Journal of Classical Analysis
Volume 18, Number 1 (2021), 17–28 doi:10.7153/jca-2021-18-02

APPLICATIONS OF BRIOT–BOUQUET DIFFERENTIAL SUBORDINATION

NISHA BOHRA, SUSHIL KUMAR AND V. RAVICHANDRAN*

Abstract. Sharp estimates on β in Briot-Bouquet differential subordination

$$p(z)+\beta zp'(z)/p(z)\prec h(z)$$

are obtained so that its solution p is subordinate to some specific Carathéodory functions. As an application, the estimates on β are obtained so that the integral operator $\beta^{-1}\int_0^z f^{1/\beta}(t)t^{-1}dt$ maps the class of starlike functions f satisfying $zf'(z)/f(z)\prec 1+z$ to various subclasses of class of starlike functions. Further, a sufficient condition is established for parabolic starlikeness.

1. Introduction

Let $\mathscr{A}$ be the class of analytic functions f in the open unit disk $\mathbb{D}:=\{z\in\mathbb{C}:|z|<1\}$ and normalized by the conditions $f(0)=0$ and $f'(0)=1$. Let $\mathscr{S}$ be the class of univalent functions in $\mathscr{A}$. For $f,g\in\mathscr{A}$, $f\prec g$ if there is an analytic function $w:\mathbb{D}\to\mathbb{D}$ with $w(0)=0$ and $|w(z)|<1$ such that $f(z)=g(w(z))$ for all $z\in\mathbb{D}$, where $\prec$ denotes the subordination. In particular, if g is univalent in $\mathbb{D}$, then the following equivalence condition holds:

$$f\prec g \iff f(0)=g(0) \text{ and } f(\mathbb{D})\subset g(\mathbb{D}).$$

By making use of subordination, Ma and Minda[13] unified various subclasses of class of starlike functions in which the quantity $zf'(z)/f(z)$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function φ with positive real part in the unit disk $\mathbb{D}$ and normalized by $\varphi(0)=1$ and $\varphi'(0)>0$. The class of Ma-Minda starlike functions consists of functions $f\in\mathscr{A}$ satisfying the subordination $zf'(z)/f(z)\prec\varphi(z)$ and is denoted by $\mathscr{S}^*(\varphi)$. In 1993, Ronning [19] introduced the class $\mathscr{S}_P^*:=\mathscr{S}^*(\varphi_{\mathscr{PAR}}(z))$ of the parabolic starlike functions, where $\varphi_{\mathscr{PAR}}(z)=1+(2/\pi^2)\left(\log((1-\sqrt{z})/(1+\sqrt{z}))\right)^2$ is the analytic function defined on $\mathbb{D}$. Analytically, it is noted that

$$f\in\mathscr{S}_P^* \quad \text{if and only if} \quad \left|\frac{zf'(z)}{f(z)}-1\right|<\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right),\ z\in\mathbb{D}.$$

Mathematics subject classification (2020): 30C45, 30C80.
Keywords and phrases: Subordination, starlike function, lemniscate of Bernoulli, sine, exponential, rational, and hypergeometric function, parabolic starlike function.

* Corresponding author.

In [22], Sokól and Stankiewicz introduced and studied the class $\mathcal{S}_L^* := \mathcal{S}^*(\sqrt{1+z})$ which is the subclass of $\mathcal{S}^*$ consisting of functions $f \in \mathcal{A}$ such that for each $z \in \mathbb{D}$, $w = zf'(z)/f(z)$ lies in the region bounded by right half of the lemniscate of Bernoulli given by $|w^2 - 1| < 1$. In 2015, Mendiratta *et al.* [16] established and studied the class $\mathcal{S}_e^* := \mathcal{S}^*(e^z)$ consisting of functions $f \in \mathcal{A}$ satisfying the condition $|\log(zf'(z)/f(z))| < 1$. In 2015, Raina and Sokół [18] studied a class $\mathcal{S}_q^* := \mathcal{S}^*(\varphi_q)$ where $\varphi_q(z) := z + \sqrt{1+z^2}$. Further, Kumar and Ravichandran [10], introduced and studied the geometric properties of the class $\mathcal{S}_R^* = \mathcal{S}^*(\varphi_0)$ associated with the rational function $\varphi_0(z) := 1 + (z/k)((k+z)/(k-z))$, $(k = \sqrt{2}+1)$. Recently, Cho *et al.*[6] discussed various geometric estimates related to the class $\mathcal{S}_s^* = \mathcal{S}^*(\varphi_s)$ where $\varphi_s(z) := 1 + \sin z$.

