}\equiv 1$.

THEOREM 2. [12] *Let f and g be two non-constant meromorphic functions such that $f^{(n)}$, $g^{(n)}$ share the value 1 IM. If*

$$5\delta(0;f)+(4n+7)\Theta(\infty;f)>4n+11 \text{ and}$$

$$5\delta(0;g)+(4n+7)\Theta(\infty;g)>4n+11,$$

then either $f\equiv g$ or $f^{(n)}.g^{(n)}\equiv 1$.

In [4] Li and Li considered the problem of replacing the derivatives by linear differential polynomials. Let f be a non-constant meromorphic function. An expression of the form

$$L(f)=f^{(n)}+a_{k-1}f^{(n-1)}+....+a_0f, \tag{1}$$

where $a_0,a_1,...,a_{n-1}$ are complex constants is called a linear differential polynomial generated by f.

Li and Li [4] proved the following theorems:

THEOREM 3. [4] *Let f and g be two non-constant entire functions. Suppose that f, g share the value 0 CM and $L(f)$, $L(g)$ share the value 1 CM and $\delta(0;f)>\frac{1}{2}$. If $\rho(f)\neq 1$, then either $f\equiv g$ or $L(f).L(g)\equiv 1$.*

THEOREM 4. [4] *Let f and g be two non-constant entire functions. Suppose that f, g share the value 0 CM and $L(f)$, $L(g)$ share the value 1 IM and $\delta(0;f)>\frac{4}{5}$. If $\rho(f)\neq 1$, then either $f\equiv g$ or $L(f).L(g)\equiv 1$.*

Recently Lahiri and Pal [5] extend the results of Li and Li [4] to homogeneous differential polynomial by including the class of entire functions of order 1.

DEFINITION 1. Let n ($\geqslant 1$) be a positive integer, p ($\geqslant 0$) be an integer and f be a non-constant meromorphic function. An expression of the form

$$P[f] = \sum_{k=1}^{n} a_k \prod_{j=0}^{p} \left(f^{(j)}\right)^{l_{kj}}, \tag{2}$$

where $a_k \in S(f)$ for $k = 1,2,......,n$ and l_{kj} ($1 \leqslant k \leqslant n$; $0 \leqslant j \leqslant p$) are non-negative integers and $d = \sum_{j=0}^{p} l_{kj}$ for $k = 1,2,......,n$, is called a homogeneous differential polynomial of degree d generated by f. Also we denote by Q the quantity $Q = \max_{1 \leqslant k \leqslant n} \sum_{j=0}^{p} jl_{kj}$.

Let f and g be two non-constant meromorphic functions. When we consider $P[f]$ and $P[g]$ are non-constant homogeneous differential polynomials of f and g respectively, then we understand that the coefficients $a_j \in S(f) \cap S(g)$.

Lahiri and Pal [5] proved the following theorem.

THEOREM 5. *Let f and g be two non-constant meromorphic functions, a ($\not\equiv 0,\infty$) $\in S(f) \cap S(g)$. Suppose $P[f]$ and $P[g]$, as defined by (2) are non-constant. If $P[f]$ and $P[g]$ share a IM, and*

$$min\left\{5\delta(0,f) + \frac{4Q+7}{d}\Theta(\infty,f), 5\delta(0,g) + \frac{4Q+7}{d}\Theta(\infty,g)\right\} > \frac{4Q+4d+7}{d},$$

then either $P[f] \equiv P[g]$ or $P[f].P[g] \equiv a^2$.

