Journal of Classical Analysis
Volume 16, Number 1 (2020), 37–44 doi:10.7153/jca-2020-16-05

GENERALIZATION OF ENESTRÖM–KAKEYA THEOREM AND ITS EXTENSION TO ANALYTIC FUNCTIONS

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Abstract. In this paper, by using standard techniques we shall obtain a result that gives regions containing all the zeros of a polynomial with real coefficients. Our result not only generalizes several well-known results concerning the location of zeros of polynomials but also yields an answer to a question raised by Professor N. K. Govil. We also obtain a similar result for analytic functions. In addition to this, we show by examples that our result gives better information about the bounds of zeros of polynomials than some known results.

1. Introduction and statement of results

Various experimental observations and investigations when translated into mathematical language lead to mathematical models. The solution of these models could lead to problems of solving algebraic polynomial equations of certain degree. The exact computation of zeros of polynomials of degree at most four made possible by virtue of algorithms having been devised for such polynomials, no such method is available for accomplishing the same task for polynomials of higher degree. The impossibility of achieving this feat, or in other words, the impossibility of solving by radicals the polynomial equations of degree 5 or greater is an important milestone in the history of mathematics, occasioned by ground breaking discoveries in algebra by N. H. Abel and E. Galois in the first quarter of the nineteenth century. This motivated the study of identifying suitable regions in the complex plane containing the zeros of a given polynomial. A classical result due to Cauchy [3] on the distribution of zeros of a polynomial may be stated as follows:

THEOREM A. *Let $P(z)=z^n+a_{n-1}z^{n-1}+\ldots+a_1z+a_0$ be a polynomial of degree n, then all the zeros of $P(z)$ lie in the disk $|z|\leqslant 1+\max_{0\leqslant j\leqslant n-1}|a_j|$.*

Although various results concerning the bounds for zeros of polynomials are available in literature [5], but the remarkable property of the bound in Theorem A which distinguishes it from other such bounds is its simplicity of computations. However, this simplicity comes at the cost of precision. The following elegant result on the location of zeros of a polynomial with restricted coefficients is known as Eneström-Kakeya Theorem (see [5], [6]) which states that:

Mathematics subject classification (2010): 30A10, 30C15.
Keywords and phrases: Polynomials, zeros, complex domain.

THEOREM B. *Let* $P(z) = a_nz^n + a_{n-1}z^{n-1} + \ldots + a_1z + a_0$ *be a polynomial of degree* n *with real coefficients satisfying* $a_n \geqslant a_{n-1} \geqslant \ldots \geqslant a_1 \geqslant a_0 \geqslant 0$, *then all the zeros of* $P(z)$ *lie in* $|z| \leqslant 1$.

Joyal, Labelle and Rahman [4] extended Theorem B to polynomials whose coefficients are monotonic but are not necessarily non-negative and proved that:

THEOREM C. *Let* $P(z) = a_nz^n + a_{n-1}z^{n-1} + \ldots + a_1z + a_0$ *be a polynomial of degree* n *with real coefficients satisfying* $a_n \geqslant a_{n-1} \geqslant \ldots \geqslant a_1 \geqslant a_0$, *then all the zeros of* $P(z)$ *lie in* $|z| \leqslant \frac{a_n - a_0 + |a_0|}{|a_n|}$.

Aziz and Zargar [2] relaxed the hypothesis in several ways and among other things they proved the following results:

THEOREM D. *Let* $P(z) = a_nz^n + a_{n-1}z^{n-1} + \ldots + a_1z + a_0$ *be a polynomial of degree* n *with real coefficients such that for some* $k \geqslant 1$, $ka_n \geqslant a_{n-1} \geqslant \ldots \geqslant a_1 \geqslant a_0$, *then all the zeros of* $P(z)$ *lie in* $|z + k - 1| \leqslant \frac{ka_n - a_0 + |a_0|}{|a_n|}$.