The class $\mathcal{P}$ consists of analytic functions $p : \mathbb{D} \to \mathbb{C}$ with positive real part in $\mathbb{D}$ and normalized by the condition $p(0) = 1$. Let β, γ be real or complex scalars and h be a univalent function defined on unit disk $\mathbb{D}$. Miller and Mocanu [14] studied the properties of Briot-Bouquet differential subordination

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z)$$

which has been used in the theory of univalent functions. For particular values of the scalars β, γ and the function h, many researchers have discussed the first order differential subordination. In 1989, Nunokawa *et al.* [17] proved that if $1 + zp'(z) \prec 1 + z$, then $p(z) \prec 1 + z$. In [2, 3], Ali *et al.* obtained some sufficient conditions for normalized analytic functions to be in some subclasses of Janowski starlike functions and lemniscate starlike functions. Recently, authors [11] computed sharp bounds on parameters involved in certain first order differential subordinations related to the functions with positive real part. For details, see [1, 4, 7, 5, 8, 12, 21].

Motivated by these works, we obtain sharp estimates on β such that the first order differential subordination

$$p(z) + \beta \frac{zp'(z)}{p(z)} \prec 1 + z$$

implies $p(z) \prec \mathcal{Q}(z)$. where $\mathcal{Q}(z) = e^z, \sqrt{1+z}, \varphi_0(z), \varphi_s(z)$ or $\varphi_q(z)$. For a starlike function satisfying $zf'(z)/f(z) \prec 1+z$, we have shown that, for $\beta \geqslant 0.291936$, the integral $\beta^{-1} \int_0^z f^{1/\beta}(t) t^{-1} dt$ belongs to the class $\mathcal{S}_e^*$. Similar bounds were obtained for the integral to belong to the classes $\mathcal{S}_L^*$, $\mathcal{S}_R^*$ and $\mathcal{S}_s^*$. Moreover, we estimate a bound on β so that $p \prec \varphi_{PAR}$ whenever the subordination relation $1 + \beta z p'(z) \prec e^z$ holds and this subordination relation provides a sufficient condition for parabolic starlikeness. Our results are proved by applying the theory of differential subordination developed by Miller and Mocanu and using some properties of confluent hypergeometric functions.

2. Subordination and hypergeometric functions

The confluent (or Kummer) hypergeometric function $\Phi(a,c;z)$ is given by the convergent power series

$$\Phi(a,c;z) = {}_1F_1(a,c;z) := 1 + \frac{a}{c}\frac{z}{1!} + \frac{a(a+1)}{c(c+1)}\frac{z^2}{2!} + \cdots, \tag{1}$$

where a and c are complex numbers with $c \neq 0,-1,-2,\ldots$. The function Φ is analytic in $\mathbb{C}$ and satisfies the Kummer's differential equation

$$zw''(z) + (c-z)w'(z) - aw(z) = 0.$$

Let $(d)_k$ denotes the Pohhammer symbol given by $(d)_k = \Gamma(d+k)/\Gamma(d) = d(d+1)\cdots(d+k-1)$ and $(d)_0 = 1$, then (1) can be written in the form

$$\Phi(a,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k}\frac{z^k}{k!} = \frac{\Gamma(c)}{\Gamma(a)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)}{\Gamma(c+k)}\frac{z^k}{k!}.$$

Also $c\Phi'(a,c;z) = a\phi(a+1,c+1;z)$ and the following integral representation of Φ [15, p. 5] given by

$$\Phi(a,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}e^{tz}dt = \int_0^1 e^{tz}d\mu(t),$$

is well known, where $d\mu(t) = \dfrac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}t^{a-1}(1-t)^{c-a-1}dt$ is a probability measure on $[0,1]$ and $\operatorname{Re} c > \operatorname{Re} a > 0$.

Let c be a complex number such that $\operatorname{Re} c > 0$, let n be a positive integer, and let

$$C_n = C_n(c) = \frac{n}{\operatorname{Re} c}\left[|c|\sqrt{1 + 2\operatorname{Re} c/n} + \operatorname{Im} c\right].$$

If $R(z)$ is the univalent function defined in $\mathbb{D}$ by $R(z) = 2C_n z/(1-z^2)$, then the open door function is defined by

$$R_{c,n}(z) = R\left(\frac{z+b}{1+\overline{b}z}\right) = 2C_n\frac{(z+b)(1+\overline{b}z)}{(1+\overline{b}z)^2 - (z+b)^2},$$

where $b = R^{-1}(c)$. We note that if $c > 0$, then the open door function reduces to

$$R_{c,n}(z) = c\frac{1+z}{1-z} + \frac{2nz}{1-z^2}. \tag{2}$$

We see that $R_{c,n}$ is univalent in $\mathbb{D}$, $R_{c,n}(0) = c$ and $R_{c,n}(\mathbb{D}) = R(\mathbb{D})$ is the complex plane with slits along the half-lines $\operatorname{Re} w = 0$ and $|\operatorname{Im} w| \geqslant C_n$.

