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ON NORMAL FUNCTIONS IN SEVERAL COMPLEX VARIABLES

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Abstract. In this paper, we generalize the conception of φ-normal to holomorphic functions of several complex variables. Extensions of some classical criteria for normality of holomorphic functions of several complex variables are also given.

1. Introduction

Let $\mathbb{D} = \{z; |z| < 1\}$ be the unit disc in the complex plane $\mathbb{C}$. A meromorphic function f in $\mathbb{D}$ is called normal if

$$\sup_{z\in\mathbb{D}}(1-|z|^2)f^\#(z) < \infty,$$

where $f^\#(z) = |f'(z)|/(1+|f(z)|^2)$ is the spherical derivative of f. Lappan [6] showed that *there exists a set E consisting of five distinct points such that if f is a meromorphic in $\mathbb{D}$ then the condition that* $\sup_{z\in f^{-1}(E)}(1-|z|^2)f^\#(z) < \infty$ *implies that* $\sup_{z\in\mathbb{D}}(1-|z|^2)f^\#(z) < \infty$ *i.e., f is a normal function.* This well-known result of Lappan is called five-point theorem. For a meromorphic function f in $\mathbb{D}$ and a positive integer k the expression $|f^{(k)}(z)|/(1+|f(z)|^{k+1})$ is an extension of the spherical derivative of f. For this expression involves higher derivatives, some interesting results related to normal functions were obtained.

THEOREM A. *([5]) If f is a normal function in $\mathbb{D}$, then for each integer $k > 0$,*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^k\frac{|f^{(k)}(z)|}{1+|f(z)|^{k+1}} < \infty.$$

THEOREM B. *([13]) Let k be a positive integer, and let f be a meromorphic function in $\mathbb{D}$, and suppose that there exists $M > 0$ such that* $\max_{1\leqslant i\leqslant k-1}|f^{(i)}(z)| \leqslant M$ *whenever $f(z) = 0$. If there exists a subset E of $\mathbb{C}\cup\{\infty\}$ containing at least $k+4$ distinct points such that*

$$\sup_{z\in f^{-1}(E)}(1-|z|^2)^k\frac{|f^{(k)}(z)|}{1+|f(z)|^{k+1}} < \infty,$$

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then f is a normal function.

In [1], R. Aulaskari and J. Rättyä introduced the concept of smoothly increasing functions and enlarged the class of normal functions. An increasing function $\varphi : [0,1) \to (0,\infty)$ is called smoothly increasing if

$$\varphi(r)(1-r) \to \infty, \quad \text{as} \quad r \to 1^-, \tag{1}$$

and

$$\mathscr{R}_a(z) := \frac{\varphi(|a+z/\varphi(|a|)|)}{\varphi(|a|)} \to 1 \quad \text{as} \quad |a| \to 1^- \tag{2}$$

uniformly on compact subsets of $\mathbb{C}$. For a given such φ, we call a function f is φ-normal if f is meromorphic in $\mathbb{D}$, and

$$\sup_{z\in\mathbb{D}} \frac{f^{\#}(z)}{\varphi(|z|)} < \infty.$$

Applying Nevanlinna theory of meromorphic functions, Xu and Qiu [14] improved Theorems A and B and establish analogues for φ-normal functions. In [12], condition (1) was replaced by a weaker one as

$$\varphi(r)(1-r) \geqslant 1, r \in [0,1). \tag{3}$$

So the function $\varphi_0(r) = \frac{1}{1-r}$ is smoothly increasing and the concept of φ_0-normal functions coincides with the concept of normal functions. In addition, the authors in [12] obtained the four-point theorem on the φ-normal criteria for meromorphic functions via bounding some quantities related to spherical derivatives of f and f'.

THEOREM C. ([12]) *Let φ: $[0,1) \to (0,\infty)$ be a smoothly increasing function, and let f be a meromorphic function in $\mathbb{D}$. Assume that there is a subset $E := \{a_1, a_2, a_3, a_4\} \subset \mathbb{C} \cup \{\infty\}$ such that*

$$\sup_{z\in f^{-1}(E)} \frac{f^{\#}(z)}{\varphi(|z|)} < \infty, \quad \textit{and} \quad \sup_{z\in f^{-1}(E\setminus\{\infty\})} (f')^{\#}(z) < \infty.$$

Then f is a φ-normal function.

2. Preliminaries and results

To state our main results, we first introduce some standard notations. Let

$$\mathbb{C}^n = \{z = (z_1, \cdots, z_n);\ z_1, \cdots, z_n \in \mathbb{C}\}$$

be the complex space of dimension n.

Denote the unit ball with respect norm $\|\cdot\|$ in $\mathbb{C}^n$ to by $\mathbb{B}_n = \{z \in \mathbb{C}^n;\ \|z\| < 1\}$. The boundary of $\mathbb{B}_n$ will be denoted by $\mathbb{S}_n$ and is called the unit sphere in $\mathbb{C}^n$. Thus $\mathbb{S}_n = \{z \in \mathbb{C}^n;\ \|z\| = 1\}$.

Let $\Omega \subset \mathbb{C}^n$ be a domain and $\mathscr{H}(\Omega)$ the collection of all holomorphic functions in Ω. Let

$$\nabla f(z) = (f_{z_1}(z), \ldots, f_{z_n}(z)),\ z \in \Omega$$

where $f_{z_i}(z) = \frac{\partial f}{\partial z_i}(z)$.