In 2019, Dilip et al [6] considered the weighted set sharing and proved the following result:

THEOREM 6. *Let f be a non-constant meromorphic function and $p(z)$ be a polynomial in z of degree n ($\geqslant 1$) with $p(0) = 0$. Let $a(z)$ ($\not\equiv 0,\infty$) be an element of $S(f)$. Let $P[f]$ be a non-constant differential polynomial of f. Suppose that $p(f)$ and $P[f]$ share the set $S_m = \{a(z), a(z)\omega, ..., a(z)\omega^{m-1}\}$ with weight l with one of the following conditions:*

(i) $l \geqslant 2$ and

$$\begin{aligned}&(mQ+3)\Theta(\infty,f) + 2n\Theta(0,p(f)) + m\overline{d}(P)\delta(0,f)\\ &> (mQ+3) + 2m\overline{d}(P) - m\underline{d}(P) - (m-2)n,\end{aligned}$$

(ii) $l = 1$ and

$$\begin{aligned}&(mQ+\frac{7}{2})\Theta(\infty,f) + \frac{5n}{2}\Theta(0,p(f)) + \overline{d}(P)\delta(0,f)\\ &> mQ + \frac{7}{2} + (m+1)\overline{d}(P) - m\underline{d}(P) + (\frac{5}{2} - m)n,\end{aligned}$$

(iii) $l = 0$ *and*

$$(2mQ+6)\Theta(\infty, f)+4n\Theta(0, p(f))+2m\overline{d}(P)\delta(0, f) \\ > 2mQ+6+4m\overline{d}(P)-2m\underline{d}(P)+(4-m)n.$$

Then $P[f] = tp(f)$ *for some* t *such that* $t^m = 1$.

To define weakly-weighted sharing we need the following definitions:

DEFINITION 2. [3] Let $N_E(r,a)$ be the counting function of all common zeros of $f-a$ and $g-a$ with the same multiplicities and $N_0(r,a)$ be the counting function of all common zeros of $f-a$ and $g-a$ ignoring multiplicities. We denote by $\overline{N}_E(r,a)$ and $\overline{N}_0(r,a)$ the reduced counting functions of f and g corresponding to the counting functions $N_E(r,a)$ and $N_0(r,a)$ respectively. If

$$\overline{N}(r,a;f)+\overline{N}(r,a;g)-2\overline{N}_E(r,a) = S(r,f)+S(r,g),$$

then we say that f and g share a "CM". If

$$\overline{N}(r,a;f)+\overline{N}(r,a;g)-2\overline{N}_0(r,a) = S(r,f)+S(r,g),$$

then we say that f and g share a "IM".

DEFINITION 3. Let p be a positive integer. Let f be a meromorphic function and $a \in S(f)$.

(i) $\overline{N}_{p)}(r,a;f)$ denotes the counting function of those a-points of f whose multiplicities are not greater than p, where each a-point is counted only once.

(ii) $\overline{N}_{(p}(r,a;f)$ denotes the counting function of those a-points of f whose multiplicities are not less than p, where each a-point is counted only once.

(iii) $N_p(r,a;f)$ denotes the counting function of those a-points of f, where an a-point of f with multiplicity m counted m times if $m \leqslant p$ and p times if $m > p$.

We denote by $\delta_p(a,f)$ the quantity

$$\delta_p(a,f) = 1-\limsup_{r\to\infty}\frac{N_p(r,a;f)}{T(r,f)}.$$

Clearly $\delta_p(a,f) \geqslant \delta(a,f)$.

Let f and g be two non-constant meromorphic functions sharing a "IM", for $a \in S(f)\cap S(g)$, and a positive integer l or ∞.

(i) $\overline{N}^E_{l)}(r,a)$ denotes the counting function of those a-points of f whose multiplicities are equal to the corresponding a-points of g, both of their multiplicities are not greater than l, where each a-point is counted only once.

(ii) $\overline{N}^{0}_{(l}(r,a)$ denotes the reduced counting function of those a-points of f which are a-points of g, both of their multiplicities are not less than l, where each a-point is counted only once.

In 2006 S. Lin and W. Lin [3] first defined the concept of weakly-weighted sharing of functions as follows.