THEOREM 1. *Let p be analytic function in $\mathbb{D}$ with $p(0)=1$. Let*

$$p(z)+\beta\frac{zp'(z)}{p(z)} \prec 1+z, \quad \beta>0.$$

Then the following are true:

(a) If $\beta \geqslant \beta_e \simeq 0.291936$, then $p(z) \prec e^z$.

(b) If $\beta \geqslant \beta_l \simeq 1.76736$, then $p(z) \prec \sqrt{1+z}$.

(c) If $\beta \geqslant \beta_s \simeq 0.322915$, then $p(z) \prec \varphi_s(z)$.

(d) If $\beta \geqslant \beta_R \simeq 4.3622$, then $p(z) \prec \varphi_0(z)$.

(e) If $\beta \geqslant \beta_L \simeq 0.391235$, then $p(z) \prec \varphi_q(z)$.

The bounds on β are sharp.

In order to prove this result, we need the following lemma due to Miller and Mocanu:

LEMMA 1. [15, Theorem 3.2j., p. 97] *Let β, $\gamma \in \mathbb{C}$ with $\beta \neq 0$, and let n be a positive integer. Let $R_{\beta a+\gamma,n}$ be as given in (2) and let h be analytic in $\mathbb{D}$, with $h(0)=a$, $\operatorname{Re}(\beta a+\gamma)>0$ and $\beta h(z)+\gamma \prec R_{\beta a+\gamma,n}(z)$. If q is the analytic solution of Briot-Bouquet differential equation*

$$q(z)+\frac{nzq'(z)}{\beta q(z)+\gamma}=h(z)$$

and if h is convex or $Q(z)=zq'(z)/(\beta q(z)+\gamma)$ is starlike, then q and h are univalent. Furthermore, if $p \in \mu[a,n]$ satisfies

$$p(z)+\frac{nzp'(z)}{\beta p(z)+\gamma} \prec h(z)$$

then $p \prec q$, and q is the best (a,n)- dominant.

Proof of Theorem 1. Consider the following Briot-Bouquet differential equation

$$p(z)+\frac{\beta zp'(z)}{p(z)}=1+z.$$

Using [15, Theorem 3.3d], its analytic solution is given by

$$q_\beta(z)=(\Phi(1,1+1/\beta,-z/\beta))^{-1}.$$

Clearly, the function $h(z)=1+z$ is convex. As $\operatorname{Re} h(z)>0$,

$$\frac{1}{\beta}h(z) \prec \frac{1}{\beta}\left(\frac{1+z}{1-z}\right)+\frac{2z}{1-z^2}=R_{1/\beta,1}(z).$$

From Lemma 1, taking $n=1$, $\gamma=0$, and $\beta \to 1/\beta$, we get $p(z) \prec q_\beta(z)$. We now obtain lower bounds on β for the subordination $q_\beta(z) \prec \mathscr{P}(z)$ to hold for different choices of $\mathscr{P}(z)$. The transitivity will then imply $p(z) \prec \mathscr{P}(z)$ for obtained values of β.

We note that $q_\beta((-1,1)) \subset \mathbb{R}$ and $\mathscr{P}((-1,1)) \subset \mathbb{R}$ for all $\beta>0$ and all choices of $\mathscr{P}$. Also $q_\beta(-1) \leqslant q_\beta(1)$. Hence the subordination $q_\beta(z) \prec \mathscr{P}(z)$ implies

$$\mathscr{P}(-1) \leqslant q_\beta(-1) \leqslant q_\beta(1) \leqslant \mathscr{P}(1).$$

The lower bounds on β are obtained by solving this inequality. Though the converse need not hold, we show that for obtained lower bounds on β, $q_\beta(z) \prec \mathscr{P}(z)$ for all our choices of function $\mathscr{P}$. We note that

$$(\Phi(1,1+1/\beta,1/\beta))^{-1} = \left(\sum_{n=0}^{\infty} \frac{1}{(1+\beta)\dots(1+n\beta)}\right)^{-1}$$

and

$$(\Phi(1,1+1/\beta,-1/\beta))^{-1} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(1+\beta)\dots(1+n\beta)}\right)^{-1}.$$

(a) For $\mathscr{P}(z)=e^z$, the inequalities $e^{-1} \leqslant q_\beta(-1)$ and $q_\beta(1) \leqslant e$ reduce to

$$f(\beta) = (\Phi(1,1+1/\beta,1/\beta))^{-1} - \frac{1}{e} \geqslant 0$$

and

$$g(\beta) = e - (\Phi(1,1+1/\beta,-1/\beta))^{-1} \geqslant 0.$$

Here $\lim_{\beta \searrow 0} f(\beta) = -1/e$ and $\lim_{\beta \nearrow \infty} f(\beta) = e-1>0$. Also $f'(\beta)>0$ for all $\beta \in (0,\infty)$. Hence f is strictly increasing in $(0,\infty)$. Let β_e denotes the unique zero of f in $(0,\infty)$. Then $f(\beta) \geqslant 0$ for every $\beta \geqslant \beta_e \simeq 0.291936$. Also $g(\beta)>0$ for all $\beta \in (0,\infty)$.

