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ON ASYMPTOTICALLY DEFERRED STATISTICAL EQUIVALENT MEASURABLE FUNCTIONS

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Abstract. The primary focus of this article is to introduce the ideas of asymptotically deferred statistical equivalence and strongly asymptotically deferred statistical equivalence by considering two nonnegative real-valued Lebesgue measurable functions on $(1,\infty)$. Additionally, we examine some main properties of these two concepts.

1. Introduction

Fast [6] extended the notion of ordinary convergence of a real sequence to the concept of statistical convergence. Also, in [16] Schoenberg presented essential properties of statistical convergence and investigated the relationship between statistical convergence and summability method. The readers should refer to the monographs [2] and [10] for the background on the sequence spaces and related topics, and to the recent papers [4], [5], [7], [11] and [17] on the statistically convergence.

On the other hand, Pobyvancts [14] presented the idea of asymptotically regular matrices. In 2003, Patterson [13] introduced the concept of asymptotically statistical equivalent sequences by combining the notion of asymptotically equivalence introduced by Marouf [9] and statistical convergence as follows:

DEFINITION 1. Two nonnegative sequences $y=(y_r)$ and $z=(z_r)$ are said to be asymptotically statistically equivalent of multiple ξ if for every $\varepsilon>0$,

$$\lim_{w\to\infty}\frac{1}{w}\left|\left\{r\leqslant w:\left|\frac{y_r}{z_r}-\xi\right|\geqslant\varepsilon\right\}\right|=0$$

and it is written as $y\overset{S_\xi}{\sim}z$.

In the other direction, Agnew [1] presented the deferred Cesáro mean by

$$(D_{\tau,\upsilon}y)_m:=\frac{1}{\upsilon(m)-\tau(m)}\sum_{r=\tau(m)+1}^{\upsilon(m)}y_r,\quad m=1,2,3,\ldots,$$

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where $\{\tau(m)\}$ and $\{\upsilon(m)\}$ are the sequences of positive natural numbers satisfying $\tau(m) < \upsilon(m)$ and $\lim_{m\to\infty} \upsilon(m) = \infty$.

Following Agnew's results, Küçükaslan and Yılmaztürk [8] introduced the notions of deferred density as follows:

DEFINITION 2. Let E be a subset of $\mathbb{N}$ and $E_d(m) = \{\zeta \in E : \tau(m) < \zeta \leqslant \upsilon(m)\}$. Then, the deferred density $\delta_d(E)$ of the set E is defined by

$$\delta_d(E) = \lim_{m\to\infty} \frac{1}{\upsilon(m) - \tau(m)} |E_d(m)|.$$

If the limit exists, then the vertical bars show the cardinality of the set $E_d(m)$.

Also, strongly summable single valued functions were introduced by Borwein [3], as follows:

DEFINITION 3. A nonnegative real-valued Lebesgue measurable function $g(\varsigma)$ on the interval $(1,\infty)$ is said to be strongly summable to ξ if,

$$\lim_{m\to\infty} \frac{1}{m} \int_1^m |g(\varsigma) - \xi| \, d\zeta = 0.$$

Afterward, Nuray [12] introduced the notion of λ-statistically convergent functions. Following his results, Savaş and Öztürk [15] presented λ-statistical asymptotically equivalence of the measurable functions. The primary focus of this article is to study the ideas of asymptotically deferred statistically equivalent functions and strongly asymptotically deferred statistical equivalent functions by using $g(\varsigma)$ and $h(\varsigma)$. Additionally, we examine some main properties of these concepts. Note that $g(\varsigma)$ and $h(\varsigma)$ shall be two measurable real valued functions on the interval $(1,\infty)$ throughout this paper.

2. The main definitions and inclusion theorems

Before we present the main results of this study, we present the following definitions:

DEFINITION 4. Let $\{\tau(m)\}$ and $\{\upsilon(m)\}$ be two sequences. Then two functions $g(\varsigma)$ and $h(\varsigma)$ are asymptotically deferred statistically equivalent of multiple ξ if for each $\varepsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{\upsilon(m) - \tau(m)} \left| \left\{ \tau(m) < \varsigma \leqslant \upsilon(m) : \left| \frac{g(\varsigma)}{h(\varsigma)} - \xi \right| \geqslant \varepsilon \right\} \right| = 0.$$

We symbolize this equivalence by

$$g(\varsigma) \overset{S_D^{\xi}}{\sim} h(\varsigma)$$

and if $\xi = 1$, then we call as asymptotically deferred statistically equivalence. If $\upsilon(m) = m$ and $\tau(m) = 0$, then the notion coincides the concept of asymptotically statistically equivalent functions of multiple ξ which was denoted by $g(\varsigma) \overset{S^{\xi}}{\sim} h(\varsigma)$. Also, if $h(\varsigma) = 1$ and for every $\varepsilon > 0$, then $g(\varsigma)$ is said to be deferred statistically convergent function to ξ.