DEFINITION 4. [3] For $a \in S(f) \cap S(g)$, if l is a positive integer or ∞, and

$$\overline{N}_{l)}(r,a;f)+\overline{N}_{l)}(r,a;g)-2\overline{N}^{E}_{l)}(r,a)=S(r,f)+S(r,g)$$

$$\overline{N}_{(l+1}(r,a;f)+\overline{N}_{(l+1}(r,a;g)-2\overline{N}^{0}_{(l+1}(r,a)=S(r,f)+S(r,g),$$

or if $l=0$ and

$$\overline{N}(r,a;f)+\overline{N}(r,a;g)-2\overline{N}_{0}(r,a)=S(r,f)+S(r,g),$$

then we say f and g weakly share a with weight l. Here, we write f, g share "(a,l)" to mean that f, g weakly share a with weight l.

Obviously if f and g share "(a,l)", then f and g share "(a,s)" for any s $(0 \leqslant s < l)$. Also, we note that f and g share a "IM" or "CM" if and only if f and g share "$(a,0)$" or "(a,∞)" respectively.

In 2006 S. Lin and W. Lin [3] proved the following theorem:

THEOREM 7. *Let n $(\geqslant 1)$ be a positive integer and let k be a non-negative integer or ∞. Let f be a non-constant meromorphic function and $a \in S(f)$ be such that $a \not\equiv 0,\infty$. If f and $f^{(n)}$ share "(a,k)" with one of the following conditions:*
(i) $2 \leqslant k \leqslant \infty$ and

$$4\Theta(\infty,f)+2\delta_{2+n}(0,f)>5,$$

(ii) $k=1$ and

$$\left(\frac{n+9}{2}\right)\Theta(\infty,f)+\frac{5}{2}\delta_{2+n}(0,f)>\frac{n}{2}+6,$$

(iii) $k=0$ and

$$(7+2n)\Theta(\infty,f)+5\delta_{2+n}(0,f)>2n+11,$$

then $f \equiv f^{(n)}$.

Later in 2011, H-Y Xu and Y Hu [13] generalize Theorem 7 by proving the following theorem:

THEOREM 8. *Let n $(\geqslant 1)$ be positive integer and let k be a non-negative integer or ∞. Let f be a non-constant meromorphic function and $a \in S(f)$ be such that $a \not\equiv 0,\infty$. Suppose $L(f)$ is defined as in (1). If f and $L(f)$ share "(a,k)" with one of the following conditions:*
(i) $2 \leqslant k \leqslant \infty$ and

$$4\Theta(\infty,f)+2\delta_{2+n}(0,f)>5,$$

(ii) $k=1$ and

$$\left(\frac{7}{2}+n\right)\Theta(\infty,f)+\frac{3}{2}\delta_2(0,f)+\delta_{n+2}(0,f)>n+5,$$

(iii) $k=0$ and

$$(6+2n)\Theta(\infty,f)+\delta_2(0,f)+2\Theta(0,f)+2\delta_{2+n}(0,f)>2n+10,$$

then $f\equiv L(f)$.

In 2019 Dilip et al [7, 8] proved uniqueness of homogeneous differential polynomials $P[f]$ and $P[g]$ when they share "(a,l)".

THEOREM 9. [7] *Let f and g be two non-constant meromorphic functions, a $(\not\equiv 0,\infty)\in S(f)\cap S(g)$. Suppose that $P[f]$ and $P[g]$, as defined by (2), are non-constant. If $P[f]$ and $P[g]$ share "(a,l)" with one of the following conditions:*
(i) $2 \leqslant l \leqslant \infty$ and

$$min\left\{2\delta(0,f)+\frac{Q+4}{d}\Theta(\infty,f),2\delta(0,g)+\frac{Q+4}{d}\Theta(\infty,g)\right\}>\frac{Q+d+4}{d},$$

(ii) $l=1$ and

$$min\left\{\frac{5}{2}\delta(0,f)+\frac{3Q+9}{2d}\Theta(\infty,f),\frac{5}{2}\delta(0,g)+\frac{3Q+9}{2d}\Theta(\infty,g)\right\}>\frac{3Q+3d+9}{2d},$$

(iii) $l=0$ and

$$min\left\{5\delta(0,f)+\frac{4Q+7}{d}\Theta(\infty,f),5\delta(0,g)+\frac{4Q+7}{d}\Theta(\infty,g)\right\}>\frac{4Q+4d+7}{d},$$

then either $P[f]\equiv P[g]$ or $P[f].P[g]\equiv a^2$.