For every function F of class $\mathscr{C}^2(\Omega)$, we define at each point $z \in \Omega$ a Hermitian form

$$L_z(F, v) = \sum_{i,j=1}^{n} \frac{\partial^2 F(z)}{\partial z_i \partial \overline{z}_j} v_i \overline{v}_j$$

and call it the Levi form of the function F at z. For a holomorphic function f in Ω, set

$$f^{\#}(z) = \sup_{|v|=1} \Big(L_z\big(\log(1+|f|^2), v\big)\Big)^{\frac{1}{2}},$$

where $v = (v_1, \cdots, v_n) \in \mathbb{C}^n, |v| = (\sum_{j=1}^{n} |v_j|^2)^{\frac{1}{2}}$.

REMARK 1. Let $\Omega \subset \mathbb{C}^n$ be a domain and $f \in \mathscr{H}(\Omega)$. Then

$$f^{\#}(z) = \sup_{|v|=1} \frac{|\langle \nabla f(z), \overline{v}\rangle|}{1+|f(z)|^2} = \frac{(\sum_{j=1}^{n} |f_{z_j}(z)|^2)^{\frac{1}{2}}}{1+|f(z)|^2},\ \ z \in \Omega.$$

where $\langle z, w\rangle = \sum_{j=1}^{n} z_j \overline{w_j}$ is the Hermitian scalar product for $z = (z_1, \ldots, z_n), w = (w_1, \ldots, w_n) \in \mathbb{C}^n$.

Proof. Since $f(z) = f(z_1, \ldots, z_n)$ is holomorphic on Ω, we have $f = f_{\overline{z}_j}(z) = 0$ for every $z \in \Omega$ and $1 \leqslant j \leqslant n$. An easy computation shows that

$$\frac{\partial}{\partial \overline{z}_j} \log(1+|f|^2)(z) = \frac{f(z)\overline{f}_{\overline{z}_j}(z)}{1+|f(z)|^2} = \frac{f(z)\overline{(f_{z_j}(z))}}{1+|f(z)|^2}$$

and

$$\frac{\partial^2}{\partial z_i \partial \overline{z}_j} \log(1+|f|^2)(z) = \frac{f f_{\overline{z}_j}}{1+|f(z)|^2} = \frac{f_{z_i}(z)\overline{(f_{z_j}(z))}}{(1+|f(z)|^2)^2}$$

for $z \in \Omega$ and $1 \leqslant i, j \leqslant n$. Hence, for each $v = (v_1, \cdots, v_n) \in \mathbb{C}^n$, we get

$$\sum_{i,j=1}^{n} \frac{\partial^2 \log(1+|f|^2)(z)}{\partial z_i \partial \overline{z}_j} v_i \overline{v}_j = \sum_{i,j=1}^{n} \frac{f_{z_i}(z) v_i \overline{(f_{z_j}(z) v_j)}}{(1+|f(z)|^2)^2} = \frac{1}{(1+|f(z)|^2)^2} |\sum_{k=1}^{n} f_{z_k}(z) v_k|^2$$
$$= \frac{|\langle \nabla f(z), \overline{v}\rangle|^2}{(1+|f(z)|^2)^2}.$$

This shows that

$$f^{\#}(z) = \sup_{|v|=1} \frac{|\langle \nabla f(z), \overline{v}\rangle|}{1+|f(z)|^2}.$$

Now we prove the second identity. If $\nabla f(z) = (0,0,\cdots,0)$, there is nothing to prove. We assume that $\nabla f(z) \neq (0,0,\cdots,0)$. We have, from Cauchy-Buniakowsky-Schwarz inequality, that $|\langle \nabla f(z), \overline{v} \rangle| \leqslant |\nabla f(z)| \cdot |\overline{v}|$. On the other hand, fix any $z \in \Omega$, we take

$$v^* = \Big(\frac{f_{z_1}(z)}{|\nabla f(z)|}, \frac{f_{z_2}(z)}{|\nabla f(z)|}, \cdots, \frac{f_{z_n}(z)}{|\nabla f(z)|} \Big)$$

It is obvious that $v^* \in \mathbb{C}^n, |v^*| = 1$ and

$$|\langle \nabla f(z), \overline{v^*} \rangle| = \sum_{i=1}^{n} f_{z_i}(z) \cdot \frac{\overline{f}_{z_i}(z)}{|\nabla f(z)|} = |\nabla f(z)|.$$

This leads that $\sup_{|v|=1} |\langle \nabla f(z), \overline{v} \rangle| = |\nabla f(z)|$. And hence,

$$\sup_{|v|=1} \frac{|\langle \nabla f(z), \overline{v} \rangle|}{1 + |f(z)|^2} = \frac{(\sum_{j=1}^{n} |f_{z_j}(z)|^2)^{\frac{1}{2}}}{1 + |f(z)|^2}, \ \ z \in \Omega.$$

We have completed the proof of Remark 1.

Let $I = (i_1, \ldots, i_n) \in \mathbb{N}^n$. We call I a multi-index and define $|I| = \sum_{\mu=1}^{n} i_\mu$. For $z \in \mathbb{C}^n$ and a multi-index I we define the partial derivative operators

$$D^I := \frac{\partial^{|I|}}{\partial z_1^{i_1} \cdots \partial z_n^{i_n}}.$$

Now we extend the concepts of smoothly increasing functions and φ-normal functions to the case of several complex variables.

DEFINITION 1. An increasing function $\varphi : [0,1) \to (0,\infty)$ is called smoothly increasing if

$$\varphi(r)(1-r) \geqslant 1, \qquad r \in [0,1), \tag{4}$$

and

$$\mathscr{R}_a(z) := \frac{\varphi(\|a + z/\varphi(\|a\|)\|)}{\varphi(\|a\|)} \to 1 \quad \text{as} \quad \|a\| \to 1^- \tag{5}$$

uniformly on compact subsets of $\mathbb{C}^n$.