In literature there exist several generalisations of above results (for reference see [5], [6], [7] and [8]). In this paper, by using standard techniques we shall obtain a result which gives regions containing all the zeros of the polynomials with real coefficients. Our result generalizes several well-known results concerning the generalization of Eneström-Kakeya Theorem. More precisely, we prove:

THEOREM 1.1. *Let* $P(z) = a_nz^n + a_{n-1}z^{n-1} + \ldots + a_1z + a_0$ *be a polynomial of degree* n *with real coefficients such that for some* $k_j \geqslant 1$, $j = 1, 2, \ldots, r$ *where* $1 \leqslant r \leqslant n$,

$$k_1a_n \geqslant k_2a_{n-1} \geqslant k_3a_{n-2} \geqslant \ldots \geqslant k_ra_{n-r+1} \geqslant a_{n-r} \geqslant \ldots \geqslant a_1 \geqslant a_0,$$

then all the zeros of $P(z)$ *lie in*

$$\begin{aligned} &|z + k_1 - 1 - (k_2 - 1)a_{n-1}/a_n| \\ &\leqslant \frac{1}{|a_n|}\left(k_1a_n - (k_2 - 1)|a_{n-1}| + 2\sum_{j=2}^{r}(k_j - 1)|a_{n-j+1}| - a_0 + |a_0|\right). \end{aligned}$$

For $r = 2$, we obtain the following result which answers the question raised by Professor N. K. Govil regarding the determination of regions containing all the zeros of the polynomial at International conference held at the University of Jammu, India, in 2007.

COROLLARY 1.1. *Let* $P(z) = a_nz^n + a_{n-1}z^{n-1} + \ldots + a_1z + a_0$ *be a polynomial of degree* n *with real coefficients such that for some* $k_1 \geqslant 1, k_2 \geqslant 1$,

$$k_1a_n \geqslant k_2a_{n-1} \geqslant a_{n-2} \geqslant a_{n-3} \geqslant \ldots \geqslant a_1 \geqslant a_0,$$

then all the zeros of $P(z)$ *lie in*

$$|z + k_1 - 1 - (k_2 - 1)a_{n-1}/a_n| \leqslant \frac{1}{|a_n|}\big(k_1a_n + (k_2 - 1)|a_{n-1}| - a_0 + |a_0|\big).$$

REMARK 1.1. On setting $k_2 = 1$, Corollary 1.1 reduces to Theorem D and if $k_1 = k_2 = 1$, Corollary 1.1 yields Theorem C.

If we assume $a_0 \geqslant 0$ in Theorem 1.1, we get the following result:

COROLLARY 1.2. *Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ be a polynomial of degree n with real coefficients such that for some $k_j \geqslant 1$, $j = 1, 2, \ldots, r$ where $1 \leqslant r \leqslant n$,*

$$k_1 a_n \geqslant k_2 a_{n-1} \geqslant k_3 a_{n-2} \geqslant \ldots \geqslant k_r a_{n-r+1} \geqslant a_{n-r} \geqslant \ldots \geqslant a_1 \geqslant a_0 \geqslant 0,$$

then all the zeros of $P(z)$ lie in

$$|z + k_1 - 1 - (k_2 - 1) a_{n-1}/a_n| \leqslant \frac{1}{a_n}\left(k_1 a_n - (k_2 - 1) a_{n-1} + 2\sum_{j=2}^{r}(k_j - 1) a_{n-j+1}\right).$$

REMARK 1.2. In Corollary 1.2 if we take $k_j = 1, j = 1, 2, 3, \ldots, r$ then we obtain the famous Eneström-Kakeya Theorem stated in Theorem B.