THEOREM 10. [8] *Let f and g be two transcendental meromorphic functions, $a=a(z)$ $(a\not\equiv 0,\infty)\in S(f)\cap S(g)$. Suppose $P[f]$ and $P[g]$, defined by (2) are non-constant. If $P[f]$ and $P[g]$ share "(a,k)" with one of the the following conditions:*
(i) $k\geqslant 2$ and

$$min\left\{(Q+4)\Theta(\infty,f)+2\delta_{2+p}(0,f),(Q+4)\Theta(\infty,g)+2\delta_{2+p}(0,g)\right\}>(6+Q-d),$$

(ii) $k = 1$ *and*

$min\left\{(3Q+9)\Theta(\infty,f)+5\delta_{2+p}(0,f),(3Q+9)\Theta(\infty,g)+5\delta_{2+p}(0,g)\right\} > 3Q+14-2d,$

(iii) $k = 0$ *and*

$min\left\{(4Q+7)\Theta(\infty,f)+5\delta_{2+p}(0,f),(4Q+7)\Theta(\infty,g)+5\delta_{2+p}(0,g)\right\} > 4Q+12-d,$

then either $P[f] \equiv P[g]$ *or* $P[f].P[g] \equiv a^2$.

In this paper we prove the following theorem:

THEOREM 11. *Let* f *and* g *be two non-constant meromorphic functions,* a $(\not\equiv 0,\infty) \in S(f)\cap S(g)$. *Suppose that* $P[f]$ *and* $P[g]$, *as defined by (2) are non-constant. If* $P[f]$ *and* $P[g]$ *share "*(a,l)*" with one of the following conditions:*
(i) $l = \infty$ *and*

$$min\left\{2\delta(0,f)+\frac{Q+4}{d}\Theta(\infty,f),2\delta(0,g)+\frac{Q+4}{d}\Theta(\infty,g)\right\} > \frac{Q+d+4}{d}, \quad (3)$$

(ii) $0 < l < \infty$ *and*

$$\begin{aligned} \min\ &\left\{\frac{(2l+1)d}{l}\delta(0,f)+\left(\frac{Q+1+2l}{l}+Q+2\right)\Theta(\infty,f),\right.\\ &\left.\frac{(2l+1)d}{l}\delta(0,g)+\left(\frac{Q+1+2l}{l}+Q+2\right)\Theta(\infty,g)\right\} \quad (4)\\ > &\ \frac{(l+1)d+Q+1}{l}+4+Q, \end{aligned}$$

(iii) $l = 0$ *and*

$$min\left\{5\delta(0,f)+\frac{4Q+7}{d}\Theta(\infty,f),5\delta(0,g)+\frac{4Q+7}{d}\Theta(\infty,g)\right\} > \frac{4Q+4d+7}{d}, \quad (5)$$

then either $P[f] \equiv P[g]$ *or* $P[f].P[g] \equiv a^2$.

Suppose F and G share "$(1,l)$" and let z_0 be a zero of $F-1$ of multiplicity p and a zero of $G-1$ of multiplicity q. We define by $\overline{N}_{G>l+1}(r,1;F)$ the reduced counting function of those 1-points of F such that $q > l+l$; $\overline{N}_{F>l+1}(r,1;G)$ is defined similarly. Also denote by $N_E^{1)}(r,1;F)$ the counting function of those 1-points of F and G where $p=q=1$ and denote by $\overline{N}_E^{(2}(r,1;F)$ the counting function of those 1-points of F and G where $p=q\geqslant 2$, where each such zero is counted only once.