DEFINITION 2. For a smoothly increasing function φ, a function $f \in \mathscr{H}(\mathbb{B}_n)$ is called φ-normal if

$$||f||_{\mathscr{N}^\varphi} := \sup_{z \in \mathbb{B}_n} \frac{f^\#(z)}{\varphi(\|z\|)} < \infty. \tag{6}$$

The class of all φ-normal functions is denoted by $\mathscr{N}^\varphi(\mathbb{B}_n)$.

We generalize Theorems A, B and C to holomorphic functions of several complex variables. More precisely, we have the following results.

THEOREM 1. *If f is a φ-normal function in $\mathbb{B}_n$, then, for each multi-index $I = (i_1, \dots, i_n)$, there exists a constant $M_I > 0$ such that*

$$\frac{1}{\varphi(\|z\|)^k} \frac{|D^I f(z)|}{1+|f(z)|^{k+1}} \leqslant M_I, \; z \in \mathbb{B}_n,$$

where $k = |I|$.

THEOREM 2. *Let φ be a smoothly increasing function, and let k be a positive integer. Suppose $f \in \mathscr{H}(\mathbb{B}_n)$ such that*

$$\sup\{|D^J f(z)|;\; z \in f^{-1}(\{0\}), J \in \mathbb{N}^n,\; 1 \leqslant |J| \leqslant k-1\} < \infty. \tag{7}$$

If there exists a set E of three distinct points in $\mathbb{C}$ such that

$$\sup\left\{\frac{1}{\varphi(\|z\|)^k} \frac{|D^I f(z)|}{1+|f(z)|^{k+1}};\; z \in f^{-1}(E)\right\} < \infty, \;\; I \in \mathbb{N}^n \text{ with } |I| = k.$$

Then f is φ-normal.

We notice that the number of points in E has nothing to do with k which is related to the order of the derivatives. In particular, when $k = 1$, we get the following corollary:

COROLLARY 1. *Let φ be a smoothly increasing function and $f \in \mathscr{H}(\mathbb{B}_n)$. If there exists a set E of three distinct points in $\mathbb{C}$ such that*

$$\sup_{z \in f^{-1}(E)} \frac{f^{\#}(z)}{\varphi(||z||)} < \infty.$$

Then f is φ-normal.

THEOREM 3. *Let φ be a smoothly increasing function and $f \in \mathscr{H}(\mathbb{B}_n)$. If there exists a subset E of $\mathbb{C}$ containing two distinct points such that*

$$\sup_{z \in f^{-1}(E)} \frac{f^{\#}(z)}{\varphi(||z||)} < \infty \text{ and } \sup_{z \in f^{-1}(E)} \left(\frac{\partial f}{\partial z_i}\right)^{\#}(z) < \infty, \quad 1 \leqslant i \leqslant n.$$

Then f is φ-normal.

3. Proof of Theorem 1

The theory of normal family is used to prove our main results. For the relationship between normal family and normal function, see [8].

DEFINITION 3. A family $\mathscr{F}$ of holomorphic functions on $\Omega \subset \mathbb{C}^n$ is normal in Ω if every sequence of functions $\{f_\mu\} \subseteq \mathscr{F}$ contains either a subsequence which converges to a limit function $f \not\equiv \infty$ uniformly on each compact subset of Ω, or a subsequence which converges uniformly to ∞ on each compact subset.

LEMMA 1. *([3]) A family $\mathscr{F}$ of functions holomorphic on $\Omega \subset \mathbb{C}^n$ is normal on Ω if and only if for each compact subset $K \subset \Omega$ there exists a constant $M(K) > 0$ such that at each point $z \in K$, $f^{\#}(z) \leqslant M(K)$ for all $f \in \mathscr{F}$.*

LEMMA 2. *Let φ be a smoothly increasing function and $f \in \mathscr{H}(\mathbb{B}_n)$. Then $f \in \mathscr{N}^{\varphi}(\mathbb{B}_n)$ if and only if for every sequence $\{a_\mu\} \subset \mathbb{B}_n$ with $\|a_\mu\| \to 1$, the family*

$$\mathscr{F} = \Big\{g_\mu(z) := f\Big(a_\mu + \frac{1}{\varphi(||a_\mu||)}z\Big);\ \mu = 1,2,\dots\Big\}$$

is normal in $\mathbb{B}_n$.

Proof. Since φ is a smoothly increasing function, from (4) we have $\frac{1}{\varphi(\|a_\mu\|)} \leqslant 1 - \|a_\mu\|$ for $\mu = 1,2,3,\dots$. Thus, for each $z \in \mathbb{B}_n$,

$$\|a_\mu + \frac{1}{\varphi(||a_\mu||)}z\| \leqslant \|a_\mu\| + \frac{\|z\|}{\varphi(||a_\mu||)} < \|a_\mu\| + \frac{1}{\varphi(||a_\mu||)} \leqslant 1.$$

Then $g_\mu(z) := f\big(a_\mu + \frac{1}{\varphi(\|a_\mu\|)}z\big)$ is well-defined and holomorphic on $\mathbb{B}_n$.