We now show that $q_\beta(z) \prec e^z$ for $\beta \geqslant \beta_e \simeq 0.291936$. First we note that $q_\beta(\mathbb{D})$ is convex, symmetrical about real axis and $q_\beta(\mathbb{D}) \subseteq q_{\beta_e}(\mathbb{D})$ for $\beta \geqslant \beta_e$. Hence it suffices to prove that $q_{\beta_e}(z) \prec e^z$. But this is equivalent to showing that

$$|\log(q_{\beta_e}(z))| \leqslant 1, \qquad z \in \overline{\mathbb{D}}$$

where log denotes the principal branch of logarithm function. In fact, proving this for boundary points is enough. For $z=e^{it}$, let $F(t) = |\log(q_{\beta_e}(e^{it}))|^2$, $-\pi \leqslant t \leqslant \pi$. We note that $F(t)=F(-t)$, so it is enough to consider the interval $0 \leqslant t \leqslant \pi$.

$$\begin{aligned} F(t) &= (\ln|q_{\beta_e}(e^{it})|)^2 + (\arg(q_{\beta_e}(e^{it}))^2 \\ &= (\ln|\Phi(1,1+1/\beta_e,-e^{it}/\beta_e)|)^2 + (\arg(\Phi(1,1+1/\beta_e,-e^{it}/\beta_e))^2. \end{aligned}$$

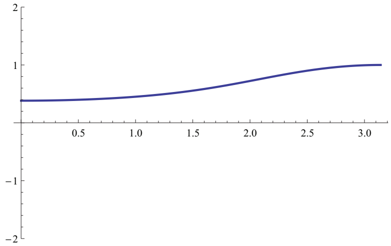

Figure 1: *Graph of* $F(t)$.

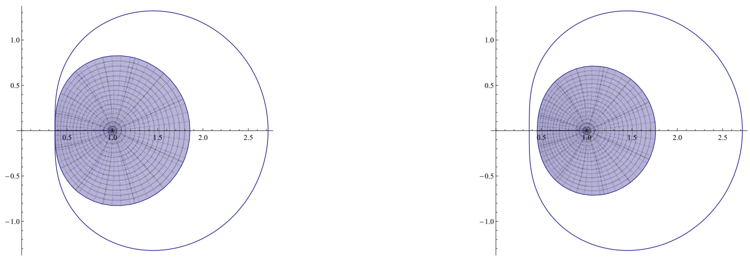

Figure 2: *Image of unit disk* $\mathbb{D}$ *under the functions* e^z *and* $q_\beta(z)$ *for* $\beta=\beta_e\simeq 0.291936$ *and for* $\beta=0.5$ *respectively.*

We see that $F(t)$ is an increasing function of t and $\max_{0\leqslant t\leqslant\pi}F(t)=F(\pi)=1$. See Figure 1.

This proves that $q_{\beta_e}(z)\prec e^z$ and hence $q_\beta(z)\prec e^z$ for $\beta\geqslant\beta_e$. Figure 2 illustrates our proof.

(b) For $\mathscr{P}(z)=\sqrt{1+z}$, the inequalities $0\leqslant q_\beta(-1)$ and $q_\beta(1)\leqslant\sqrt{2}$ reduce to

$$f(\beta)=(\Phi(1,1+1/\beta,1/\beta))^{-1}\geqslant 0$$

and

$$g(\beta)=\sqrt{2}-(\Phi(1,1+1/\beta,-1/\beta))^{-1}\geqslant 0.$$

Clearly, $f(\beta)\geqslant 0$ for $\beta>0$. Also, we see that $\lim_{\beta\searrow 0}g(\beta)<0$ and $\lim_{\beta\nearrow\infty}g(\beta)=\sqrt{2}-1>0$. Also $g'(\beta)>0$ for all $\beta\in(0,\infty)$. Let β_l denotes the unique zero of g in $(0,\infty)$. Then we show that $q_\beta(z)\prec\mathscr{P}(z)$ for $\beta\geqslant\beta_l\simeq 1.76736$.