2. Lemmas

Let F and G be two non-constant meromorphic functions. We shall define by H the following function

$$H=\left(\frac{F^{(2)}}{F^{(1)}}-2\frac{F^{(1)}}{F-1}\right)-\left(\frac{G^{(2)}}{G^{(1)}}-2\frac{G^{(1)}}{G-1}\right).$$

LEMMA 1. [5] *Let f be a non-constant meromorphic function and $P[f]$ be defined by (2), then*

$$(i)\ T(r,P) \leqslant dT(r,f) + Q\overline{N}(r,\infty;f) + S(r,f).$$
$$(ii)\ N(r,0;P) \leqslant T(r,P) - dT(r,f) + dN(r,0;f) + S(r,f),$$
$$\leqslant Q\overline{N}(r,\infty;f) + dN(r,0;f) + S(r,f).$$

LEMMA 2. [2] *Let f be a transcendental meromorphic function, $P[f]$ be a homogeneous differential polynomial of degree $d \geqslant 1$. Then*

$$dT(r,f) \leqslant \overline{N}(r,\infty;f) + \overline{N}(r,1;F) + N(r,0;f^d) - N_0(r,0;(P(f))^{(1)}) + S(r,f),$$

where $N_0(r,0;(P[f])^{(1)})$ denotes the counting function corresponding to the zeros of $(P[f])^{(1)}$ which are not the zeros of $P[f]$ and $P[f]-1$.

LEMMA 3. [9] *Let f be a non-constant meromorphic function and let*

$$p(f) = a_n f^n + a_{n-1} f^{n-1} + \ldots + a_1 f + a_0,$$

where $a_i \in S(f)$ for $i = 0,\ 1\ldots\ n,$ $a_n\ (\neq 0)$ be a polynomial of degree n. Then $T(r,p(f)) = nT(r,f) + S(r,f)$.

LEMMA 4. *If F and G be non-constant meromorphic functions share "$(1,l)$" where l is positive integer and $H \not\equiv 0$. Then*

$$T(r,F) \leqslant (1+\frac{1}{l})N(r,0;F) + (2+\frac{1}{l})\overline{N}(r,\infty;F) + N(r,0;G) + 2\overline{N}(r,\infty;G)$$
$$+S(r,F) + S(r,G).$$

Proof. Let $H \not\equiv 0$. Then by a simple calculation we see that

$$N_E^{1)}(r,1;F) \leqslant N(r,0;H) \leqslant T(r,H) + O(1)$$
$$\leqslant N(r,\infty;H) + S(r,F) + S(r,G). \tag{6}$$

Since F and G share "$(1,l)$", we have

$$N(r,\infty;H) \leqslant \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \overline{N}_{G>l+1}(r,1;F) + \overline{N}_{F>l+1}(r,1;G) + \overline{N}_{(2}(r,0;F)$$
$$+\overline{N}_{(2}(r,0;G) + \overline{N}_0(r,0;F^{(1)}) + \overline{N}_0(r,0;G^{(1)}) + S(r,F) + S(r,G). \tag{7}$$

By Nevanlinna's second fundamental theorem, we have

$$T(r,F) + T(r,G) \leqslant \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1;F) + \overline{N}(r,0;G)$$
$$+\overline{N}(r,\infty;G) + \overline{N}(r,1;G) - N_0(r,0;F^{(1)})$$
$$-N_0(r,0;G^{(1)}) + S(r,F) + S(r,G), \tag{8}$$

where $N_0(r,0;F^{(1)})$ denotes the counting function corresponding to the zeros of $F^{(1)}$ which are not the zeros of F and $F-1$. Similarly defined $N_0(r,0;G^{(1)})$.

