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CERTAIN PROPERTIES OF SPIRALLIKE SAKAGUCHI TYPE FUNCTIONS CONNECTED WITH q–HYPERGEOMETRIC SERIES

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Abstract. We discuss the properties like coefficient estimation, subordination results and Fekete-Szegő problem for certain subclass of spirallike Sakaguchi type functions associated with q–hypergeometric series.

1. Introduction

Let $\mathscr{A}$ be the class of all functions of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \tag{1}$$

which are holomorphic in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$ and denote $\mathscr{S}$ the subclass of $\mathscr{A}$ consisting of functions that are univalent in $\mathbb{U}$.

A function $f \in \mathscr{A}$ is said to be in the class of α–spirallike functions of order β in $\mathbb{U}$ which we denote $\mathscr{SP}(\alpha,\beta)$ if

$$\Re\left\{e^{i\alpha}\frac{zf'(z)}{f(z)}\right\} > \beta\cos\alpha, \qquad z \in \mathbb{U} \tag{2}$$

for $0 \leqslant \beta < 1$ and some real α with $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$.

The class $\mathscr{SP}(\alpha,\beta)$ was studied by Libera [1] and Keogh and Merkes [2]. Note that $\mathscr{SP}(\alpha,0)$ is the class of spirallike functions introduced by Spacek [3], $\mathscr{SP}(0,\beta) = S^*(\beta)$ is the class of starlike functions of order β and $\mathscr{SP}(0,0) = S^*$ is the class of starlike functions.

Let $\mathscr{B}$ be the class of analytic functions $\omega \in \mathscr{A}$ that satisfy the conditions $\omega(0) = 0$ and $|\omega(z)| < 1$, for $|z| < 1$.

For functions $f, g \in \mathscr{A}$ given by (1) we define its Hadamard product by

$$(f * g)(z) = z + a_2 b_2 z^2 + a_3 b_3 z^3 + \cdots, \qquad z \in \mathbb{U}. \tag{3}$$

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Jackson [4] reintroduced and started a systematic study of the q–difference operator. Namely, by him for $q \in (0,1)$, the Jackson's q-derivative operator or q–difference operator for a function $f \in \mathscr{A}$ is defined by

$$D_q(f(z)) = \begin{cases} \dfrac{f(qz)-f(z)}{qz-z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases}, \tag{4}$$

which is now sometimes referred to as Euler–Jackson derivative or simply q–derivative.

For the power function $h(z) = z^n$, observe that

$$D_q(h(z)) = D_q(z^n) = \frac{1-q^n}{1-q} z^{n-1} =: [n]_q z^{n-1}.$$

Note that

$$\lim_{q \to 1} D_q(h(z)) = \lim_{q \to 1} [n]_q z^{n-1} = n z^{n-1} = h'(z),$$

where $h'(z)$ is the ordinary derivative and $[n]_q$ is the q–integer or basic number n.

The q–shifted factorial is defined for $a \in \mathbb{C}$ as a product of n factors by

$$(a;q)_n = \begin{cases} 1, & n = 0 \\ (1-a)(1-aq)\ldots(1-aq^{n-1}), & n \in \mathbb{N} := \{1,2,\ldots\} \end{cases}.$$

For further properties and applications of q-calculus one can refer to [4, 5, 6, 7, 8, 9] and to the reference cited therein.

The q-hypergeometric function is a power series in one complex variable z, which coefficients depend on l upper and m lower parameters and are built by products of q–shifted factorials. Namely,

$$\psi_m^l(a_1,\ldots;a_l,b_1,\ldots,b_m;q,z) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n\ldots(a_l;q)_n}{(q;q)_n(b_1;q)_n\ldots(b_m;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+m-l} z^n.$$

Here $b_j \neq 0,-1,-2,\ldots; j = 1,2,\ldots,m$ and $q \neq 0$, $l > m+1$, $l,m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, while $z \in \mathbb{U}$. For $q \in (0,1)$ and $l = m+1$, the q–hypergeometric series takes the form

$$\psi_m^{m+1}(a_1,\ldots,a_{m+1},b_1,\ldots,b_m;q,z) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n\ldots(a_{m+1};q)_n}{(q;q)_n(b_1;q)_n\ldots(b_m;q)_n} z^n,$$

which converges absolutely in the open unit disk $\mathbb{U}$.

Now, for functions $f \in \mathscr{A}$ and for real a,b parameters we introduce the linear operator $\mathscr{F}_m^{m+1}(q,z) : \mathscr{A} \mapsto \mathscr{A}$ in the form

$$\begin{aligned} \mathscr{F}_m^{m+1}(q,z) &= z\psi_m^{m+1}(a_1,a_2,\ldots,a_{m+1};b_1,b_2,\ldots,b_m;q,z) \\ &= z + \sum_{n=2}^{\infty} \frac{(a_1;q)_{n-1}(a_2;q)_{n-1}\ldots(a_{m+1};q)_{n-1}}{(q;q)_{n-1}(b_1;q)_{n-1}\ldots(b_m;q)_{n-1}} z^n \\ &= z + \sum_{n=2}^{\infty} \varphi_m^{m+1}(n,q) z^n, \end{aligned}$$

where

$$\varphi_m^{m+1}(n,q) = \frac{(a_1;q)_{n-1}(a_2;q)_{n-1}\dots(a_{m+1};q)_{n-1}}{(q;q)_{n-1}(b_1;q)_{n-1}\dots(b_m;q)_{n-1}}. \tag{5}$$