Suppose that $f \in \mathscr{N}^{\varphi}(\mathbb{B}_n)$. Then $||f||_{\mathscr{N}^\varphi} < \infty$. An easy computation shows that

$$\begin{aligned} g_\mu^{\#}(z) &= \frac{1}{\varphi(\|a_\mu\|)} f^{\#}\Big(a_\mu + \frac{z}{\varphi(\|a_\mu\|)}\Big) = \frac{\varphi(\|a_\mu + \frac{z}{\varphi(\|a_\mu\|)}\|)}{\varphi(\|a_\mu\|)} \cdot \frac{f^{\#}(a_\mu + \frac{z}{\varphi(\|a_\mu\|)})}{\varphi(\|a_\mu + \frac{z}{\varphi(\|a_\mu\|)}\|)} \\ &\leqslant \frac{\varphi(\|a_\mu + \frac{z}{\varphi(\|a_\mu\|)}\|)}{\varphi(\|a_\mu\|)} \cdot \| f \|_{\mathscr{N}^\varphi} = \mathscr{R}_{a_\mu}(z) \cdot \| f \|_{\mathscr{N}^\varphi} \end{aligned}$$

Together with (5), this implies that $\{g_\mu(z)\}$ is bounded uniformly on compact subsets of $\mathbb{B}^n$. Hence, it follows from Lemma 1 that $\{g_\mu(z)\}$ is a normal family in $\mathbb{B}_n$.

Conversely assume, to the contrary, that $f \notin \mathscr{N}^{\varphi}(\mathbb{B}_n)$. Then by (2), there exist $\{b_\mu\} \subset \mathbb{B}_n$ with $\|b_\mu\| \to 1$, such that

$$\lim_{\mu\to\infty} \frac{f^{\#}(b_\mu)}{\varphi(\|b_\mu\|)} = \infty. \tag{8}$$

Now, we investigate the family

$$\mathscr{F} = \Big\{g_\mu(z) := f\Big(b_\mu + \frac{1}{\varphi(||b_\mu||)}z\Big);\ \mu = 1,2,\dots\Big\}.$$

It follows from (8) that

$$g_\mu^{\#}(0) = \frac{f^{\#}(b_\mu)}{\varphi(\|b_\mu\|)} \to \infty$$

as $\mu \to \infty$. Because of Lemma 1, we get the family $\{g_\mu(z) = f\big(b_\mu + \frac{1}{\varphi(\|b_\mu\|)}z\big);\ \mu = 1,2,\dots\}$ is not normal in $\mathbb{B}_n$.

REMARK 2. If the function φ satisfies condition (1) instead of (4), that is $\lim_{r\to 1^-}\varphi(r)(1-r)=\infty$, we have a similar fact. Take $R>0$. It follows from $\|a_\mu\|\to 1^-$ that $\frac{1}{\varphi(\|a_\mu\|)}<\frac{1-\|a_\mu\|}{R}$ for sufficiently large μ. Thus, we have

$$\|a_\mu+\frac{1}{\varphi(||a_\mu||)}z\|\leqslant\|a_\mu\|+\frac{\|z\|}{\varphi(||a_\mu||)}\leqslant\|a_\mu\|+\frac{R}{\varphi(\|a_\mu\|)}<1$$

for $\|z\|\leqslant R$ and sufficiently large μ. Therefore, for each compact set K in $\mathbb{C}^n$, $g_\mu(z):=f\big(a_\mu+\frac{1}{\varphi(\|a_\mu\|)}z\big)$ is well-defined and holomorphic on K for sufficiently large μ. By the proof of Lemma 2, we obtain $f\in\mathscr{N}^\varphi(\mathbb{B}_n)$ *if and only if for every sequence* $\{a_\mu\}\subset\mathbb{B}_n$ *with* $\|a_\mu\|\to 1$, *the family* $\{g_\mu(z):=f\big(a_\mu+\frac{1}{\varphi(||a_\mu||)}z\big);\ \mu=1,2,\dots\}$ *is normal in* $\mathbb{C}^n$.

Proof of Theorem 1. If $k=1$, there really isn't anything to do once we notice the definition of f as a φ-normal function. It suffices to prove the theorem in the case where $k\geqslant 2$. Suppose the conclusion is not valid, then there exists a sequence $\{z_\mu\}\subset\mathbb{B}_n$, such that

$$\frac{1}{\varphi(\|z_\mu\|)^k}\frac{|D^I f(z_\mu)|}{1+|f(z_\mu)|^{k+1}}\to\infty,\quad \mu\to\infty. \tag{9}$$

Since f is φ-normal in $\mathbb{B}_n$, by Lemma 2, we get

$$\Big\{g_\mu(z)=f\Big(z_\mu+\frac{1}{\varphi(\|z_\mu\|)}z\Big),\ z\in\mathbb{B}_n\Big\}$$

is a normal family. Then, for each sequence $\{g_\mu\}$, in view of Definition 3, there exists a subsequence of $\{g_\mu\}$ (without loss of generality, we still denote by $\{g_\mu\}$ for convenience) which either converges locally uniformly to holomorphic function $g(z)$ or tends locally uniformly to infinity in $\mathbb{B}_n$.

We distinguish two cases.

Case 1. $g(z)\in\mathscr{H}(\mathbb{B}_n)$.

Then $g(z)$ is holomorphic in $\mathbb{B}_{r_0}=\{z:||z||<r_0\}$, where $0<r_0<1$. Weierstrass Theorem of several complex variables (see [11], p.16) implies that

$$D^I g_\mu(z)\to D^I g(z),\quad z\in\mathbb{B}_{r_0}.$$

Then, we have

$$\frac{|D^I g_\mu(z)|}{1+|g_\mu(z)|^{k+1}}\to\frac{|D^I g(z)|}{1+|g(z)|^{k+1}},\quad z\in\mathbb{B}_{r_0}.$$

Since g is holomorphic, then $|g(z)|$ and $|D^I g(z)|$ is bounded in $\overline{\mathbb{B}}_{r_0}=\{z:||z||\leqslant r_0\}$, obviously, there exists $Q>0$ such that

$$\max_{z\in\overline{\mathbb{B}}_{r_0}}\frac{|D^I g(z)|}{1+|g(z)|^{k+1}}\leqslant Q.$$

Then, for sufficiently large μ, we obtain

$$\max_{z\in\overline{\mathbb{B}}_{r_0}} \frac{|D^I g_\mu(z)|}{1+|g_\mu(z)|^{k+1}} \leqslant Q+1.$$

In particular, for sufficiently large μ, taking $z=0$, we get

$$\frac{|D^I g_\mu(0)|}{1+|g_\mu(0)|^{k+1}} = \frac{1}{\varphi(\|z_\mu\|)^k} \frac{|D^I f(z_\mu)|}{1+|f(z_\mu)|^{k+1}} \leqslant Q+1.$$

we get a contradiction with (9).