Dynamical systems are frequently used as models for the temporal evolution of technical processes. In this case, the stability analysis for an equilibrium state of the system naturally leads to the problem of finding all the zeros or at least bounds on the set of zeros of a certain analytic function $f : \mathbb{C} \to \mathbb{C}$. For instance, if there is no temporal delay to be taken into account in the underlying system then one has to consider the spectrum of the Jacobian of a vector field evaluated at the equilibrium. This can e.g., be done by using efficient eigenvalue solvers. However, if the technical process has to be modelled by a delay differential equation then the zero set of a non-polynomial but holomorphic function has to be approximated.
Motivated by this application, we propose in this paper to study the zeros of a class of analytic functions. In this direction, we prove the following result:

THEOREM 1.2. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j \not\equiv 0$ be analytic in $|z| \leqslant 1$. If for some $k_1, k_2 \geqslant 1$,*

$$k_1 a_0 \geqslant k_2 a_1 \geqslant a_2 \geqslant a_3 \geqslant \ldots, \qquad a_j > 0,\ j = 1, 2, 3 \ldots,$$

then $f(z)$ does not vanish in the region

$$\left| z - \frac{(k_1 - 1) a_0 - (k_2 - 1) a_1}{(2k_1 - 1)(a_0 + 2(k_2 - 1) a_1)} \right| < \frac{k_1 a_0 + (k_2 - 1) a_1}{(2k_1 - 1)(a_0 + 2(k_2 - 1) a_1)}.$$

For $k_2 = 1$, Theorem 1.2 yields the following result:

COROLLARY 1.3. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j \not\equiv 0$ be analytic in $|z| \leqslant 1$. If for some $k \geqslant 1$,*

$$k a_0 \geqslant a_1 \geqslant a_2 \geqslant a_3 \geqslant \ldots, \qquad a_j > 0,\ j = 1, 2, 3 \ldots,$$

then $f(z)$ *does not vanish in the region*

$$\left|z-\frac{k-1}{2k-1}\right|<\frac{k}{2k-1}.$$

The above result was also proved by Aziz and Shah [1].

2. Computations and analysis

In this section, we give some examples of polynomials to show that Theorem 1.1 gives better information about the location of the zeros than Theorem A. It is worth mentioning that all existing Eneström-Kakeya type results are not applicable for these polynomials.

EXAMPLE 2.1. Let $P(z) = 19z^4 + 18z^3 + 20z^2 + 15z + 2$

Results	Radius of Disk	Area of Disk
Theorem A	2.053	13.2
Theorem 1.1	1.157	4.2
Actual Bound	0.94	2.78

By taking $r = 2$ with $k_1 = 20/19$ and $k_2 = 10/9$ in Theorem 1.1, it is evident from the above table that the Theorem 1.1 gives better bound, with 68% improvement in the area over Theorem A.

EXAMPLE 2.2. Let $P(z) = 3z^4 + 2.8z^3 + 2.6z^2 + 3.2z + 1$

Results	Radius of Disk	Area of Disk
Theorem A	2.06	13.32
Theorem 1.1	1.6	8.03
Actual Bound	0.97	2.96

By taking $r = 3$ with $k_1 = 16/15$, $k_2 = 8/7$ and $k_3 = 16/13$ in Theorem 1.1, it is evident from the above table that the Theorem 1.1 gives better bound, with 40% improvement in the area over Theorem A.

EXAMPLE 2.3. Let $P(z) = 4z^5 + 3.8z^4 + 3.5z^3 + 4z^2 + 2z + 1$

Results	Radius of Disk	Area of Disk
Theorem A	2	12.56
Theorem 1.1	1.3	5.3
Actual Bound	0.91	2.61

By taking $r = 3$ with $k_1 = 1$, $k_2 = 20/19$ and $k_3 = 8/7$ in Theorem 1.1, it is evident from the above table that the Theorem 1.1 gives better bound, with 58% improvement in the area over Theorem A.