Throughout our study for $f \in \mathscr{A}$, we write

$$\mathscr{F}_q f(z) = \mathscr{F}_m^{m+1}(q,z) * f(z) = z + \sum_{n=2}^{\infty} \varphi_m^{m+1}(n,q) a_n z^n. \tag{6}$$

From (4) and (6), we get

$$D_q[\mathscr{F}_q f(z)] = 1 + \sum_{n=2}^{\infty} \varphi_m^{m+1}(n,q)[n]_q a_n z^{n-1} \tag{7}$$

where $\varphi_m^{m+1}(n,q)$ is given by (5).

Making use of the generalized q-hypergeometric differential operator $\mathscr{F}_q f(z)$, we introduce a new subclass of spirallike functions as follows.

DEFINITION 1. For $0 \leqslant \lambda \leqslant 1$, $0 \leqslant \beta < 1$ and $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, $|t| \leqslant 1$, $t \neq 1$, we let $\mathscr{H}_q(\alpha,\beta,\lambda,t)$ be the subclass of $\mathscr{A}$ consisting of functions $f \in \mathscr{A}$ of the form (1) and satisfying the analytic condition:

$$\Re\left\{ e^{i\alpha} \frac{[(1-t)z]^{1-\lambda} D_q[\mathscr{F}_q f(z)]}{[\mathscr{F}_q f(z) - \mathscr{F}_q f(tz)]^{1-\lambda}} \right\} > \beta\cos\alpha, \qquad z \in \mathbb{U},$$

where $D_q[\mathscr{F}_q f(z)]$ is given by (7).

EXAMPLE 1. For $t = 0$, we let $\mathscr{H}_q(\alpha,\beta,\lambda,0) = \mathscr{S}\mathscr{P}_q(\alpha,\beta,\lambda)$ be the subclass of $\mathscr{A}$ consisting of functions $f \in \mathscr{A}$ of the form (1) and satisfying the analytic condition:

$$\Re\left\{ e^{i\alpha} \frac{z^{1-\lambda} D_q[\mathscr{F}_q f(z)]}{\mathscr{F}_q f(z)^{1-\lambda}} \right\} > \beta\cos\alpha, \qquad z \in \mathbb{U},$$

where $D_q[\mathscr{F}_q f(z)]$ is given by (7).

EXAMPLE 2. For $\lambda = 0$ and $t = 0$, we let $\mathscr{H}_q(\alpha,\beta,0,0) = \mathscr{S}\mathscr{P}_q(\alpha,\beta)$ be the subclass of $\mathscr{A}$ consisting of functions $f \in \mathscr{A}$ of the form (1) and satisfying the analytic condition:

$$\Re\left\{ e^{i\alpha} \frac{z D_q[\mathscr{F}_q f(z)]}{\mathscr{F}_q f(z)} \right\} > \beta\cos\alpha, \qquad z \in \mathbb{U}, \tag{8}$$

where $D_q[\mathscr{F}_q f(z)]$ is given by (7).

EXAMPLE 3. For $\lambda = 1$, we let $\mathscr{H}_q(\alpha,\beta,1,t) = \mathscr{S}\mathscr{P}_q(\alpha,\beta,t)$ be the subclass of $\mathscr{A}$ consisting of functions $f \in \mathscr{A}$ of the form (1) and satisfying the analytic condition:

$$\Re\left\{ e^{i\alpha} D_q[\mathscr{F}_q f(z)] \right\} > \beta\cos\alpha, \qquad z \in \mathbb{U}, \tag{9}$$

where $D_q[\mathscr{F}_q f(z)]$ is given by (7).

EXAMPLE 4. For $\lambda = 1$ and $\alpha = 0$, we let $\mathscr{H}_q(0,\beta,1,t) = \mathscr{S}\mathscr{P}_q(\beta,t)$ be the subclass of $\mathscr{A}$ consisting of functions $f \in \mathscr{A}$ of the form (1) and satisfying the analytic condition:

$$\Re\left\{D_q[\mathscr{F}_q f(z)]\right\} > \beta, \qquad z \in \mathbb{U},$$

where $D_q[\mathscr{F}_q f(z)]$ is given by (7).

The motivation of the current work is to examine the coefficient estimates and subordination properties for the class of functions $\mathscr{H}_q(\alpha,\beta,\lambda,t)$. Related consequences of the earned results are also presented.

Throughout this paper, unless otherwise stated, we assume that

$$0 \leqslant \lambda \leqslant 1, \quad 0 \leqslant \beta < 1, \quad -\frac{\pi}{2} < \alpha < \frac{\pi}{2}, \quad |t| \leqslant 1, \quad t \neq 1.$$

2. Coefficient estimates

In this section we obtain sufficient conditions for a function $f \in \mathscr{A}$ belonging to the class $\mathscr{H}_q(\alpha,\beta,\lambda,t)$.