Again as done in part (a), it suffices to show that $q_{\beta_l}(z)\prec\sqrt{1+z}$. This is equivalent to showing that

$$|(q_{\beta_l}(z))^2-1|\leqslant 1, z\in\overline{\mathbb{D}}. \tag{3}$$

We prove it for boundary points. For $z=e^{it}$, $q_{\beta_l}(z)=(\Phi(1,1+1/\beta_l,-e^{it}/\beta_l))^{-1}$, hence inequality (3) is equivalent to showing that

$$(2\operatorname{Re}\Phi(1,1+1/\beta_l,-e^{it}/\beta_l)^2-1)\geqslant 0.$$

We define

$$G(t) = \operatorname{Re} \Phi(1, 1+1/\beta_l, -e^{it}/\beta_l)^2, \quad t \in [0, \pi].$$

We see that $G(t)$ is an increasing function with $\min_{0 \leqslant t \leqslant \pi} G(t) = G(0) = 1/2$. See Figure 3.

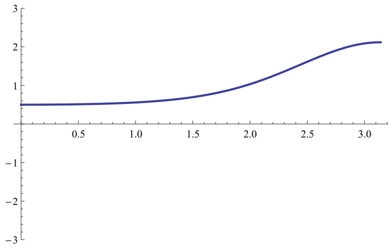

Figure 3: *Graph of* $G(t)$.

(c) For $\mathscr{P}(z) = \varphi_s(z)$, the inequalities $1 - \sin 1 \leqslant q_\beta(-1)$ and $q_\beta(1) \leqslant 1 + \sin 1$ reduce to

$$f(\beta) = (\Phi(1, 1+1/\beta, 1/\beta))^{-1} - 1 + \sin 1 \geqslant 0$$

and

$$g(\beta) = 1 + \sin 1 - (\Phi(1, 1+1/\beta, -1/\beta))^{-1} \geqslant 0.$$

Here we see that both $f(\beta)$ and $g(\beta)$ have unique root in $(0, \infty)$. In fact, $f(\beta) \geqslant 0$ for $\beta \geqslant \beta_1 \simeq 0.04435$ and $g(\beta) \geqslant 0$ for $\beta \geqslant \beta_2 \simeq 0.322915$. Let $\beta_s = \max\{\beta_1, \beta_2\} = \beta_2$.We show that $q_\beta(z) \prec 1 + \sin z$ for $\beta \geqslant \beta_s$. As done in previous parts, it suffices to show that $q_{\beta_s}(z) \prec 1 + \sin z$. This is equivalent to showing that $|\arcsin(q_{\beta_s}(z) - 1)| \leqslant 1$, $z \in \overline{\mathbb{D}}$. We define

$$H(t) = |\arcsin(q_{\beta_s}(e^{it}) - 1)|, \qquad t \in [0, \pi].$$

Then $H(t)$ is decreasing function of t with maximum value 1 attained at $t = 0$. See Figure 4.

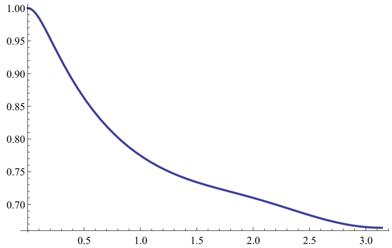

Figure 4: *Graph of* $H(t)$.

(d) For $\mathscr{P}(z)=\varphi_0(z)$, consider the inequalities $\varphi_0(-1)\leqslant q_\beta(-1)$ and $q_\beta(1)\leqslant \varphi_0(1)$. Let

$$f(\beta)=(\Phi(1,1+1/\beta,1/\beta))^{-1}-2(\sqrt{2}-1)\geqslant 0$$

and

$$g(\beta)=2-(\Phi(1,1+1/\beta,-1/\beta))^{-1}\geqslant 0.$$

Since $\lim_{\beta\searrow 0}f(\beta)=-2(\sqrt{2}-1)<0$, $\lim_{\beta\nearrow\infty}f(\beta)=3-2\sqrt{2}>0$ and $f'(\beta)>0$ for all $\beta\in(0,\infty)$, $f(\beta)$ has a unique zero in $(0,\infty)$ say $\beta_R\simeq 4.3622$. Also we see that $g(\beta)\geqslant 0$ for $\beta>0$. We show that

$$q_\beta(z)\prec\varphi_0(z)\quad\text{ for }\quad \beta\geqslant\beta_R.$$

Again it suffices to show that $q_{\beta_R}(z)\prec\varphi_0(z)$. This is equivalent to showing that

$$\frac{k}{2}\left|-q_{\beta_R}(z)+|q_{\beta_R}(z)-2|\right|\leqslant 1,\qquad z\in\overline{\mathbb{D}},$$

where $k=\sqrt{2}+1$. It is enough to show for boundary points. We define

$$M(t)=\left|-q_{\beta_R}(e^{it})+|q_{\beta_R}(e^{it})-2|\right|,\qquad t\in[0,\pi].$$

Then we see that $\max_{0\leqslant t\leqslant 1}M(t)=M(0)=0.405592$ which is less than $2/k=0.828427$. See Figure 5.