3. Proof of the theorems

Proof of Theorem 1.1. Consider the polynomial

$$\begin{aligned}F(z)=&(1-z)P(z)\\ =&-a_nz^{n+1}+(a_n-a_{n-1})z^n+\ldots+(a_{n-r}-a_{n-r-1})z^{n-r}+\ldots+(a_1-a_0)z+a_0\\ =&-a_nz^{n+1}+(k_1a_n-k_2a_{n-1}-(k_1-1)a_n+(k_2-1)a_{n-1})z^n\\ &+(k_2a_{n-1}-k_3a_{n-2}-(k_2-1)a_{n-1}+(k_3-1)a_{n-2})z^{n-1}\\ &+\ldots+(k_{r-1}a_{n-r+2}-k_ra_{n-r+1}-(k_{r-1}-1)a_{n-r+2}+(k_r-1)a_{n-r+1})z^{n-r+2}\\ &+(k_ra_{n-r+1}-a_{n-r}-(k_r-1)a_{n-r+1})z^{n-r+1}+(a_{n-r}-a_{n-r-1})z^{n-r}+\ldots\\ &+(a_1-a_0)z+a_0,\end{aligned}$$

which implies,

$$\begin{aligned}|F(z)|=|&-a_nz^{n+1}-(k_1-1)a_nz^n+(k_1a_n-k_2a_{n-1})z^n+(k_2-1)a_{n-1}z^n\\ &+(k_2a_{n-1}-k_3a_{n-2})z^{n-1}-(k_2-1)a_{n-1}z^{n-1}+(k_3-1)a_{n-2}z^{n-1}+\ldots\\ &+(k_{r-1}a_{n-r+2}-k_ra_{n-r+1})z^{n-r+2}-(k_{r-1}-1)a_{n-r+2}z^{n-r+2}\\ &+(k_r-1)a_{n-r+1}z^{n-r+2}+(k_ra_{n-r+1}-a_{n-r})z^{n-r+1}-(k_r-1)a_{n-r+1}z^{n-r+1}\\ &+(a_{n-r}-a_{n-r-1})z^{n-r}+\ldots+(a_1-a_0)z+a_0|,\end{aligned}$$

that is,

$$\begin{aligned}|F(z)|\geqslant|z|^n\big[&|(z+k_1-1)a_n-(k_2-1)a_{n-1}|-\big(|k_1a_n-k_2a_{n-1}|+|k_2a_{n-1}-k_3a_{n-2}|/|z|\\ &+|k_2-1||a_{n-1}|/|z|+|k_3-1||a_{n-2}|/|z|+\ldots\\ &+|k_{r-1}a_{n-r+2}-k_ra_{n-r+1}|/|z|^{r-2}+|k_{r-1}-1||a_{n-r+2}|/|z|^{r-2}\\ &+|k_r-1||a_{n-r+1}|/|z|^{r-2}+|k_ra_{n-r+1}-a_{n-r}|/|z|^{r-1}\\ &+|k_r-1||a_{n-r+1}|/|z|^{r-1}+|a_{n-r}-a_{n-r-1}|/|z|^r+\ldots+|a_1-a_0|/|z|^{n-1}\\ &+|a_0|/|z|^n\big)\big].\end{aligned}$$

By using hypothesis, we have for $|z|>1$,

$$\begin{aligned}|F(z)|\geqslant|z|^n\big[&|(z+k_1-1)a_n-(k_2-1)a_{n-1}|-\big(k_1a_n-k_2a_{n-1}+k_2a_{n-1}-k_3a_{n-2}\\ &+(k_2-1)|a_{n-1}|+(k_3-1)|a_{n-2}|+\ldots+k_{r-1}a_{n-r+2}-k_ra_{n-r+1}\\ &+(k_{r-1}-1)|a_{n-r+2}|+(k_r-1)|a_{n-r+1}|+k_ra_{n-r+1}-a_{n-r}\\ &+(k_r-1)|a_{n-r+1}|+a_{n-r}-a_{n-r-1}+\ldots+a_1-a_0+|a_0|\big)\big],\end{aligned}$$