THEOREM 1. *Let $f \in \mathscr{A}$ and let σ be a real number with $0 \leqslant \sigma < 1$. If*

$$\left|\frac{[(1-t)z]^{1-\lambda} D_q[\mathscr{F}_q f(z)]}{[\mathscr{F}_q f(z) - \mathscr{F}_q f(tz)]^{1-\lambda}} - 1\right| \leqslant 1-\sigma, \qquad z \in \mathbb{U}, \tag{10}$$

then $f \in \mathscr{H}_q(\alpha,\beta,\lambda,t)$, provided that $|\alpha| \leqslant \cos^{-1}\left(\frac{1-\sigma}{1-\beta}\right)$.

Proof. From (10), it follows that

$$\frac{[(1-t)z]^{1-\lambda} D_q[\mathscr{F}_q f(z)]}{[\mathscr{F}_q f(z) - \mathscr{F}_q f(tz)]^{1-\lambda}} = (1-\sigma)w(z) + 1,$$

where $w(z) \in \mathscr{B}$. We have

$$\begin{aligned}\Re\left\{e^{i\alpha}\frac{[(1-t)z]^{1-\lambda} D_q[\mathscr{F}_q f(z)]}{[\mathscr{F}_q f(z) - \mathscr{F}_q f(tz)]^{1-\lambda}}\right\} &= \Re\{e^{i\alpha}(1-\sigma)w(z) + e^{i\alpha}\} \\ &= (1-\sigma)\Re\{e^{i\alpha}w(z)\} + \cos\alpha \geqslant -(1-\sigma)|e^{i\alpha}w(z)| + \cos\alpha \\ &= -(1-\sigma) + \cos\alpha = \beta\cos\alpha,\end{aligned}$$

provided $|\alpha| \leqslant \cos^{-1}\left(\frac{1-\sigma}{1-\beta}\right)$. Thus the proof is completed. □

COROLLARY 1. *Let $f \in \mathscr{A}$ and assume $\sigma = 1-(1-\beta)\cos\alpha$. If*

$$\left|\frac{[(1-t)z]^{1-\lambda} D_q[\mathscr{F}_q f(z)]}{[\mathscr{F}_q f(z) - \mathscr{F}_q f(tz)]^{1-\lambda}} - 1\right| \leqslant (1-\beta)\cos\alpha, \qquad z \in \mathbb{U}, \tag{11}$$

then $f \in \mathscr{H}_q(\alpha,\beta,\lambda,t)$.

Now, we present another sufficient condition for $f \in \mathscr{A}$ to be in $\mathscr{H}_q(\alpha,\beta,\lambda,t)$.

THEOREM 2. *A function $f \in \mathscr{A}$ of the form (1) belongs to $\mathscr{H}_q(\alpha,\beta,\lambda,t)$ when*

$$\sum_{n=2}^{\infty}\{([n]_q-(1-\lambda)u_n)\sec\alpha+(1-\beta)(1-\lambda)u_n\}\varphi_m^{m+1}(n,q)\,|a_n|\leqslant 1-\beta, \tag{12}$$

where $u_n=\frac{1-t^n}{1-t}$.

Proof. In view of above Corollary 1, we have

$$\left|\frac{[(1-t)z]^{1-\lambda}\left\{1+\sum_{n=2}^{\infty}\varphi_m^{m+1}(n,q)a_n[n]_q z^{n-1}\right\}}{[(1-t)z]^{1-\lambda}\left\{1+\sum_{n=2}^{\infty}\varphi_m^{m+1}(n,q)a_n u_n z^{n-1}\right\}^{1-\lambda}}-1\right|\leqslant(1-\beta)\cos\alpha$$

$$\left|\frac{1+\sum_{n=2}^{\infty}\varphi_m^{m+1}(n,q)a_n[n]_q z^{n-1}}{1+(1-\lambda)\sum_{n=2}^{\infty}\varphi_m^{m+1}(n,q)a_n u_n z^{n-1}}-1\right|\leqslant(1-\beta)\cos\alpha$$

$$\frac{\sum_{n=2}^{\infty}([n]_q-(1-\lambda)u_n)\varphi_m^{m+1}(n,q)|a_n|}{1-\sum_{n=2}^{\infty}\varphi_m^{m+1}(n,q)(1-\lambda)u_n|a_n|}\leqslant(1-\beta)\cos\alpha,$$

that is,

$$\sum_{n=2}^{\infty}([n]_q-(1-\lambda)u_n)\varphi_m^{m+1}(n,q)|a_n|\leqslant(1-\beta)\cos\alpha\left\{1-\sum_{n=2}^{\infty}\varphi_m^{m+1}(n,q)(1-\lambda)u_n|a_n|\right\},$$

which means

$$\sum_{n=2}^{\infty}\left\{([n]_q-(1-\lambda)u_n)\sec\alpha+(1-\beta)(1-\lambda)u_n\right\}\varphi_m^{m+1}(n,q)|a_n|\leqslant 1-\beta.$$