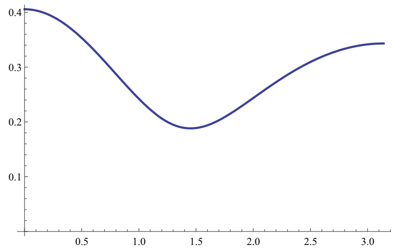

Figure 5: *Graph of $M(t)$.*

(e) For $\mathscr{P}(z)=\varphi_q(z)$, the inequalities $-1+\sqrt{2}\leqslant q_\beta(-1)$ and $q_\beta(1)\leqslant 1+\sqrt{2}$ reduce to

$$f(\beta)=(\Phi(1,1+1/\beta,1/\beta))^{-1}+1-\sqrt{2}\geqslant 0$$

and

$$g(\beta)=1+\sqrt{2}-(\Phi(1,1+1/\beta,-1/\beta))^{-1}\geqslant 0.$$

Here we see that $g(\beta)\geqslant 0$ for $\beta>0$ and $f(\beta)$ have unique root in $(0,\infty)$. In fact, $f(\beta)\geqslant 0$ for

$$\beta\geqslant\beta_L\simeq 0.391235.$$

We show that $q_\beta(z) \prec \varphi_q(z)$ for $\beta \geqslant \beta_L$. It suffices to show that

$$\left| \frac{(q_{\beta_L}(z))^2 - 1}{2q_{\beta_L}(z)} \right| \leqslant 1, \qquad z \in \overline{\mathbb{D}}.$$

We define

$$N(t) = \left| \frac{1 - \Phi(1, 1 + 1/\beta_L, -e^{it}/\beta_L)^2}{2\Phi(1, 1 + 1/\beta_L, -e^{it}/\beta_L)} \right|, \qquad t \in [0, \pi].$$

Then it can be seen that $N(t)$ is an increasing function of t with $\max_{0 \leqslant t \leqslant \pi} N(t) = 1$. See Figure 6.

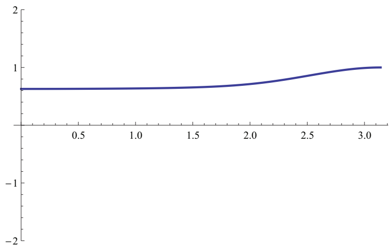

Figure 6: *Graph of $N(t)$.*

□

The Briot-Bouquet differential equations and the integral operator F defined by

$$F(z) = \frac{1}{\beta} \int_0^z f^{1/\beta}(t) t^{-1} dt. \tag{4}$$

are closely related. If we set $p(z) = zF'(z)/F(z)$, then

$$p(z) + \frac{\beta z p'(z)}{p(z)} = \frac{z f'(z)}{f(z)}.$$

Using this relation we have the following corollary:

COROLLARY 1. *Let F be as defined in (4). If $zf'(z)/f(z) \prec 1 + z$, then the following hold:*

(a) If $\beta \geqslant \beta_e \simeq 0.291936$, then $zF'(z)/F(z) \prec e^z$.

(b) If $\beta \geqslant \beta_L \simeq 1.76736$, then $zF'(z)/F(z) \prec \sqrt{1+z}$.

(c) If $\beta \geqslant \beta_S \simeq 0.322915$, then $zF'(z)/F(z) \prec \varphi_s(z)$.

(d) If $\beta \geqslant \beta_R \simeq 4.3622$, then $zF'(z)/F(z) \prec \varphi_0(z)$.

The bounds on β are sharp.

In [20], authors obtained a condition on β so that $p \prec \varphi_{PAR}$ whenever $1+\beta zp'(z)$ is subordinate to $\sqrt{1+z}$. To determine the estimate on β so that $p \prec \varphi_{PAR}$ whenever $1+\beta zp'(z) \prec e^z$, following lemmas are needed.

LEMMA 2. [9, Lemma 4, p. 3] *Let z be a complex number. Then,*

$$|\log(1+z)| \geqslant 1 \quad \textit{if and only if} \quad |z| \geqslant e-1.$$

LEMMA 3. [15, Theorem 3.4h, p. 132] *Let q be analytic in $\mathbb{D}$ and let ψ and ν be analytic in a domain U containing $q(\mathbb{D})$ with $\psi(w) \neq 0$ when $w \in q(\mathbb{D})$. Set $Q(z) := zq'(z)\psi(q(z))$ and $h(z) := \nu(q(z)) + Q(z)$. Suppose that*

(i) either h is convex, or Q is starlike univalent in $\mathbb{D}$ and

(ii) $\operatorname{Re}(zh'(z)/Q(z)) > 0$ for $z \in \mathbb{D}$.