Note that F and G share "$(1,l)$" and using (6), (7)

$$\begin{aligned}
\overline{N}(r,1;F)+\overline{N}(r,1;G) &\leqslant N_E^{1)}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)\\
&\quad +N_E^{(2}(r,1;F)+\overline{N}(r,1;G)\\
&= N_E^{1)}(r,1;F)+\overline{N}_{G>l+1}(r,1;F)+\overline{N}_{F>l+1}(r,1;G)\\
&\quad +N_E^{(2}(r,1;F)+\overline{N}(r,1;G)\\
&\leqslant \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_{(2}(r,0;F)+\overline{N}_{(2}(r,0;G)\\
&\quad +2\overline{N}_{G>l+1}(r,1;F)+2\overline{N}_{F>l+1}(r,1;G)+N_E^{(2}(r,1;F)+\overline{N}(r,1;G)\\
&\quad +\overline{N}_0(r,0;F^{(1)})+\overline{N}_0(r,0;G^{(1)})+S(r,F)+S(r,G)\\
&\leqslant \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_{(2}(r,0;F)+\overline{N}_{(2}(r,0;G)\\
&\quad +\overline{N}_{G>l+1}(r,1;F)+\overline{N}_{F>l+1}(r,1;G)+N(r,1;G)+\overline{N}_0(r,0;F^{(1)})\\
&\quad +\overline{N}_0(r,0;G^{(1)})+S(r,F)+S(r,G)\\
&\leqslant \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_{(2}(r,0;F)+\overline{N}_{(2}(r,0;G)\\
&\quad +\overline{N}_{G>l+1}(r,1;F)+\overline{N}_{F>l+1}(r,1;G)+T(r,G)+\overline{N}_0(r,0;F^{(1)})\\
&\quad +\overline{N}_0(r,0;G^{(1)})+S(r,F)+S(r,G). \qquad (9)
\end{aligned}$$

Also we have,

$$\begin{aligned}
\overline{N}_{G>l+1}(r,1;F)+\overline{N}_{F>l+1}(r,1;G) &\leqslant \frac{1}{l}N(r,0;F^{(1)})+S(r,F)\\
&\leqslant \frac{1}{l}N(r,0;F)+\frac{1}{l}\overline{N}(r,\infty;F)+S(r,F). \qquad (10)
\end{aligned}$$

Now using (9), (10) in (8) we get

$$\begin{aligned}
T(r,F)+T(r,G) &\leqslant 2\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)+\overline{N}(r,0;F)+\overline{N}_{(2}(r,0;F)\\
&\quad +\overline{N}(r,0;G)+\overline{N}_{(2}(r,0;G)+\frac{1}{l}N(r,0;F)+\frac{1}{l}\overline{N}(r,\infty;F)\\
&\quad +T(r,G)+S(r,F)+S(r,G).
\end{aligned}$$

Since $\overline{N}(r,0;F)+\overline{N}_{(2}(r,0;F)\leqslant N(r,0;F)$, we have

$$\begin{aligned}
T(r,F) &\leqslant (1+\frac{1}{l})N(r,0;F)+(2+\frac{1}{l})\overline{N}(r,\infty;F)+N(r,0;G)+2\overline{N}(r,\infty;G)\\
&\quad +S(r,F)+S(r,G).
\end{aligned}$$

3. Proof of the main theorem

Proof of Theorem 11. Let

$$F=\frac{P[f]}{a},\ G=\frac{P[g]}{a}.$$

Since $P[f]$ and $P[g]$ share "(a,l)", it follows that F, G share "$(1,l)$" except at the zeros and poles of a.
Now we consider the following cases:
Case 1: $l=\infty$. By (i) of Theorem 9 we get the result.
Case 2: $0<l<\infty$. Suppose $H\not\equiv 0$. From Lemma 4 we get

$$\begin{aligned}T(r,F)\leqslant{}&(1+\frac{1}{l})N(r,0;F)+(2+\frac{1}{l})\overline{N}(r,\infty;F)+N(r,0;G)+2\overline{N}(r,\infty;G)\\&+S(r,F)+S(r,G).\end{aligned}$$