Thus the proof is completed. □

COROLLARY 2. *A function $f \in \mathscr{A}$ of the form (1) is in $\mathscr{SP}_q(\alpha,\beta,\lambda)$ if*

$$\sum_{n=2}^{\infty}\{([n]_q-1+\lambda)\sec\alpha+(1-\beta)(1-\lambda)\}\varphi_m^{m+1}(n,q)\,|a_n|\leqslant 1-\beta.$$

COROLLARY 3. *The function $f \in \mathscr{A}$ of the form (1) is in the class $\mathscr{SP}_q(\alpha,\beta)$, if*

$$\sum_{n=2}^{\infty}\left\{([n]_q-1)\sec\alpha+(1-\beta)\right\}\varphi_m^{m+1}(n,q)\,|a_n|\leqslant 1-\beta.$$

COROLLARY 4. *The function $f \in \mathscr{A}$ of the form (1) is in $\mathscr{SP}_q(\alpha,\beta,t)$ if,*

$$\sum_{n=2}^{\infty}[n]_q\sec\alpha\,\varphi_m^{m+1}(n,q)\,|a_n|\leqslant 1-\beta.$$

COROLLARY 5. *The function* $f \in \mathscr{A}$ *of the form* (1) *belongs to* $\mathscr{S}\mathscr{P}_q(\beta,t)$ *if,*

$$\sum_{n=2}^{\infty} [n]_q \, \varphi_m^{m+1}(n,q) \, |a_n| \leqslant 1-\beta.$$

For the sake of brevity throughout this paper we will use the shorthand

$$d_n(\alpha,\beta,\lambda) = ([n]_q - (1-\lambda)u_n) \sec\alpha + (1-\beta)(1-\lambda)u_n \tag{13}$$

Our next result gives the coefficient estimates for functions in the class $\mathscr{H}_q(\alpha,\beta,\lambda,t)$.

THEOREM 3. *If* $f \in \mathscr{H}_q(\alpha,\beta,\lambda,t)$, *then*

$$|a_n| \leqslant \frac{1-\beta}{d_n(\alpha,\beta,\lambda)\varphi_m^{m+1}(n,q)}, \qquad n = 2,3,4,\ldots. \tag{14}$$

The result if sharp for the functions $f_n(z)$ *given by*

$$f_n(z) = z + \frac{1-\beta}{d_n(\alpha,\beta,\lambda)\varphi_m^{m+1}(n,q)} z^n, \qquad n = 2,3,4,\ldots.$$

Proof. If $f \in \mathscr{H}_q(\alpha,\beta,\lambda,t)$, then we have for each $n \geqslant 2$,

$$d_n(\alpha,\beta,\lambda)\varphi_m^{m+1}(n,q)|a_n| \leqslant \sum_{n=2}^{\infty} d_n(\alpha,\beta,\lambda)\varphi_m^{m+1}(n,q) \, |a_n| \leqslant 1-\beta.$$

So the estimate (14). Since

$$f_n(z) = z + \frac{1-\beta}{d_n(\alpha,\beta,\lambda)\varphi_m^{m+1}(n,q)} z^n, \qquad n = 2,3,4,\ldots$$

satisfies the conditions of Theorem 2, $f_n(z) \in \mathscr{H}_q(\alpha,\beta,\lambda,t)$ and the equality is attained for this function. □

REMARK 1. We observe that Corollary 3 yields the result of Silverman [10] for the special values of α and β.

3. Subordination results

In order to obtain subordination results for the class $\mathscr{H}_q(\alpha,\beta,\lambda,t)$, we need the following definitions and the lemma due to Wilf [11].

DEFINITION 2. Let $g,h \in \mathscr{A}$. The function g is said to be subordinate to the function h, denoted by $g \prec h$, if there exists a function $\omega \in \mathscr{B}$ such that $g(z) = h(\omega(z))$, for all $z \in \mathbb{U}$.

DEFINITION 3. Let $\mathscr{C}$ be the subclasses of $\mathscr{S}$ consisting of convex function, defined analytically the equivalences

$$f \in \mathscr{C} \iff \Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > 0.$$

DEFINITION 4. [11] A sequence $\{c_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

is regular, univalent and convex in $\mathbb{U}$, we have

$$\sum_{n=1}^{\infty} b_n c_n z^n \prec g(z), \qquad z \in \mathbb{U}. \tag{15}$$

LEMMA 1. [11] *The sequence* $\{c_n\}_{n=1}^{\infty}$ *is a subordinating factor sequence* **iff**

$$\Re\left\{1+2\sum_{n=1}^{\infty} c_n z^n\right\} > 0, \qquad z \in \mathbb{U}. \tag{16}$$

THEOREM 4. *Let* $f \in \mathscr{H}_q(\alpha,\beta,\lambda,t)$ *and* $g \in \mathscr{C}$, *then*

$$\frac{d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)}{2\{1-\beta+d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)\}}(f*g)(z) \prec g(z), \tag{17}$$

where

$$d_2(\alpha,\beta,\lambda) = \{[2]_q-(1-\lambda)u_2\}\sec\alpha + (1-\beta)(1-\lambda)u_2 \tag{18}$$

and

$$\Re\{f(z)\} > -\frac{1-\beta+d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)}{d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)}, \qquad z \in \mathbb{U}. \tag{19}$$

The constant factor $\frac{d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)}{2\{1-\beta+d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)\}}$ *in* (17) *cannot be replaced by a larger number.*