Case 2. $g(z)\equiv\infty$.

Then $\frac{1}{g}\equiv 0$ in $\mathbb{B}_n$. For sufficiently large μ, $\frac{1}{g_\mu}$ is holomorphic and $\frac{1}{g_\mu}\to 0$ in $\mathbb{B}_n$. Next we prove that $\frac{D^I g_\mu}{g_\mu^{k+1}}\to 0$ by using induction on $k=|I|$.

If $k=1$, set $D^I=\frac{\partial}{\partial z_i}$ for some $i\in\{1,2,\cdots,n\}$, we deduce that $\frac{1}{g_\mu^2}\cdot\frac{\partial g_\mu}{\partial z_i} = -\frac{\partial \frac{1}{g_\mu}}{\partial z_i}\to 0$, $i=1,2,\cdots,n$, $z\in\mathbb{B}_n$.

By the induction principle, we have to prove that $\frac{D^I g_\mu}{g_\mu^{k+1}}\to 0$ when $|I|=m$ under the induction hypothesis that $\frac{D^I g_\mu}{g_\mu^{k+1}}\to 0$ when $1\leqslant|k|\leqslant m-1$. It is easy to check that for each I with $|I|=m$,

$$\frac{D^I g_\mu}{g_\mu^{m+1}} = -\frac{D^I(\frac{1}{g_\mu})}{g_\mu^{m-1}} + \text{a polynomial of } \frac{D^J g_\mu}{g_\mu^{|J|+1}},\ 1\leqslant|J|\leqslant m-1.$$

Hence $\frac{D^I g_\mu}{g_\mu^{m+1}}\to 0$, $z\in\mathbb{B}_n$, $|I|=m$.

Obviously,

$$\frac{|D^I g_\mu(z)|}{1+|g_\mu(z)|^{k+1}} \leqslant \frac{|D^I g_\mu(z)|}{|g_\mu(z)|^{k+1}} \to 0,\ z\in\mathbb{B}_n.$$

Taking $z=0$, we obtain

$$\frac{|D^I g_\mu(0)|}{1+|g_\mu(0)|^{k+1}} = \frac{1}{\varphi(\|z_\mu\|)^k} \frac{|D^I f(z_\mu)|}{1+|f(z_\mu)|^{k+1}} \to 0,$$

which is also a contradiction with (9).

4. Proof of Theorem 2

Zalcman's Rescalling Lemma in several complex variables plays an important role in the proofs of Theorems 2 and 3 .

LEMMA 3. *([4]) Suppose that a family* $\mathscr{F}$ *of functions holomorphic on* $\Omega \subset \mathbb{C}^n$ *is not normal at some point* $z_0 \in \Omega$. *Then there exist sequences* $\{f_\mu\} \in \mathscr{F}$, $z_\mu \to z_0$, $\rho_\mu = 1/f_\mu^\#(z_\mu) \to 0$ *such that the sequence*

$$g_\mu(z) = f_\mu(z_\mu + \rho_\mu z)$$

converges locally uniformly in $\mathbb{C}^n$ *to a non-constant entire function* g *satisfying* $g^\#(z) \leqslant g^\#(0) = 1$.

LEMMA 4. *([9, 10]) Let* $\Omega \subseteq \mathbb{C}^n$ *be open. Let* f_j *be holomorphic functions on* Ω *for* $j = 1, 2, \cdots, n$. *Suppose that* f *is holomorphic on* Ω *and that* $f_j \to f$ *normally. If each* f_j *is zero-free, then prove that either* f *is zero-free or* $f \equiv 0$ *on* Ω.

Proof. Because the reference [9] is written in Chinese, we give here a detailed proof of Lemma 4. Assume that $f \not\equiv 0$. For any $a \in \Omega$, we prove $f(a) \neq 0$. Take polydisc $P(a,r) \subset \Omega$, and take $\alpha_1, \cdots, \alpha_n$ and λ such that $|\alpha_j| < r$, $|\lambda| < 1$. Then $(a_1 + \alpha_1\lambda, \cdots, a_n + \alpha_n\lambda) \in P(a,r) \subset \Omega$. Select a group of a_j that satisfies the above conditions so that $\psi(\lambda) = f(a_1 + \alpha_1\lambda, \cdots, a_n + \alpha_n\lambda) \not\equiv 0$ in $|\lambda| < 1$. This can be done, otherwise $f \equiv 0$ in $P(a,r)$. Let

$$\psi_k(\lambda) = f_k(a_1 + \alpha_1\lambda, \cdots, a_n + \alpha_n\lambda).$$

Thus, ψ_k converges locally uniformly to ψ in $|\lambda| < 1$, and $\psi \not\equiv 0$. From Hurwitz's theorem of one complex variable, ψ is not equal to 0 everywhere in $|\lambda| < 1$. In particular, $\psi(0) \neq 0$, that is, $f(a) \neq 0$.