DEFINITION 5. Let $\{\tau(m)\}$ and $\{\upsilon(m)\}$ be two sequences. Then two functions $g(\varsigma)$ and $h(\varsigma)$ are strongly deferred asymptotically equivalent of multiple ξ if for every $\varepsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{\upsilon(m) - \tau(m)} \int_{\zeta=\tau(m)+1}^{\upsilon(m)} \left| \frac{g(\varsigma)}{h(\varsigma)} - \xi \right| d\varsigma = 0.$$

We symbolize it by

$$g(\varsigma) \overset{V_D^{\xi}}{\sim} h(\varsigma)$$

and if $\xi = 1$ then we call it a simply strongly deferred asymptotically equivalent. If $\upsilon(m) = m$ and $\tau(m) = 0$, then the concept coincides the notion of strongly asymptotically equivalent functions of multiple ξ which was denoted by $g(\varsigma) \overset{V^{\xi}}{\sim} h(\varsigma)$, and also if $h(\varsigma) = 1$, for every $\varepsilon > 0$, then $g(\varsigma)$ is said to be strongly $D_f^{\tau,\upsilon}$-convergent to ξ.

THEOREM 1. *If* $\left\{\frac{\upsilon(m)}{\upsilon(m)-\tau(m)}\right\}$ *is bounded, then* $g(\varsigma) \overset{S^{\xi}}{\sim} h(\varsigma)$ *implies* $g(\varsigma) \overset{S_D^{\xi}}{\sim} h(\varsigma)$*)*.

Proof. Let $g(\varsigma) \overset{S^{\xi}}{\sim} h(\varsigma)$. Since $\tau(m) < \upsilon(m)$ and $\lim_{m\to\infty} \upsilon(m) = +\infty$, if

$$\lim_{m\to\infty} \frac{1}{m} \left| \left\{ \varsigma \leqslant m : \left| \frac{g(\varsigma)}{h(\varsigma)} - \xi \right| \geqslant \varepsilon \right\} \right| = 0,$$

then

$$\lim_{m\to\infty} \frac{1}{\upsilon(m)} \left| \left\{ \varsigma \leqslant \upsilon(m) : \left| \frac{g(\varsigma)}{h(\varsigma)} - \xi \right| \geqslant \varepsilon \right\} \right| = 0.$$

Since $\left\{\frac{\upsilon(m)}{\upsilon(m)-\tau(m)}\right\}$ is bounded then there is a number M such that $\frac{\upsilon(m)}{\upsilon(m)-\tau(m)} \leqslant M$. For a given $\varepsilon > 0$, we obtain

$$\left\{ \zeta \leqslant \upsilon(m) : \left| \frac{g(\varsigma)}{h(\varsigma)} - \xi \right| \geqslant \varepsilon \right\} \supseteq \left\{ \tau(m) < \zeta \leqslant \upsilon(m) : \left| \frac{g(\varsigma)}{h(\varsigma)} - \xi \right| \geqslant \varepsilon \right\}$$

and therefore

$$\left| \left\{ \zeta \leqslant \upsilon(m) : \left| \frac{g(\varsigma)}{h(\varsigma)} - \xi \right| \geqslant \varepsilon \right\} \right| \geqslant \left| \left\{ \tau(m) < \zeta \leqslant \upsilon(m) : \left| \frac{g(\varsigma)}{h(\varsigma)} - \xi \right| \geqslant \varepsilon \right\} \right|.$$

Consequently,

$$\begin{aligned}&\frac{1}{\upsilon(m)-\tau(m)}\left|\left\{\tau(m)<\zeta\leqslant\upsilon(m):\left|\frac{g(\varsigma)}{h(\varsigma)}-\xi\right|\geqslant\varepsilon\right\}\right|\\&\leqslant\frac{\upsilon(m)}{\upsilon(m)-\tau(m)}\frac{1}{\upsilon(m)}\left|\left\{\zeta\leqslant\upsilon(m):\left|\frac{g(\varsigma)}{h(\varsigma)}-\xi\right|\geqslant\varepsilon\right\}\right|.\end{aligned}$$