Proof. Let $f \in \mathscr{H}_q(\alpha,\beta,\lambda,t)$ satisfy the coefficient inequality (12) and suppose that $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathscr{C}$. Then by Definition 4, the subordination (17) of our theorem will hold true if the sequence

$$\left\{\frac{d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)}{2\{1-\beta+d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)\}}a_n\right\}_{n=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. By Lemma 1, it is evident to prove

$$\Re\left\{1+2\sum_{n=1}^{\infty}\frac{d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)}{2\{1-\beta+d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)\}}a_nz^n\right\}>0$$

In view of (12), where $|z| = r < 1$, we obtain

$$\begin{aligned}
&\Re\left\{1+\frac{d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)}{\{1-\beta+d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)\}}\sum_{n=1}^{\infty}a_nz^n\right\}\\
&\geqslant 1-\frac{d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)}{1-\beta+d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)}\left|\sum_{n=1}^{\infty}a_nz^n\right|\\
&=1-\frac{d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)}{1-\beta+d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)}r-\frac{\sum_{n=2}^{\infty}d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)|a_n|r}{1-\beta+d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)}\\
&=1-\frac{1-\beta+d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)}{1-\beta+d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)}r>0.
\end{aligned}$$

Hence, equation (17) holds. Now, the inequality (19) follows from (17) by taking $g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n \in \mathscr{C}$. Therefore, we get

$$\begin{aligned}
\frac{d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)}{2\{1-\beta+d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)\}}f(z)&\prec g(z)\\
\frac{d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)}{2\{1-\beta+d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)\}}\Re\{f(z)\}&>\Re\{g(z)\}\\
\frac{d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)}{2\{1-\beta+d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)\}}\Re\{f(z)\}&>-\frac{1}{2}
\end{aligned}$$

which implies (19). The sharpness of the multiplying factor in (17) can be established by considering a function

$$F(z)=z-\frac{1-\beta}{1-\beta+d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)}z^2.$$

Clearly $F \in \mathscr{H}_q(\alpha,\beta,\lambda,t)$. By (17), we infer that

$$\frac{d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)}{2\{1-\beta+d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)\}}(F*g)(z)\prec\frac{z}{1-z}$$

and it follows that,

$$\min\left[\Re\left\{\frac{d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)}{2\{1-\beta+d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)\}}F(z)\right\}\right]=-\frac{1}{2},\quad z\in\mathbb{U}$$

This shows that the constant $\frac{d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)}{2\{1-\beta+d_2(\alpha,\beta,\lambda)\varphi_m^{m+1}(2,q)\}}$ cannot be replaced by any larger one. □

In view of Examples 2 and 3, we state the following corollaries for the Theorem 4.

COROLLARY 6. *If* $f \in \mathscr{SP}_q(\alpha,\beta)$ *and* $g \in \mathscr{C}$*, then*

$$\frac{[q\sec\alpha+(1-\beta)]\varphi_m^{m+1}(2,q)}{2\{1-\beta+[q\sec\alpha+(1-\beta)]\varphi_m^{m+1}(2,q)\}}(f*g)(z) \prec g(z) \tag{20}$$

where

$$\Re\{f(z)\} > -\frac{1-\beta+[q\sec\alpha+(1-\beta)]\varphi_m^{m+1}(2,q)}{[q\sec\alpha+(1-\beta)]\varphi_m^{m+1}(2,q)}, \qquad z \in \mathbb{U}.$$

The constant factor $\dfrac{[q\sec\alpha+(1-\beta)]\varphi_m^{m+1}(2,q)}{2\{1-\beta+[q\sec\alpha+(1-\beta)]\varphi_m^{m+1}(2,q)\}}$ *in* (20) *cannot be replace by a larger one.*

COROLLARY 7. *If* $f \in \mathscr{SP}_q(\alpha,\beta,t)$ *and* $g \in \mathscr{C}$*, then*

$$\frac{(1+q)\sec\alpha\ \varphi_m^{m+1}(2,q)}{2\{1-\beta+(1+q)\sec\alpha\ \varphi_m^{m+1}(2,q)\}}(f*g)(z) \prec g(z) \tag{21}$$

where

$$\Re\{f(z)\} > -\frac{1-\beta+(1+q)\sec\alpha\ \varphi_m^{m+1}(2,q)}{(1+q)\sec\alpha\ \varphi_m^{m+1}(2,q)}, \qquad z \in \mathbb{U}.$$

The constant factor $\dfrac{(1+q)\sec\alpha\ \varphi_m^{m+1}(2,q)}{2\{1-\beta+(1+q)\sec\alpha\ \varphi_m^{m+1}(2,q)\}}$ *in* (21) *cannot be replaced by a larger one.*

4. The Fekete-Szegő problem

The Fekete-Szegö consists form deriving sharp upper bounds for the functional $|a_3-\mu a_2^2|$ for various subclasses of $\mathscr{A}$ (see [12, 13]). In order to obtain sharp upper bounds for $\left|a_3-\mu a_2^2\right|$ for the class $\mathscr{H}_q(\alpha,\beta,\lambda,t)$ the following lemma is required (see, eg. [14, p. 108]).