If p is analytic in $\mathbb{D}$, with $p(0) = q(0)$, $p(\mathbb{D}) \subseteq U$ and $\nu(p(z)) + zp'(z)\psi(p(z)) \prec \nu(q(z)) + zq'(z)\psi(q(z))$, then $p(z) \prec q(z)$, and q is best dominant.

THEOREM 2. *Let the function $p \in \mathscr{P}$ satisfying following subordination relation*

$$1+\beta zp'(z) \prec e^z \quad \textit{for } \beta \geqslant \pi(e-1).$$

Then

$$p(z) \prec \varphi_{PAR}(z).$$

Proof. Let the function $q : \mathbb{D} \to \mathbb{C}$ be defined by $q(z) = \varphi_{PAR}(z)$ that is univalent. Let $\nu(w) = 1$ and $\psi(w) = \beta$ be two functions. Since the function q is convex univalent, then the function

$$Q(z) = zq'(z)\psi(q(z)) = \beta zq'(z) = \beta \frac{4\sqrt{z}}{\pi^2(1-z)} \log\frac{1+\sqrt{z}}{1-\sqrt{z}}$$

is starlike in $\mathbb{D}$. The function h is defined on $\mathbb{D}$ as

$$h(z) := \nu(q_\beta(z)) + Q(z) = 1 + \beta \frac{4\sqrt{z}}{\pi^2(1-z)} \log\frac{1+\sqrt{z}}{1-\sqrt{z}}$$

satisfies the inequality $\operatorname{Re}(zh'(z)/Q(z)) > 0$ in $\mathbb{D}$. Note that the function h is univalent. Using Lemma 3, we note that $1+\beta zp'(z) \prec 1+\beta zq'(z)$ implies $p \prec q$. We set $\rho(z) = e^z$ so that $\rho(\mathbb{D}) = \{w \in \mathbb{C} : |\log w| \leqslant 1\}$. Therefore the required subordination $\rho \prec h$ holds if $\partial h(\mathbb{D}) \subset \mathbb{C} \setminus \overline{\rho(\mathbb{D})}$ or $|\log h(z)| > 1$ or

$$\left|\log\left(1+\beta \frac{4\sqrt{z}}{\pi^2(1-z)} \log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right| > 1$$

holds. In view of Lemma 2, the last inequality is equivalent to

$$\left|\beta \frac{4\sqrt{z}}{\pi^2(1-z)} \log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right|^2 > (e-1)^2. \tag{5}$$

For $z = e^{i\theta}$ and $\theta \in [-\pi, \pi]$, consider

$$\begin{aligned}\left|\beta \frac{4e^{i\theta/2}}{\pi^2(1-e^{i\theta})} \log \frac{1+e^{i\theta/2}}{1-e^{i\theta/2}}\right|^2 &= \left|\beta \frac{2i}{\pi^2} \csc(\theta/2)\log(i\cot(\theta/4))\right|^2 \\ &= \frac{\beta^2}{\pi^2}\csc^2(\theta/2) + \frac{4\beta^2}{\pi^4}\csc^2(\theta/2)(\log(\cot(\theta/4)))^2 \\ &=: \beta^2 g(\theta),\end{aligned} \tag{6}$$

where

$$g(\theta) = \frac{1}{\pi^2}\csc^2(\theta/2)\left(1 + \frac{4}{\pi^2}(\log(\cot(\theta/4)))^2\right).$$

Using second derivative test, we note that the function g has minimum value at $\theta = \pi$. Therefore, inequality (6) becomes

$$\left|\beta \frac{4e^{i\theta/2}}{\pi^2} \log \frac{1+e^{i\theta/2}}{1-e^{i\theta/2}}\right|^2 = \beta^2 g(\theta) \geqslant \beta^2 g(\pi) = \frac{\beta^2}{\pi^2} \tag{7}$$

In view of inequalities (5) and (7), we conclude that the desired subordination for $\beta \geqslant \pi(e-1)$. $\square$

As an application of Theorem 2, we have following sufficient condition for the parabolic starlike function:

COROLLARY 2. *Let $\beta \geqslant \pi(e-1)$. If the function $f \in \mathscr{S}$ satisfies following subordination relation*

$$1 + \beta \frac{zf'(z)}{f(z)}\left(\frac{(zf'(z))'}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec e^z$$

then $f \in \mathscr{S}_P^*$.

REFERENCES

[1] O. P. AHUJA, S. KUMAR AND V. RAVICHANDRAN, *Applications of first order differential subordination for functions with positive real part*, Stud. Univ. Babeş-Bolyai Math, **63** (2018), no. 3, 303–311.

