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A PROBLEM CONCERNING RIEMANN SUMS

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Abstract. An open problem concerning Riemann sums, posed by O. Furdui, is considered.

Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function. For natural n, let

$$x_n := \sum_{k=1}^{n-1} f\Big(\frac{k}{n}\Big) \quad \text{and} \quad y_n := x_{n+1} - x_n. \tag{1}$$

One may note here that x_n/n is a Riemann sum approximating the integral $\int_0^1 f(x)\,dx$.

Part (a) of Problem 1.32 in the book [3] is to find $\lim_{n\to\infty} y_n$ if the function f is continuously differentiable. It is not hard to do a bit more:

PROPOSITION 1. *Whenever f is absolutely continuous, one has*

$$\lim_{n\to\infty} y_n = \int_0^1 f(x)\,dx. \tag{2}$$

On the other hand, it is even easier to show this:

PROPOSITION 2. *Whenever $\lim_{n\to\infty} y_n$ exists, equality (2) holds.*

Propositions 1 and 2 will be proved at the end of this note.

Part (b) of Problem 1.32 in [3] is the following question, which has so far remained apparently unanswered:

What is the limit [in (2)] *when f is only continuous?*

By Proposition 2, this limit, if it exists, may only be $\int_0^1 f(x)\,dx$. However, we have

THEOREM 3. *There are continuous functions $f\colon [0,1] \to \mathbb{R}$ for which $\lim_{n\to\infty} y_n$ does not exist.*

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This is a pure existence theorem, and its proof, given below, is non-constructive. So, the problem of explicitly constructing a continuous function for which $\lim_{n\to\infty} y_n$ does not exist remains open.

Proof of Theorem 3. It may come as a surprise that this proof uses a probabilistic method. Let f be the random function W that is a standard Wiener process (Brownian motion) over the interval $[0,1]$. Then the probability that f is continuous everywhere on $[0,1]$ is 1; see e.g. [4]. In fact, without loss of generality we may assume that all realizations of the random function $f = W$ are everywhere continuous.

Informally, the idea of this proof of Theorem 3 is that, while all realizations of W are everywhere continuous, they are rather non-smooth, not only in the sense of being nowhere differentiable, but also not being Hölder continuous with any exponent $\geqslant 1/2$, in view of the (local) law of the iterated logarithm [4].

Now back to the formal proof: actually, Theorem 3 follows immediately from

LEMMA 4. *For $f = W$, the distribution of the random variable $y_{4s} - y_{2s}$ converges to the centered normal distribution with variance* $1/4$.

(The convergence here and in the rest of the proof of Theorem 3 is as $\mathbb{N} \ni s \to \infty$.)

Indeed, if Theorem 3 were false, we would have $y_{4s} - y_{2s} \to 0$ almost surely and hence in distribution, which would contradict Lemma 4. So, to complete the proof of Theorem 3, it remains to prove the lemma.

Proof of Lemma 4. Since $y_{4s} - y_{2s}$ is a centered normal random variable, it suffices to show that

$$\mathsf{E}(y_{4s} - y_{2s})^2 \overset{(?)}{\longrightarrow} 1/4. \tag{3}$$

The proof of (3) consists in direct calculations, which are somewhat involved, though, as we have to deal carefully enough with the discreteness in the definition of x_n. In carrying out this task, the choice of indices, $4s$ and $2s$, in the statement of Lemma 4 turns out to be sufficiently convenient.

The just mentioned calculations are based on the formula

$$\mathsf{E}W(u)W(v) = u \wedge v$$

for all u, v in $[0,1]$. By (1), for $f = W$,

$$\mathsf{E}x_n^2 = \sum_{j,k=1}^{n-1} \mathsf{E}W\Big(\frac{j}{n}\Big)W\Big(\frac{k}{n}\Big) = \sum_{j,k=1}^{n-1}\Big(\frac{j}{n} \wedge \frac{k}{n}\Big) = \frac{2n^2 - 3n + 1}{6}. \tag{4}$$

Somewhat similarly,

$$\begin{aligned} \mathsf{E}x_n x_{n+1} &= \sum_{j=1}^{n-1}\sum_{k=1}^{n}\Big(\frac{j}{n} \wedge \frac{k}{n+1}\Big) \\ &= \sum_{j=1}^{n-1}\sum_{k=j+1}^{n}\frac{j}{n} + \sum_{j=1}^{n-1}\sum_{k=1}^{j}\frac{k}{n+1} = \frac{2n^2 - n - 1}{6}. \end{aligned} \tag{5}$$