LEMMA 2. *Let the function* $\omega \in \mathscr{B}$ *be given by*

$$\omega(z) = \sum_{n=1}^{\infty} \omega_n z^n, \qquad z \in \mathbb{U}. \tag{22}$$

Then $|\omega_1| \leqslant 1$, $|\omega_2| \leqslant 1-|\omega_1|^2$ *and*

$$\left|\omega_2 - s\omega_1^2\right| \leqslant \max\{1,|s|\} \tag{23}$$

for any complex number s. *The function* $\omega(z)=z$ *and* $\omega(z)=z^2$ *or one of their rotations show that both inequalities* (22) *and* (23) *are sharp.*

For the constants α, β, with $0 \leqslant \beta < 1$ and $|\alpha| < \frac{\pi}{2}$ denote

$$p_{\alpha,\beta}(z) = \frac{1 + e^{-i\alpha}[e^{-i\alpha} - 2\beta\cos\alpha]z}{1-z}, \qquad z \in \mathbb{U}. \tag{24}$$

The function $p_{\alpha,\beta}(z)$ maps the open unit disk onto the half-plane

$$H_{\alpha,\beta} = \{z \in \mathbb{C} : \Re\{e^{i\alpha}z\} > \beta\cos\alpha\}.$$

If $p_{\alpha,\beta}(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, then it is easy to check that

$$p_n = 2e^{-i\alpha}(1-\beta)\cos\alpha \tag{25}$$

for all $n \geqslant 1$. First we obtain sharp upper bounds for the Fekete-Szegö functional $\left|a_3 - \mu a_2^2\right|$ with μ real parameter.

THEOREM 5. *Let $f \in \mathscr{H}_q(\alpha,\beta,\lambda,t)$ be given by (1) and let μ be a real number. Then*

$$\left|a_3 - \mu a_2^2\right| \leqslant$$
$$\begin{cases} \frac{2(1-\beta)\cos\alpha}{\psi_3(\lambda)\varphi_m^{m+1}(3,q)}\left[\frac{2(1-\beta)(1-\lambda)u_2\psi_2(\frac{\lambda}{2})}{[\psi_2(\lambda)]^2} - \frac{2\mu(1-\beta)\psi_3(\lambda)\varphi_m^{m+1}(3,q)}{[\psi_2(\lambda)]^2[\varphi_m^{m+1}(2,q)]^2} + 1\right] & \textit{if } \mu \leqslant \delta_1 \\ \frac{2(1-\beta)\cos\alpha}{\psi_3(\lambda)\varphi_m^{m+1}(3,q)} & \textit{if } \delta_1 \leqslant \mu \leqslant \delta_2 \\ \frac{2(1-\beta)\cos\alpha}{\psi_3(\lambda)\varphi_m^{m+1}(3,q)}\left[\frac{2\mu(1-\beta)\psi_3(\lambda)\varphi_m^{m+1}(3,q)}{[\psi_2(\lambda)]^2[\varphi_m^{m+1}(2,q)]^2} - \frac{2(1-\beta)(1-\lambda)u_2\psi_2(\frac{\lambda}{2})}{[\psi_2(\lambda)]^2} - 1\right] & \textit{if } \mu \geqslant \delta_2 \end{cases} \tag{26}$$

where

$$\delta_1 = \frac{(1-\lambda)u_2[1+q-(1-\lambda)u_2]}{[1+q+q^2-(1-\lambda)u_3]}\frac{[\varphi_m^{m+1}(2,q)]^2}{\varphi_m^{m+1}(3,q)}, \tag{27}$$

$$\delta_2 = \frac{[1+q-(1-\lambda)u_2][1+q-(1-\lambda)\beta u_2]}{(1-\beta)[1+q+q^2-(1-\lambda)u_3]}\frac{[\varphi_m^{m+1}(2,q)]^2}{\varphi_m^{m+1}(3,q)}, \tag{28}$$

$$q_n = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}, \tag{29}$$

$$\psi_n(\lambda) = [q_n - (1-\lambda)u_n], \quad \textit{for } n = 2,3 \tag{30}$$

and $\varphi_m^{m+1}(2,q)$, $\varphi_m^{m+1}(3,q)$ are defined by (5) with $n=2$ and $n=3$, respectively. All estimates are sharp.

Proof. Suppose that $f \in \mathscr{H}_q(\alpha,\beta,\lambda,t)$ is given by (1). Then, from the definition of the class $\mathscr{H}_q(\alpha,\beta,\lambda,t)$, there exists $\omega \in \mathscr{B}$, $\omega(z) = \omega_1 z + \omega_2 z^2 + \omega_3 z^3 + \cdots$ such that

$$\frac{[(1-t)z]^{1-\lambda}D_q[\mathscr{F}_q f(z)]}{[\mathscr{F}_q f(z) - \mathscr{F}_q f(tz)]^{1-\lambda}} = p_{\alpha,\beta}(\omega(z)) = 1 + p_1 z + p_2 z^2 + \cdots, \qquad z \in \mathbb{U}. \tag{31}$$