Using Lemma 1 we obtain

$$\begin{aligned}dT(r,f)\leqslant{}&\frac{Q+1+2l}{l}\overline{N}(r,\infty;f)+(2+Q)\overline{N}(r,\infty;g)+\frac{(l+1)d}{l}N(r,0;f)\\&+dN(r,0;g)+S(r,f)+S(r,g).\end{aligned}\tag{11}$$

Similarly,

$$\begin{aligned}dT(r,g)\leqslant{}&\frac{Q+1+2l}{l}\overline{N}(r,\infty;g)+(2+Q)\overline{N}(r,\infty;f)+\frac{(l+1)d}{l}N(r,0;g)\\&+dN(r,0;f)+S(r,f)+S(r,g).\end{aligned}\tag{12}$$

Adding (11) and (12) we get

$$\begin{aligned}&dT(r,f)+dT(r,g)\\\leqslant{}&\left(\frac{Q+1+2l}{l}+Q+2\right)\overline{N}(r,\infty;f)+\frac{(2l+1)d}{l}N(r,0;f)\\&+\left(\frac{Q+1+2l}{l}+Q+2\right)\overline{N}(r,\infty;g)+\frac{(2l+1)d}{l}N(r,0;g)+S(r,f)+S(r,g).\\&\left\{\frac{(2l+1)d}{l}\delta(0,f)+\left(\frac{Q+1+2l}{l}+Q+2\right)\Theta(\infty,f)-\frac{(l+1)d+Q+1}{l}-4-Q\right\}T(r,f)\\&+\left\{\frac{(2l+1)d}{l}\delta(0,g)+\left(\frac{Q+1+2l}{l}+Q+2\right)\Theta(\infty,g)-\frac{(l+1)d+Q+1}{l}-4-Q\right\}T(r,g)\\\leqslant{}&S(r,f)+S(r,g),\end{aligned}$$

which contradict assumption (4).
Thus $H\equiv 0$. That is

$$\left(\frac{F^{(2)}}{F^{(1)}}-2\frac{F^{(1)}}{F-1}\right)=\left(\frac{G^{(2)}}{G^{(1)}}-2\frac{G^{(1)}}{G-1}\right).$$

By integrating twice we get

$$\frac{1}{G-1}=\frac{A}{F-1}+B,$$

where $A\ (\neq 0)$ and B are constants.
Thus

$$G=\frac{(B+1)F+(A-B-1)}{BF+(A-B)} \tag{13}$$

and

$$F=\frac{(B-A)G+(A-B-1)}{BG-(B+1)}. \tag{14}$$

Next we consider following three subcases:
Subcase 2.1: $B\neq 0,-1$. Then from (14) we have

$$\overline{N}\left(r,\frac{B+1}{B};G\right)=\overline{N}(r,\infty;F).$$

By Nevanlinna second fundamental theorem and (ii) of Lemma 1 we get

$$\begin{aligned}T(r,G) &\leqslant \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}\left(r,\frac{B+1}{B};G\right)+S(r,G)\\ &\leqslant \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+S(r,G)\\ &\leqslant \overline{N}(r,\infty;G)+T(r,G)-dT(r,g)+dN(r,0;g)+\overline{N}(r,\infty;F)+S(r,G)\\ \Rightarrow dT(r,g) &\leqslant \overline{N}(r,\infty;f)+dN(r,0;g)+\overline{N}(r,\infty;g)+S(r,f)+S(r,g). \end{aligned}\tag{15}$$

If $A-B-1\neq 0$, then it follows from (13) that

$$N\left(r,\frac{-A+B+1}{B+1};F\right)=N(r,0;G).$$

Again by Nevanlinna second fundamental theorem and Lemma 1 we have

$$\begin{aligned}&T(r,F)\leqslant \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}\left(r,\frac{-A+B+1}{B+1};F\right)+S(r,F)\\ &\Rightarrow dT(r,f)\leqslant \overline{N}(r,\infty;f)+dN(r,0;f)+Q\overline{N}(r,\infty;g)+dN(r,0;g)+S(r,f)+S(r,g).\end{aligned}\tag{16}$$