Proof of Theorem 2. Suppose f is not φ-normal. Then, by Lemma 2, the family $\mathscr{F} = \{g_\mu(z)\}$ is not normal at some point $z_0 \in \mathbb{B}_n$. In view of Lemma 3, there exist sequences $\{g_\mu(z)\} \subset \mathscr{F}$ (we still denote by $\{g_\mu\}$ for convenience), a sequence $\{z_\mu\} \subset \mathbb{B}^n$ with $z_\mu \to z_0$, $\rho_\mu \to 0$ such that

$$G_\mu(z) = g_\mu(z_\mu + \rho_\mu z) = f\left(a_\mu + \frac{z_\mu}{\varphi(\|a_\mu\|)} + \frac{\rho_\mu}{\varphi(\|a_\mu\|)} z\right) \to G(z) \tag{10}$$

uniformly on compact subsets of $\mathbb{C}^n$, where $G(z)$ is a nonconstant holomorphic function on $\mathbb{C}^n$. Therefore, for each $J \in \mathbb{N}^n$,

$$D^J G_\mu(z) = \left(\frac{\rho_\mu}{\varphi(\|a_\mu\|)}\right)^{|J|} D^J f\left(a_\mu + \frac{z_\mu}{\varphi(\|a_\mu\|)} + \frac{\rho_\mu}{\varphi(\|a_\mu\|)} z\right) \to D^J G(z) \tag{11}$$

uniformly on compact subsets of $\mathbb{C}^n$.

Let K be a compact set containing z_0 and assume that $G(z_0) = 0$. Lemma 4 implies that there exists a sequence $z_\mu^* \to z_0$ such that

$$f\left(a_\mu + \frac{z_\mu}{\varphi(\|a_\mu\|)} + \frac{\rho_\mu}{\varphi(\|a_\mu\|)} z_\mu^*\right) = G_\mu(z_\mu^*) = 0.$$

For brevity, set $\hat{z}_\mu = a_\mu + \frac{z_\mu}{\varphi(\|a_\mu\|)} + \frac{\rho_\mu}{\varphi(\|a_\mu\|)} z_\mu^*$. Since $\rho_\mu \to 0$, $\hat{z}_\mu \in \mathbb{B}_n$ for sufficiently large μ. Then, by the hypothesis, there exists $M > 0$ such that

$$|D^J f(\hat{z}_\mu)| \leqslant M$$

for $0 \leqslant |J| \leqslant k-1$. Since $\varphi : [0,1) \to (0,\infty)$ is a smoothly increasing function, we obtain

$$D^J G_\mu(z_\mu^*) = \left(\frac{\rho_\mu}{\varphi(\|a_\mu\|)}\right)^{|J|} D^J f(\hat{z}_\mu) \leqslant \left(\frac{\rho_\mu}{\varphi(0)}\right)^{|J|} D^J f(\hat{z}_\mu).$$

This and (11) imply that $D^J G(z_0) = 0$ for $0 \leqslant |J| \leqslant k-1$. Thus all zeros of $G(z)$, if any, have multiplicity at least k, and $D^I G \not\equiv 0$. Suppose $z_0 \in \mathbb{C}^n$ such that $G(z_0) = a \in E$, then by (10) and applying Lemma 4 (see [10], p.316), there exists $z_\mu^* \to z_0$ such that

$$f\left(a_\mu + \frac{z_\mu}{\varphi(\|a_\mu\|)} + \frac{\rho_\mu}{\varphi(\|a_\mu\|)} z_\mu^*\right) = G_\mu(z_\mu^*) = a.$$

Then, by the assumption, there exists $M > 0$ such that

$$\frac{1}{\varphi(\|\hat{z}_\mu\|)^k} \frac{|D^I f(\hat{z}_\mu)|}{1 + |f(\hat{z}_\mu)|^{k+1}} \leqslant M$$

for sufficiently large μ. Thus, we obtain

$$\frac{|D^I G_\mu(z_\mu^*)|}{1 + |G_\mu(z_\mu^*)|^{k+1}} = \left(\frac{\rho_\mu}{\varphi(\|a_\mu\|)}\right)^k \frac{|D^I f(\hat{z}_\mu)|}{1 + |f(\hat{z}_\mu)|^{k+1}} \leqslant (\rho_\mu)^k \left(\frac{\varphi(\|\hat{z}_\mu\|)}{\varphi(\|a_\mu\|)}\right)^k M$$

for sufficiently large μ. From (5) and letting $\mu \to \infty$, we obtain $\frac{|D^I G(z_0)|}{1+|G(z_0)|^{k+1}} = 0$. It implies that z_0 is a zero of $D^I G(z)$. Thus, $D^I G(z) = 0, \quad z \in f^{-1}(E), \quad |I| = k$.

We next prove that $G(z)$ is constant. For any $b \in \mathbb{C}^n$, we define

$$g_b(\xi) := G(\xi b) = G(\xi b_1, \xi b_2, \cdots, \xi b_n), \quad \xi \in \mathbb{C}.$$

Then, the zero multiplicity of $g_b(\xi)$ is at least k and $g_b^{(k)} \not\equiv 0$. For $I = (i_1, i_2, \cdots, i_n)$,

$$g_b^{(k)}(\xi) = \sum_{i_1, i_2, \cdots, i_n = 1}^{n} b_{i_1} b_{i_2} \cdots b_{i_n} \frac{\partial^{|I|} G}{\partial \xi_{i_1} \partial \xi_{i_2} \cdots \partial \xi_{i_n}}(\xi b), \quad |I| = k.$$

Suppose that $G(\xi_0 b) \in E$, then $g_b(\xi_0) \in E$. From $D^I G(\xi_0 b) = 0$ with $|I| = k$, we get $g_b^{(k)}(\xi_0) = 0$. This implies that

$$\sum_{i=1}^{3} \overline{N}\left(r, \frac{1}{g_b - a_i}\right) \leqslant \overline{N}\left(r, \frac{1}{g_b^{(k)}}\right)$$