By taking limit as $m\to\infty$, we are granted $g(\varsigma)\overset{S_D^{\xi}}{\sim}h(\varsigma)$. □

COROLLARY 1. *Let $\{\upsilon(m)\}$ be an arbitrary sequence with $\upsilon(m)<m$ for all $m\in\mathbb{N}$ and $\left\{\frac{m}{\upsilon(m)-\tau(m)}\right\}$ be a bounded sequence. Then, $g(\varsigma)\overset{S^{\xi}}{\sim}h(\varsigma)$ implies $g(\varsigma)\overset{S_D^{\xi}}{\sim}h(\varsigma)$.*

THEOREM 2. *If $\left\{\frac{\tau(m)}{\upsilon(m)-\tau(m)}\right\}$ is bounded then $g(\varsigma)\overset{V^{\xi}}{\sim}h(\varsigma)$ implies $g(\varsigma)\overset{V_D^{\xi}}{\sim}h(\varsigma)$.*

Proof. Since $\left\{\frac{\tau(m)}{\upsilon(m)-\tau(m)}\right\}$ is bounded there is a positive number κ such that $\frac{\tau(m)}{\upsilon(m)-\tau(m)}\leqslant\kappa$. Presume that $g(\varsigma)\overset{V^{\xi}}{\sim}h(\varsigma)$. Then, we obtain the following

$$\begin{aligned}&\frac{1}{\upsilon(m)-\tau(m)}\int_{\zeta=\tau(m)+1}^{\upsilon(m)}\left|\frac{g(\varsigma)}{h(\varsigma)}-\xi\right|d\varsigma\\
&=\frac{1}{\upsilon(m)-\tau(m)}\left[\int_{\zeta=1}^{\upsilon(m)}\left|\frac{g(\varsigma)}{h(\varsigma)}-\xi\right|d\varsigma-\int_{\zeta=1}^{\tau(m)}\left|\frac{g(\varsigma)}{h(\varsigma)}-\xi\right|d\varsigma\right]\\
&\leqslant\frac{1}{\upsilon(m)-\tau(m)}\left[\int_{\zeta=1}^{\upsilon(m)}\left|\frac{g(\varsigma)}{h(\varsigma)}-\xi\right|d\varsigma+\int_{\zeta=1}^{\tau(m)}\left|\frac{g(\varsigma)}{h(\varsigma)}-\xi\right|d\varsigma\right]\\
&=\frac{\tau(m)}{\upsilon(m)-\tau(m)}\frac{1}{\tau(m)}\int_{\zeta=1}^{\tau(m)}\left|\frac{g(\varsigma)}{h(\varsigma)}-\xi\right|d\varsigma\\
&\quad+\frac{\upsilon(m)}{\upsilon(m)-\tau(m)}\frac{1}{\upsilon(m)}\int_{\zeta=1}^{\upsilon(m)}\left|\frac{g(\varsigma)}{h(\varsigma)}-\xi\right|d\varsigma\\
&=\frac{\kappa}{\tau(m)}\int_{\zeta=1}^{\tau(m)}\left|\frac{g(\varsigma)}{h(\varsigma)}-\xi\right|d\varsigma+(1+\kappa)\frac{1}{\upsilon(m)}\int_{\zeta=1}^{\upsilon(m)}\left|\frac{g(\varsigma)}{h(\varsigma)}-\xi\right|d\varsigma.\end{aligned}$$

Therefore, $g(\varsigma)\overset{V_D^{\xi}}{\sim}h(\varsigma)$. □

COROLLARY 2. *Let* $\{\tau(m)\}$ *and* $\{\upsilon(m)\}$ *be two sequences. If* $\left\{\frac{\upsilon(m)+\tau(m)}{\upsilon(m)-\tau(m)}\right\}$ *is bounded, then* $g(\varsigma) \overset{V^{\xi}}{\sim} h(\varsigma)$ *implies* $g(\varsigma) \overset{V_D^{\xi}}{\sim} h(\varsigma)$.