implies,

$$\begin{aligned}|F(z)|\geqslant|a_n||z|^n\Bigg[&|z+k_1-1-(k_2-1)a_{n-1}/a_n|-\frac{1}{|a_n|}\Bigg(k_1a_n-(k_2-1)|a_{n-1}|\\ &+2\sum_{j=2}^{r}(k_j-1)|a_{n-j+1}|-a_0+|a_0|\Bigg)\Bigg]>0\end{aligned}$$

if

$$\begin{aligned}&|z+k_1-1-(k_2-1)a_{n-1}/a_n|\\ &>\frac{1}{|a_n|}\left(k_1a_n-(k_2-1)|a_{n-1}|+2\sum_{j=2}^{r}(k_j-1)|a_{n-j+1}|-a_0+|a_0|\right).\end{aligned}$$

This shows that those zeros of $F(z)$ whose modulus is greater than 1 lie in

$$\begin{aligned}&|z+k_1-1-(k_2-1)a_{n-1}/a_n|\\ &\leqslant\frac{1}{|a_n|}\left(k_1a_n-(k_2-1)|a_{n-1}|+2\sum_{j=2}^{r}(k_j-1)|a_{n-j+1}|-a_0+|a_0|\right).\end{aligned}$$

But those zeros of $F(z)$ whose modulus is less than or equal to 1 already lie in this region. Hence it follows that all the zeros of $F(z)$ and therefore of $P(z)$ lie in

$$\begin{aligned}&|z+k_1-1-(k_2-1)a_{n-1}/a_n|\\ &\leqslant\frac{1}{|a_n|}\left(k_1a_n-(k_2-1)|a_{n-1}|+2\sum_{j=2}^{r}(k_j-1)|a_{n-j+1}|-a_0+|a_0|\right).\end{aligned}$$

This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Since $f(z)=\sum_{j=0}^{\infty}a_jz^j$ is analytic in $|z|\leqslant 1$ and it is easy to observe that $\lim_{j\to\infty}a_j=0$. Now consider the function

$$\begin{aligned}F(z)=&(z-1)f(z)\\ =&(z-1)\left(a_0+a_1z+a_2z^2+a_3z^3+\dots\right)\\ =&-a_0+(a_0-a_1)z+(a_1-a_2)z^2+(a_2-a_3)z^3+\dots\\ =&-a_0+(k_1a_0-k_2a_1)z+((1-k_1)a_0+(k_2-1)a_1)z+((k_2a_1-a_2)-(k_2-1)a_1)z^2\\ &+(a_2-a_3)z^3+\dots\\ =&-a_0-((k_1-1)a_0-(k_2-1)a_1)z+\phi(z),\end{aligned}$$

where $\phi(z)=(k_1a_0-k_2a_1)z+((k_2a_1-a_2)-(k_2-1)a_1)z^2+\sum_{j=3}^{\infty}(a_{j-1}-a_j)z^j$. Clearly $\phi(z)$ is analytic for $|z|\leqslant 1$ with $\phi(0)=0$. Moreover, for $|z|=1$

$$\begin{aligned}|\phi(z)|\leqslant&|k_1a_0-k_2a_1||z|+|(k_2a_1-a_2)-(k_2-1)a_1||z|^2+\sum_{j=3}^{\infty}|a_{j-1}-a_j||z|^j\\ \leqslant&k_1a_0-k_2a_1+k_2a_1-a_2+(k_2-1)a_1+a_2-a_3+a_3-a_4+\dots\\ =&k_1a_0+(k_2-1)a_1.\end{aligned}$$