[2] R. M. ALI, V. RAVICHANDRAN AND N. SEENIVASAGAN, *Sufficient conditions for Janowski starlikeness*, Int. J. Math. Math. Sci. **2007**, Art. ID 62925, 7 pp.

[3] R. M. ALI, N. E. CHO, V. RAVICHANDRAN AND S. S. KUMAR, *Differential subordination for functions associated with the lemniscate of Bernoulli*, Taiwanese J. Math. **16** (2012), no. 3, 1017–1026.

[4] N. BOHRA, S. KUMAR, AND V. RAVICHANDRAN, *Some special differential subordinations*, Hacet. J. Math. Stat., 48 (4) (2019), 1017–1034.

[5] T. BULBOACĂ, *Differential Subordinations and Superordinations. Recent Results*, House of Scientific Book Publ., Cluj-Napoca, 2005.

[6] N. E. CHO, V. KUMAR, S. S. KUMAR, V. RAVICHANDRAN, *Radius problems for starlike functions associated with the sine function*, Bull. Iranian Math. Soc. **45** (2019), no. 1, 213–232.

[7] F. R. KEOGH AND E. P. MERKES, *Preservation of subordination*, J. Analyse Math. **36** (1979), 179–183 (1980).

[8] I. H. KIM, Y. J. SIM AND N. E. CHO, *New criteria for Carathéodory functions*, J. Inequal. Appl. **2019**, Paper No. 13, 16 pp.

[9] S. S. KUMAR AND P. GOEL, *Starlike functions and higher order differential subordinations*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **114**, 192 (2020).

[10] S. KUMAR AND V. RAVICHANDRAN, *A subclass of starlike functions associated with a rational function*, Southeast Asian Bull. Math. **40** (2016), no. 2, 199–212.

[11] S. KUMAR AND V. RAVICHANDRAN, *Subordinations for functions with positive real part*, Complex Anal. Oper. Theory **12** (2018), no. 5, 1179–1191.

[12] J.-L. LIU, *Applications of differential subordinations for generalized Bessel functions*, Houston J. Math. **45** (2019), no. 1, 71–85.

[13] W. C. MA AND D. MINDA, *A unified treatment of some special classes of univalent functions*, in *Proceedings of the Conference on Complex Analysis (Tianjin, 1992)*, 157–169, Conf. Proc. Lecture Notes Anal., I, Int. Press, Cambridge, MA.

[14] S. S. MILLER AND P. T. MOCANU, *Univalent solutions of Briot- Bouquet differential equations*, J. Differential Equations **56** (1985), no. 3, 297–309.

[15] S. S. MILLER AND P. T. MOCANU, *Differential subordinations*, Monographs and Textbooks in Pure and Applied Mathematics, 225, Marcel Dekker, Inc., New York, 2000.

[16] R. MENDIRATTA, S. NAGPAL AND V. RAVICHANDRAN, *On a subclass of strongly starlike functions associated with exponential function*, Bull. Malays. Math. Sci. Soc. **38** (2015), no. 1, 365–386.

[17] M. NUNOKAWA, M. OBRADOVIĆ AND S. OWA, *One criterion for univalency*, Proc. Amer. Math. Soc. **106** (1989), no. 4, 1035–1037.

[18] R. K. RAINA AND J. SOKÓŁ, *Some properties related to a certain class of starlike functions*, C. R. Math. Acad. Sci. Paris, **353** (2015), no. 11, 973–978.

[19] F. RØNNING, *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc. **118** (1993), no. 1, 189–196.

[20] K. SHARMA AND V. RAVICHANDRAN, *Applications of subordination theory to starlike functions*, Bull. Iranian Math. Soc. **42** (2016), no. 3, 761–777.

[21] Y. J. SIM AND O. S. KWON, *The Briot-Bouquet differential subordination associated with vertical strip domains*, Honam Math. J. **39** (2017), no. 4, 503–514.

[22] J. SOKOL AND J. STANKIEWICZ, *Radius of convexity of some subclasses of strongly starlike functions*, Zeszyty Nauk. Politech. Rzeszowskiej Mat. No. 19 (1996), 101–105.

(Received February 26, 2021)

Nisha Bohra
Department of Mathematics, Sri Venkateswara College
University of Delhi
Delhi–110 021, India
e-mail: nishib89@gmail.com

Sushil Kumar
Bharati Vidyapeeth's college of Engineering
Delhi-110063, India
e-mail: sushilkumar16n@gmail.com

V. Ravichandran
Department of Mathematics
National Institute of Technology
Tiruchirappalli–620015, India
e-mail: ravic@nitt.edu; vravi68@gmail.com