Suppose that the entire function g_b is not constant, by standard symbols and fundamental results of Nevanlinna theory (for details, see for example [16]), we obtain

$$\begin{aligned} 2T(r, g_b) &\leqslant \sum_{i=1}^{3} \overline{N}\left(r, \frac{1}{g_b - a_i}\right) + S(r, g_b) \leqslant \overline{N}\left(r, \frac{1}{g_b^{(k)}}\right) + S(r, g_b) \\ &\leqslant T(r, g_b^{(k)}) + S(r, g_b) \leqslant T(r, g_b) + S(r, g_b). \end{aligned} \tag{12}$$

So, $T(r,g_b) \leqslant S(r,g_b)$, which is a contradiction. Thus,

$$g_b(\xi) = C(b),$$

where $C(b)$ is constant with respect to ξ (but depends on b). Therefore

$$C(b) = g_b(\xi) = g_b(0) = G(0), \quad \xi \in \mathbb{C}.$$

In particular, $G(b) = g_b(1) = G(0)$. Since $b \in \mathbb{C}^n$ is taken arbitrarily, we then have $G(z) \equiv G(0)$, a contradiction.

5. Proof of Theorem 3

In order to prove Theorem 3, we first give the following lemma.

LEMMA 5. *Let $f(z)$ be a holomorphic function in $\mathbb{C}^n$, and the integer $k = 2$ or 3. If there exists a subset E of $\mathbb{C}$ containing $5-k$ distinct points such that*

$$D^I f(z) = 0, \ \ |I| = 1, \cdots, k-1, \quad z \in f^{-1}(E).$$

Then f is constant.

Proof. For any $b \in \mathbb{C}^n$, we define

$$g_b(\xi) := f(\xi b) = f(\xi b_1, \xi b_2, \cdots, \xi b_n), \quad \xi \in \mathbb{C}.$$

Suppose that f is not constant and $f(\xi_0 b) \in E$, then g_b is not constant and $g_b(\xi_0) \in E$. Moreover,

$$g_b'(\xi_0) = \sum_{i=1}^{n} b_i \frac{\partial f}{\partial \xi_i}(\xi_0 b), \tag{13}$$

$$g_b''(\xi_0) = \sum_{i,j=1}^{n} b_i b_j \frac{\partial^2 f}{\partial \xi_i \partial \xi_j}(\xi_0 b). \tag{14}$$

If $k = 2$, by the assumption $D^I f(\xi_0 b) = 0$, $|I| = 1$, together with (13), we have

$$g_b'(\xi_0) = 0.$$

This means that ξ_0 is a a-point of g_b with multiplicity at least 2. Applying Nevanlinna theory for meromorphic functions, it is clear that

$$\begin{aligned} 2T(r,g_b) &\leqslant \sum_{i=1}^{3} \overline{N}\Big(r, \frac{1}{g_b - a_i}\Big) + S(r,g_b) \leqslant \frac{1}{2}\sum_{i=1}^{3} N\Big(r, \frac{1}{g_b - a_i}\Big) + S(r,g_b) \\ &\leqslant \frac{3}{2} T(r,g_b) + S(r,g_b). \end{aligned}$$

So, $\frac{1}{2}T(r,g_b) \leqslant S(r,g_b)$, which is a contradiction. Thus,

$$g_b(\xi) = C(b),$$

where $C(b)$ is constant with respect to ξ (but depends on b). Therefore, $C(b) = g_b(\xi) = g_b(0) = f(0)$. Similar to the argument in Proof of Theorem 2 we have f is constant.

If $k = 3$, by the assumption $D^I f(\xi_0 b) = 0,\ |I| = 1,2$, combination with (13) and (14), it implies that

$$g_b'(\xi_0) = g_b''(\xi_0) = 0.$$

Then ξ_0 is a a-point of g_b with multiplicity at least 3. Similarly, applying Nevanlinna theory for meromorphic functions, we obtain

$$\begin{aligned} T(r,g_b) &\leqslant N(r,g) + \sum_{i=1}^{2} \overline{N}\Big(r, \frac{1}{g_b - a_i}\Big) + S(r,g_b) = \sum_{i=1}^{2} \overline{N}\Big(r, \frac{1}{g_b - a_i}\Big) + S(r,g_b) \\ &\leqslant \frac{1}{3}\sum_{i=1}^{2} N\Big(r, \frac{1}{g_b - a_i}\Big) + S(r,g_b) \leqslant \frac{2}{3}T(r,g_b) + S(r,g_b). \end{aligned}$$

So, $\frac{1}{3}T(r,g_b) \leqslant S(r,g_b)$, which is a contradiction. Thus, $g_b(\xi) = C(b)$, where $C(b)$ is constant. Again, similar to the argument in Proof of Theorem 2 we get f is constant.

Proof of Theorem 3. Assume for a contradiction. If f is not φ-normal in $\mathbb{B}_n$. From Lemma 2, the family $\mathscr{F} = \{g_\mu(z)\}$ is not normal at $z_0 \in \mathbb{B}_n$, by Lemma 3, there exist sequences $\{g_\mu(z)\}$ (without loss of generality, we still denote by $\{g_\mu\}$ for convenience) $\in \mathscr{F}$, $z_\mu \to z_0$, $\rho_\mu \to 0$ such that

$$G_\mu(z) = g_\mu(z_\mu + \rho_\mu z) = f\Big(a_\mu + \frac{z_\mu}{\varphi(\|a_\mu\|)} + \frac{\rho_\mu}{\varphi(\|a_\mu\|)} z\Big) \to G(z) \tag{15}$$

uniformly on compact subsets of $\mathbb{C}^n$, where $G(z)$ is a nonconstant holomorphic function on $\mathbb{C}^n$. Therefore, for $1 \leqslant i \leqslant n$,

$$\frac{\partial G_\mu(z)}{\partial z_i} = \frac{\rho_\mu}{\varphi(\|a_\mu\|)} \frac{\partial f}{\partial z_i}\Big(a_\mu + \frac{z_\mu}{\varphi(\|a_\mu\|)} + \frac{\rho_\mu}{\varphi(\|a_\mu\|)} z\Big) \to \frac{\partial G(z)}{\partial z_i} \tag{16}$$

uniformly on compact subsets of $\mathbb{C}^n$.