Therefore, by the Schwarz Lemma,

$$|\phi(z)|\leqslant(k_1a_0+(k_2-1)a_1)|z|\qquad\text{for}\quad|z|<1.$$

Hence, for $|z|<1$,

$$|F(z)| \geqslant |a_0+((k_1-1)a_0-(k_2-1)a_1)z|-|k_1a_0+(k_2-1)a_1||z|>0$$

if

$$|((k_1-1)a_0-(k_2-1)a_1)z+a_0|>|k_1a_0+(k_2-1)a_1||z|,$$

that is, $F(z)$ and therefore $f(z)$ does not vanish in

$$(k_1a_0+(k_2-1)a_1)|z|<|((k_1-1)a_0-(k_2-1)a_1)z+a_0|,$$

which is precisely the region

$$\left|z-\frac{(k_1-1)a_0-(k_2-1)a_1}{(2k_1-1)(a_0+2(k_2-1)a_1)}\right|<\frac{k_1a_0+(k_2-1)a_1}{(2k_1-1)(a_0+2(k_2-1)a_1)}.$$

That completes the proof of Theorem 1.2.

4. Concluding remarks

Applying Theorem 1.1 to the polynomial $P(tz)$, we obtain the following result:

COROLLARY 4.1. *Let* $P(z)=a_nz^n+a_{n-1}z^{n-1}+\ldots+a_1z+a_0$ *be a polynomial of degree* n *with real coefficients such that for some* $t>0$ *and* $k_j\geqslant 1,\ j=1,2,\ldots,r$ *where* $1\leqslant r\leqslant n$,

$$k_1t^na_n\geqslant k_2t^{n-1}a_{n-1}\geqslant k_3t^{n-2}a_{n-2}\geqslant\ldots\geqslant k_ra_{n-r+1}t^{n-r+1}\geqslant a_{n-r}t^{n-r}\geqslant\ldots\geqslant a_1t\geqslant a_0,$$

then all the zeros of $P(z)$ *lie in*

$$|z+(k_1-1)t-(k_2-1)a_{n-1}/a_n| \leqslant\frac{1}{|a_n|}\left(k_1ta_n-(k_2-1)|a_{n-1}|+2\sum_{j=2}^{r}(k_j-1)\frac{|a_{n-j+1}|}{t^{j-2}}+\frac{|a_0|-a_0}{t^{n-1}}\right).$$

Taking $r=2$ and $a_0\geqslant 0$ in the above Corollary, we obtain the following result:

COROLLARY 4.2. *Let* $P(z)=a_nz^n+a_{n-1}z^{n-1}+\ldots+a_1z+a_0$ *be a polynomial of degree* n *with real coefficients such that for some* $k_1\geqslant 1,\ k_2\geqslant 1$ *and* $t>0$,

$$k_1t^na_n\geqslant k_2t^{n-1}a_{n-1}\geqslant t^{n-2}a_{n-2}\geqslant\ldots\geqslant a_1t\geqslant a_0\geqslant 0,$$

then all the zeros of $P(z)$ *lie in*

$$|z+(k_1-1)t-(k_2-1)a_{n-1}/a_n|\leqslant k_1t+\frac{(k_2-1)a_{n-1}}{a_n}.$$

Similarly applying Theorem 1.2 to the function $f(tz)$, we obtain the following result:

COROLLARY 4.3. *Let* $f(z) = \sum_{j=0}^{\infty} a_j z^j \not\equiv 0$ *be analytic in* $|z| \leqslant t$. *If for some* $k_1, k_2 \geqslant 1$,

$$k_1 a_0 \geqslant k_2 t a_1 \geqslant t^2 a_2 \geqslant t^3 a_3 \geqslant \ldots, \qquad a_j > 0,\ j = 1, 2, 3 \ldots,$$

then $f(z)$ *does not vanish in the region*

$$\left| z - \frac{(k_1-1)a_0 t - (k_2-1)a_1 t^2}{(2k_1-1)(a_0+2(k_2-1)a_1 t)} \right| < \frac{k_1 a_0 t + (k_2-1)a_1 t^2}{(2k_1-1)(a_0+2(k_2-1)a_1 t)}.$$

For different choices of t, k_1 and k_2 several interesting results can be obtained.

Acknowledgement. We are grateful to the referees for their valuable suggestions, which has certainly enhanced the presentation of this paper.

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(Received March 25, 2020)

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