Combining (15) and (16)

$$\begin{aligned}&T(r,f)+T(r,g)\\ &\leqslant N(r,0;f)+\frac{2}{d}\overline{N}(r,\infty;f)+2N(r,0;g)+\frac{Q+1}{d}\overline{N}(r,\infty;g)+S(r,f)+S(r,g),\\ &\left\{\delta(0,f)+\frac{2}{d}\Theta(\infty,f)-\frac{2}{d}\right\}T(r,f)+\left\{2\delta(0,g)+\frac{Q+1}{d}\Theta(\infty,g)-\frac{Q+d+1}{d}\right\}T(r,g)\\ &\leqslant S(r,f)+S(r,g),\end{aligned}$$

which contradict (4). Therefore $A-B-1=0$. Then by (13)

$$\overline{N}(r,0;F+\frac{1}{B})=\overline{N}(r,\infty;G).$$

Again by Nevanlinna second fundamental theorem

$$\begin{aligned}T(r,F)&\leqslant \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,0;F+\frac{1}{B})+S(r,F),\\&\leqslant \overline{N}(r,\infty;f)+T(r,F)-dT(r,f)+dN(r,0;f)+\overline{N}(r,\infty;g)+S(r,f)+S(r,g),\\ \Rightarrow dT(r,f)&\leqslant \overline{N}(r,\infty;f)+dN(r,0;f)+\overline{N}(r,\infty;g)+S(r,f)+S(r,g). \qquad (17)\end{aligned}$$

Combining (15) and (17)

$$T(r,f)+T(r,g)\leqslant N(r,0;f)+\frac{2}{d}\overline{N}(r,\infty;f)+N(r,0;g)+\frac{2}{d}\overline{N}(r,\infty;g)+S(r,f)+S(r,g),$$

$$\left\{\delta(0,f)+\frac{2}{d}\Theta(\infty,f)-\frac{2}{d}\right\}T(r,f)+\left\{\delta(0,g)+\frac{2}{d}\Theta(\infty,g)-\frac{2}{d}\right\}T(r,g)$$
$$\leqslant S(r,f)+S(r,g),$$

which violates assumption (4).
Subcase 2.2: $B=-1$. Then

$$G=\frac{A}{A+1-F}$$

and

$$F=\frac{(1+A)G-A}{G}.$$

If $A+1\neq 0$,

$$\overline{N}(r,A+1;F)=\overline{N}(r,\infty;G)$$

$$\overline{N}(r,\frac{A}{A+1};G)=\overline{N}(r,0;F).$$

By similar argument as Subcase 2.1 we have a contradiction.
Therefore $A+1=0$ then

$$FG=1\Rightarrow P[f].P[g]\equiv a^2.$$

Subcase 2.3: $B=0$. Then (13) and (14) gives $G=\frac{F+A-1}{A}$ and $F=AG+1-A$
If $A-1\neq 0$, $\overline{N}(r,0;A-1+F)=\overline{N}(r,0;G)$ and $\overline{N}(r,\frac{A-1}{A};G)=\overline{N}(r,0;F)$. Proceeding similarly as in Subcase 2.1 we get a contradiction.
Therefore, $A-1=0$ then $F\equiv G$ i.e.,

$$P[f]\equiv P[g].$$

Case 3: $l=0$. By (iii) of Theorem 9 we get the result.
This completes the proof.

For further study we should consider the following question: if we have $P[f] \equiv P[g]$, can we find more exact relation between f and g?

Acknowledgement. Authors would like to thanks the referees for their valuable comments and suggestions towards the improvement of this paper.

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(Received March 6, 2020)

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