It follows from (1), (4), and (5) that

$$\mathsf{E}\,y_n^2 = \mathsf{E}\,x_{n+1}^2 + \mathsf{E}\,x_n^2 - 2\,\mathsf{E}\,x_n x_{n+1} = 1/2. \tag{6}$$

Now take any natural s. Similarly to (5), we have

$$\begin{aligned}
\mathsf{E}\,x_{4s}x_{2s} &= \sum_{j=1}^{4s-1}\sum_{k=1}^{2s-1}\Big(\frac{j}{4s}\wedge\frac{k}{2s}\Big) \\
&= \sum_{k=1}^{2s-1}\sum_{j=1}^{2k}\frac{j}{4s} + \sum_{k=1}^{2s-1}\sum_{j=2k+1}^{4s-1}\frac{k}{2s} = \frac{32s^2-18s+1}{12}, \\
\mathsf{E}\,x_{4s+1}x_{2s+1} &= \sum_{j=1}^{4s}\sum_{k=1}^{2s}\Big(\frac{j}{4s+1}\wedge\frac{k}{2s+1}\Big) \\
&= \sum_{k=1}^{2s}\sum_{j=1}^{2k-1}\frac{j}{4s+1} + \sum_{k=1}^{2s}\sum_{j=2k}^{4s-1}\frac{k}{2s+1} = \frac{32s^3+14s^2}{12s+3},
\end{aligned}$$

$$\begin{aligned}
\mathsf{E}\,x_{4s+1}x_{2s} &= \sum_{j=1}^{4s}\sum_{k=1}^{2s-1}\Big(\frac{j}{4s+1}\wedge\frac{k}{2s}\Big) \\
&= \sum_{k=1}^{2s-1}\sum_{j=1}^{2k}\frac{j}{4s+1} + \sum_{k=1}^{2s-1}\sum_{j=2k+1}^{4s}\frac{k}{2s} = \frac{32s^3-2s^2-5s-1}{12s+3}, \\
\mathsf{E}\,x_{4s}x_{2s+1} &= \sum_{j=1}^{4s-1}\sum_{k=1}^{2s}\Big(\frac{j}{4s}\wedge\frac{k}{2s+1}\Big) \\
&= \sum_{k=1}^{s}\sum_{j=1}^{2k-1}\frac{j}{4s} + \sum_{k=1}^{s}\sum_{j=2k}^{4s-1}\frac{k}{2s+1} \\
&\quad + \sum_{k=s+1}^{2s}\sum_{j=1}^{2k-2}\frac{j}{4s} + \sum_{k=s+1}^{2s}\sum_{j=2k-1}^{4s-1}\frac{k}{2s+1} = \frac{64s^3+28s^2-7s-1}{24s+12}.
\end{aligned}$$

So,

$$\begin{aligned}
\mathsf{E}\,y_{4s}y_{2s} &= \mathsf{E}\,x_{4s}x_{2s} + \mathsf{E}\,x_{4s+1}x_{2s+1} - \mathsf{E}\,x_{4s+1}x_{2s} - \mathsf{E}\,x_{4s}x_{2s+1} \\
&= \frac{12s^2+9s+2}{32s^2+24s+4} \longrightarrow \frac{12}{32}.
\end{aligned}$$

Thus, in view of (6),

$$\mathsf{E}(y_{4s}-y_{2s})^2 = \mathsf{E}\,y_{4s}^2 + \mathsf{E}\,y_{2s}^2 - 2\,\mathsf{E}\,y_{4s}y_{2s} \longrightarrow \frac{1}{2}+\frac{1}{2}-2\times\frac{12}{32} = \frac{1}{4},$$

so that (3) is verified, which completes the proof of Lemma 4. □

The proof of Theorem 3 is now complete as well. □

To conclude this note, it remains to prove Propositions 1 and 2.