From (25) we have

$$p_1 = p_2 = 2e^{-i\alpha}(1-\beta)\cos\alpha.$$

Now LHS of (31) is given by

$$1+\varphi_m^{m+1}(2,q)\{1+q-(1-\lambda)u_2\}a_2 z + \left\{ \begin{array}{l} \varphi_m^{m+1}(3,q)[1+q+q^2-(1-\lambda)u_3]a_3 \\ -[\varphi_m^{m+1}(2,q)]^2(1-\lambda)u_2\left[1+q-\frac{(2-\lambda)}{2}u_2\right]a_2^2 \end{array} \right\} z^2+\cdots$$

$$1+\varphi_m^{m+1}(2,q)\{q_2-(1-\lambda)u_2\}a_2 z + \left\{ \begin{array}{l} \varphi_m^{m+1}(3,q)[q_3-(1-\lambda)u_3]a_3 \\ -[\varphi_m^{m+1}(2,q)]^2(1-\lambda)u_2\left[q_2-(1-\frac{\lambda}{2})u_2\right]a_2^2 \end{array} \right\} z^2+\cdots$$

$$1+\varphi_m^{m+1}(2,q)\psi_2(\lambda)a_2 z + \left\{ \begin{array}{l} \varphi_m^{m+1}(3,q)\psi_3(\lambda)a_3 \\ -[\varphi_m^{m+1}(2,q)]^2(1-\lambda)u_2\psi_2\left(\frac{\lambda}{2}\right)a_2^2 \end{array} \right\} z^2+\cdots \tag{32}$$

where q_n and $\psi_n(\lambda)$ are given by (29) and (30) respectively.

Equating the coefficient of z and z^2 on both sides of (31) and taking (32) in account, we obtain

$$a_2 = \frac{p_1\omega_1}{\psi_2(\lambda)\varphi_m^{m+1}(2,q)}$$

and

$$a_3 = \frac{1}{\psi_3(\lambda)\varphi_m^{m+1}(3,q)}\left[p_1\omega_2+\left(p_2+\frac{(1-\lambda)u_2\psi_2\left(\frac{\lambda}{2}\right)}{[\psi_2(\lambda)]^2}p_1^2\right)\omega_1^2\right]$$

and thus we obtain

$$a_2 = \frac{2e^{-i\alpha}(1-\beta)\cos\alpha}{\psi_2(\lambda)\varphi_m^{m+1}(2,q)}\omega_1 \tag{33}$$

$$a_3 = \frac{2e^{-i\alpha}(1-\beta)\cos\alpha}{\psi_3(\lambda)\varphi_m^{m+1}(3,q)}\left[\omega_2+\left(1+\frac{2e^{-i\alpha}(1-\beta)(1-\lambda)u_2\cos\alpha\,\psi_2\left(\frac{\lambda}{2}\right)}{[\psi_2(\lambda)]^2}p_1^2\right)\omega_1^2\right] \tag{34}$$

Now

$$\begin{aligned} &\left|a_3-\mu a_2^2\right| \\ &\leqslant \frac{2(1-\beta)\cos\alpha}{\psi_3(\lambda)\varphi_m^{m+1}(3,q)}\Bigg[|\omega_2| \\ &\quad+\left|1+\frac{2e^{-i\alpha}(1-\beta)\cos\alpha}{[\psi_2(\lambda)]^2}\left(\psi_2\left(\frac{\lambda}{2}\right)(1-\lambda)u_2-\mu\psi_3(\lambda)\frac{\varphi_m^{m+1}(3,q)}{\varphi_m^{m+1}(2,q)}\right)\right||\omega_1|^2\Bigg] \\ &\leqslant \frac{2(1-\beta)\cos\alpha}{\psi_3(\lambda)\varphi_m^{m+1}(3,q)}\left[(1-|\omega_1|^2)+\left|1+Me^{-i\alpha}\cos\alpha\right||\omega_1|^2\right] \end{aligned}$$

where
$$M = \frac{2(1-\beta)}{[\psi_2(\lambda)]^2}\left(\psi_2\left(\frac{\lambda}{2}\right)(1-\lambda)u_2 - \mu\psi_3(\lambda)\frac{\varphi_m^{m+1}(3,q)}{\varphi_m^{m+1}(2,q)}\right). \tag{35}$$
Now
$$\begin{aligned}|a_3-\mu a_2^2| &\leqslant \frac{2(1-\beta)\cos\alpha}{\psi_3(\lambda)\varphi_m^{m+1}(3,q)}\left[1+(|1+Me^{-i\alpha}\cos\alpha|-1)|\omega_1|^2\right]\\ &\leqslant \frac{2(1-\beta)\cos\alpha}{\psi_3(\lambda)\varphi_m^{m+1}(3,q)}\left[1+\left(\sqrt{1+M[2+M\cos\alpha]\cos\alpha}-1\right)|\omega_1|^2\right]\\ &=: \frac{2(1-\beta)\cos\alpha}{\psi_3(\lambda)\varphi_m^{m+1}(3,q)}F(x,y), \qquad (x,y)\in[0,1]^2, \end{aligned} \tag{36}$$

where $F(x,y) = 1+\left(\sqrt{1+M[2+Mx]x}-1\right)y^2$, $x=\cos\alpha$, $y=|\omega_1|$.