THEOREM 3. *If* $g(\varsigma) \overset{V_D^{\xi}}{\sim} h(\varsigma)$ *, then* $g(\varsigma) \overset{S_D^{\xi}}{\sim} h(\varsigma)$.

Proof. Presume that $g(\varsigma) \overset{V_D^{\xi}}{\sim} h(\varsigma)$ and let us denote the set

$$\left\{\tau(m) \leqslant \zeta \leqslant \upsilon(m) : \left|\frac{g(\varsigma)}{h(\varsigma)} - \xi\right| \geqslant \varepsilon\right\}$$

by $E(\varepsilon)$. Thus,

$$\begin{aligned}
&\frac{1}{\upsilon(m)-\tau(m)} \int\limits_{\zeta=\tau(m)+1}^{\upsilon(m)} \left|\frac{g(\varsigma)}{h(\varsigma)} - \xi\right| d\varsigma \\
&\geqslant \frac{1}{\upsilon(m)-\tau(m)} \int\limits_{\zeta=\tau(m)+1,\zeta\in E(\varepsilon)}^{\upsilon(m)} \left|\frac{g(\varsigma)}{h(\varsigma)} - \xi\right| d\varsigma \\
&\geqslant \varepsilon \frac{1}{\upsilon(m)-\tau(m)} |E(\varepsilon)|
\end{aligned}$$

holds. After taking limits when $m \to \infty$, we obtain the following

$$\lim_{m\to\infty} \frac{1}{\upsilon(m)-\tau(m)} \left|\left\{\tau(m) \leqslant \zeta \leqslant \upsilon(m) : \left|\frac{g(\varsigma)}{h(\varsigma)} - \xi\right| \geqslant \varepsilon\right\}\right| = 0$$

the proof of theorem is obtained. □

THEOREM 4. *Let* $g(\varsigma)$ *and* $h(\varsigma)$ *are bounded measurable functions. If* $g(\varsigma) \overset{S_D^{\xi}}{\sim} h(\varsigma)$*, then* $g(\varsigma) \overset{V_D^{\xi}}{\sim} h(\varsigma)$.

Proof. Let us denote $E^c(\varepsilon)$ by

$$E^c(\varepsilon) := \left\{\zeta : \tau(m) \leqslant \zeta \leqslant \upsilon(m) : \left|\frac{g(\varsigma)}{h(\varsigma)} - \xi\right| < \varepsilon\right\},$$

and we suppose that $g(\varsigma)$ and $h(\varsigma)$ are bounded and $g(\varsigma) \overset{S_D^{\xi}}{\sim} h(\varsigma)$. It is obvious that there exists a positive real number M such that $\left|\frac{g(\varsigma)}{h(\varsigma)} - \xi\right| \leqslant M$ for $\zeta \in \mathbb{N}$. Hence, we get

$$
\begin{aligned}
&\frac{1}{\upsilon(m)-\tau(m)} \int_{\zeta=\tau(m)+1}^{\upsilon(m)} \left|\frac{g(\varsigma)}{h(\varsigma)}-\xi\right| d\varsigma \\
&= \frac{1}{\upsilon(m)-\tau(m)} \left[\int_{\zeta=\tau(m)+1, \zeta\in E(\varepsilon)}^{\upsilon(m)} \left|\frac{g(\varsigma)}{h(\varsigma)}-\xi\right| d\varsigma + \int_{\zeta=\tau(m)+1, \zeta\notin E(\varepsilon)}^{\upsilon(m)} \left|\frac{g(\varsigma)}{h(\varsigma)}-\xi\right| d\varsigma \right] \\
&\leqslant \frac{1}{\upsilon(m)-\tau(m)} \left[\int_{\zeta=\tau(m)+1, \zeta\in E(\varepsilon)}^{\upsilon(m)} M + \int_{\zeta=\tau(m)+1, \zeta\notin E(\varepsilon)}^{\upsilon(m)} \varepsilon \right] \\
&\leqslant \frac{1}{\upsilon(m)-\tau(m)} \left[M\left|E(\varepsilon)\right| + \varepsilon\left|E^c(\varepsilon)\right| \right].
\end{aligned}
$$

Therefore, we have

$$
\lim_m \frac{|E(\varepsilon)|}{\upsilon(m)-\tau(m)} = 0 \quad \text{and} \quad \lim_m \frac{|E^c(\varepsilon)|}{\upsilon(m)-\tau(m)} = 1.
$$

This proves the theorem. □

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