Proof of Proposition 1. Since f is absolutely continuous, there is a function $g \in L^1[0,1]$ such that

$$f(x) = f(0) + \int_0^x g(u)\,du = f(0) + \int_0^1 g(u)\,\mathrm{I}\{u < x\}\,du \tag{7}$$

for all $x \in [0,1]$, where $\mathrm{I}\{\cdot\}$ denotes the indicator. So, by (1),

$$x_n = (n-1)f(0) + \int_0^1 g(u) \sum_{k=1}^{n-1} \mathrm{I}\Big\{u < \frac{k}{n}\Big\}\,du$$

and hence

$$y_n = f(0) + I_n(g) - J_n(g), \tag{8}$$

where

$$I_n(g) := \int_0^1 g(u)\,\mathrm{I}\Big\{u < \frac{n}{n+1}\Big\}\,du, \quad J_n(g) := \int_0^1 g(u) h_n(u)\,du,$$

and

$$h_n(u) := \sum_{k=1}^{n-1} \mathrm{I}\Big\{\frac{k}{n+1} \leqslant u < \frac{k}{n}\Big\}. \tag{9}$$

Clearly,

$$I_n(g) = f\Big(\frac{n}{n+1}\Big) - f(0) \longrightarrow f(1) - f(0). \tag{10}$$

Here and in the rest of this proof, the convergence is as $n \to \infty$.

To deal with $J_n(g)$, take any real $\varepsilon > 0$. Since $g \in L^1[0,1]$, by [1, Corollary 4.2.2], $\int_0^1 |g(u) - \tilde{g}(u)|\,du \leqslant \varepsilon$ for some continuous function $\tilde{g}\colon [0,1] \to \mathbb{R}$. Note also that $0 \leqslant h_n \leqslant 1$, since $[\frac{k}{n+1}, \frac{k}{n}) \subset [\frac{k-1}{n}, \frac{k}{n})$ for $k = 1, \dots, n-1$. So,

$$|J_n(g) - J_n(\tilde{g})| \leqslant \int_0^1 |g(u) - \tilde{g}(u)|\,du \leqslant \varepsilon. \tag{11}$$

Introduce now the function $\tilde{g}_n$ by the formula

$$\tilde{g}_n(u) := \sum_{k=1}^{n-1} \tilde{g}\Big(\frac{k}{n}\Big)\,\mathrm{I}\Big\{\frac{k}{n+1} \leqslant u < \frac{k}{n}\Big\}$$

for $u \in [0,1]$. Since the function $\tilde{g}$ is continuous, it is uniformly continuous on $[0,1]$, so that, in view of (9), $\|\tilde{g}h_n - \tilde{g}_n h_n\|_\infty = \|\tilde{g}h_n - \tilde{g}_n\|_\infty \to 0$ and hence

$$|J_n(\tilde{g}) - J_n(\tilde{g}_n)| \longrightarrow 0. \tag{12}$$

On the other hand, using the continuity of $\tilde{g}$ and integration by parts, we have

$$J_n(\tilde{g}_n) = \sum_{k=1}^{n-1} \tilde{g}\Big(\frac{k}{n}\Big) \frac{k}{n} \frac{1}{n} \frac{n}{n+1} \longrightarrow \int_0^1 \tilde{g}(u) u\,du = \tilde{f}(1) - \int_0^1 \tilde{f}(u)\,du \tag{13}$$

where $\tilde{f}(x) := f(0) + \int_0^x \tilde{g}(u)\,du$ for $x \in [0,1]$. By (7) and the second inequality in (11), we have $|\tilde{f} - f| \leqslant \varepsilon$ and hence $|\int_0^1 \tilde{f}(u)\,du - \int_0^1 f(u)\,du| \leqslant \varepsilon$. Collecting now (8), (10), (11), (12), and (13), we see that

$$\limsup_{n\to\infty} \left| y_n - \int_0^1 f(u)\,du \right| \leqslant 3\varepsilon$$

for any real $\varepsilon > 0$, which completes the proof of Proposition 1. $\square$

Proof of Proposition 2. The Stolz–Cesàro theorem ([5, pages 173–175] and [2, page 54]) states the following: if (a_n) and (b_n) are sequences of real numbers such that b_n is strictly increasing to ∞ and $\frac{a_{n+1}-a_n}{b_{n+1}-b_n} \longrightarrow \ell \in \mathbb{R}$, then $\frac{a_n}{b_n} \longrightarrow \ell$. Now Proposition 2 follows immediately by applying the Stolz–Cesàro theorem with $a_n = x_n$ and $b_n = n$, since $\frac{x_n}{n} \longrightarrow \int_0^1 f(x)\,dx$. $\square$

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