Simple calculation shows that the function $F(x,y)$ does not have a local maximum at any interior point of the open rectangle $(0,1)^2$. Thus, the maximum must be attained at the boundary point. Since $F(x,0)=1$, $F(0,y)=1$ and $F(1,1)=|1+M|$, it follows that the maximal value of $F(x,y)$ may be $F(0,0)=1$ or $F(1,1)=|1+M|$. So,
$$\left|a_3-\mu a_2^2\right| \leqslant \frac{2(1-\beta)\cos\alpha}{\psi_3(\lambda)\varphi_m^{m+1}(3,q)}\max\{1,|1+M|\}. \tag{37}$$

Case 1: If $\mu \leqslant \delta_1$, where δ_1 is given by (27), then $M \geqslant 0$ implies $|1+M| \geqslant 1$. Now from (37) we get
$$\begin{aligned}\left|a_3-\mu a_2^2\right| &\leqslant \frac{2(1-\beta)\cos\alpha}{\psi_3(\lambda)\varphi_m^{m+1}(3,q)}\\ &\times\left[1+\frac{2(1-\beta)(1-\lambda)u_2\psi_2\left(\frac{\lambda}{2}\right)}{[\psi_2(\lambda)]^2} - \frac{2\mu(1-\beta)\psi_3(\lambda)\varphi_m^{m+1}(3,q)}{[\psi_2(\lambda)]^2[\varphi_m^{m+1}(2,q)]^2}\right]\end{aligned}$$

which is the first part of the inequality (26).

Case 2: If $\delta_1 \leqslant \mu \leqslant \delta_2$, where δ_2 is given by (28), then $|1+M| \leqslant 1$, thus from (37), we obtain
$$\left|a_3-\mu a_2^2\right| \leqslant \frac{2(1-\beta)\cos\alpha}{\psi_3(\lambda)\varphi_m^{m+1}(3,q)}$$
which is the second part of the inequality (26).

Case 3: If $\mu \geqslant \delta_2$, where δ_2 is given by (28), then $M \leqslant -2$ and it follows from (37)
$$\begin{aligned}\left|a_3-\mu a_2^2\right| &\leqslant \frac{2(1-\beta)\cos\alpha}{\psi_3(\lambda)\varphi_m^{m+1}(3,q)}\\ &\times\left[\frac{2\mu(1-\beta)\psi_3(\lambda)\varphi_m^{m+1}(3,q)}{[\psi_2(\lambda)]^2[\varphi_m^{m+1}(2,q)]^2} - \frac{2(1-\beta)(1-\lambda)u_2\psi_2(\frac{\lambda}{2})}{[\psi_2(\lambda)]^2} - 1\right],\end{aligned}$$

which is the third part of the inequality (26). □

In view of Lemma 2, the results are sharp for $\omega(z)=z$ and $\omega(z)=z^2$ or one of their rotations.

Next, we consider the Fekete-Szegő problem for the class $\mathscr{H}_q(\alpha,\beta,\lambda,t)$ with μ complex parameter.

THEOREM 6. *Let $f\in\mathscr{H}_q(\alpha,\beta,\lambda,t)$ be given by (1) and let μ be a complex number. Then,*

$$\left|a_3-\mu a_2^2\right| \leqslant \frac{2(1-\beta)\cos\alpha}{\psi_3(\lambda)\varphi_m^{m+1}(3,\alpha)}\max\{1,|S|\}, \tag{38}$$

where

$$S=\frac{2e^{-i\alpha}(1-\beta)\cos\alpha}{[\psi_2(\lambda)]^2}\left(\mu\psi_3(\lambda)\frac{\varphi_m^{m+1}(3,q)}{[\varphi_m^{m+1}(2,q)]^2}-\psi_2\left(\frac{\lambda}{2}\right)(1-\lambda)u_2\right)-1. \tag{39}$$

The result is sharp.

Proof. Assume that $f\in\mathscr{H}_q(\alpha,\beta,\lambda,t)$. Making use of (31) and (32) we obtain

$$\begin{aligned}|a_3-\mu a_3^2| \leqslant \frac{2(1-\beta)\cos\alpha}{\psi_3(\lambda)\varphi_m^{m+1}(3,\alpha)}\Bigg|\omega_2-\Bigg\{\frac{2e^{-i\alpha}(1-\beta)\cos\alpha}{[\psi_2(\lambda)]^2}\Bigg(-\psi_2\left(\frac{\lambda}{2}\right)(1-\lambda)u_2\\ +\mu\psi_3(\lambda)\frac{\varphi_m^{m+1}(3,q)}{[\varphi_m^{m+1}(2,q)]^2}\Bigg)-1\Bigg\}\omega_1^2\Bigg| \leqslant \frac{2(1-\beta)\cos\alpha}{\psi_3(\lambda)\varphi_m^{m+1}(3,\alpha)}|\omega_2-S\omega_1^2|,\end{aligned}$$

where S is given by (39). □

The inequality (38) follows by appying Lemma 2.

REMARK 2. By specializing the parameter $\lambda=0$ and $\lambda=1$, one can state the above discussed results for function f in the subclasses defined in Example 2 and 3, respectively.

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