Let K be a compact set containing z_0. Suppose $z_0 \in \mathbb{C}^n$ such that $G(z_0) = a \in E$, then by (15) and applying Lemma 4, there exists a sequence $\{z_\mu^*\} \to z_0$ such that

$$f\Big(a_\mu + \frac{z_\mu}{\varphi(\|a_\mu\|)} + \frac{\rho_\mu}{\varphi(\|a_\mu\|)} z_\mu^*\Big) = G_\mu(z_\mu^*) = a.$$

For brevity, set $\hat{z}_\mu = a_\mu + \frac{z_\mu}{\varphi(\|a_\mu\|)} + \frac{\rho_\mu}{\varphi(\|a_\mu\|)} z_\mu^*$. Clearly, $\hat{z}_\mu \in \mathbb{B}_n$ for sufficiently large μ. Then, by the assumption, for sufficiently large μ, there exists $M > 0$ such that

$$\sup_{z \in f^{-1}(E)} \frac{f^{\#}(\hat{z}_\mu)}{\varphi(\|\hat{z}_\mu\|)} \leqslant M. \tag{17}$$

Thus,

$$G_\mu^\#(z_\mu^*) = \frac{\rho_\mu}{\varphi(\|a_\mu\|)} f^\#(\hat{z}_\mu) = \rho_\mu \frac{f^\#(\hat{z}_\mu)}{\varphi(\|\hat{z}_\mu\|)} \cdot \frac{\varphi(\|\hat{z}_\mu\|)}{\varphi(\|a_\mu\|)} \leqslant \rho_\mu M \frac{\varphi(\|\hat{z}_\mu\|)}{\varphi(\|a_\mu\|)}.$$

From (5), taking the limit, we have $G^\#(z_0) = \lim_{\mu\to\infty} G_\mu^\#(z_\mu^*) = 0$. Hence, $\frac{\partial G}{\partial z_i}(z_0) = \lim_{\mu\to\infty} \frac{\partial G_\mu}{\partial z_i}(z_\mu^*) = 0$ for all $1 \leqslant i \leqslant n$. By the definition of $f^\#$ and (17), we have

$$\left|\frac{\partial f}{\partial z_i}(\hat{z}_\mu)\right| \leqslant (1+|f(\hat{z}_\mu)|^2) f^\#(\hat{z}_\mu) \leqslant M(1+\max_{b\in E}|b|^2)\varphi(\|\hat{z}_\mu\|). \tag{18}$$

Therefore,

$$\begin{aligned}
\frac{|\frac{\partial^2 G_\mu}{\partial z_i \partial z_j}(z_\mu^*)|}{1+\left|\frac{\partial G_\mu}{\partial z_i}(z_\mu^*)\right|^2} &= \frac{\rho_\mu^2}{\varphi^2(\|a_\mu\|)} \cdot \frac{|\frac{\partial^2 f}{\partial z_i \partial z_j}(\hat{z}_\mu)|}{1+\frac{\rho_\mu^2}{\varphi^2(\|a_\mu\|)}|\frac{\partial f}{\partial z_i}(\hat{z}_\mu)|^2} \\
&= \frac{\rho_\mu^2}{\varphi^2(\|a_\mu\|)} \cdot \frac{|\frac{\partial^2 f}{\partial z_i \partial z_j}(\hat{z}_\mu)|}{1+|\frac{\partial f}{\partial z_i}(\hat{z}_\mu)|^2} \cdot \frac{1+|\frac{\partial f}{\partial z_i}(\hat{z}_\mu)|^2}{1+\frac{\rho_\mu^2}{\varphi^2(\|a_\mu\|)}|\frac{\partial f}{\partial z_i}(\hat{z}_\mu)|^2} \\
&\leqslant M \frac{\rho_\mu^2}{\varphi^2(\|a_\mu\|)} \left(1+\left|\frac{\partial f}{\partial z_i}(\hat{z}_\mu)\right|^2\right) \\
&\leqslant M \frac{\rho_\mu^2}{\varphi^2(\|a_\mu\|)} \left(1+[M(1+\max_{b\in E}|b|^2)]^2 \varphi^2(\|\hat{z}_\mu)\|)\right) \\
&\leqslant {\rho_\mu}^2 M\left(1+[M(1+\max_{b\in E}|b|^2)]^2\right)\left(\frac{\varphi(\|\hat{z}_\mu\|)}{\varphi(\|a_\mu\|)}\right)^2
\end{aligned}$$

for all $1 \leqslant i, j \leqslant n$. From (5) and (16), it implies that

$$\left(\frac{\partial G}{\partial z_i}\right)^\#(z_0) = \lim_{\mu\to\infty}\left(\frac{\partial G_\mu}{\partial z_i}\right)^\#(z_\mu^*) = 0, \quad 1 \leqslant i \leqslant n.$$

Hence, $\frac{\partial^2 G}{\partial z_i \partial z_j}(z_0) = \lim_{\mu\to\infty} \frac{\partial^2 G_\mu}{\partial z_i \partial z_j}(z_\mu^*) = 0$ for all $1 \leqslant i, j \leqslant n$. It follows from Lemma 5 that $G(z)$ is constant, a contradiction.

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