Journal of Classical Analysis
Volume 18, Number 1 (2021), 39–47 doi:10.7153/jca-2021-18-04

UNIFORM NORM ESTIMATES OF BERNSTEIN–TYPE FOR LACUNARY–TYPE COMPLEX POLYOMIALS

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Abstract. In this paper, we prove some inequalities for the uniform norm of the derivative and polar derivative of lacunary-type complex polynomials having zeros in closed exterior of a circle of arbitrary positive radius. The results obtained besides extend some classical Bernstein-type inequalities also include several interesting generalizations and refinements of some known inequalities involving the polar derivative.

1. Introduction

Let $P(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n in the complex plane and $P'(z)$ be its derivative. The study of Bernstein-type inequalities that relate the norm of a polynomial to that of its derivative and their various versions are a classical topic in analysis. One basic result is that: for $P(z)$ to be a polynomial of degree n, it is true that

$$\max_{|z|=1} |P'(z)| \leqslant n \max_{|z|=1} |P(z)|, \tag{1}$$

where as concerning the maximum modulus of $P(z)$ on the circle $|z| = R \geqslant 1$, we have (for reference see [11], p. 346),

$$\max_{|z|=R} |P(z)| \leqslant R^n \max_{|z|=1} |P(z)|. \tag{2}$$

Equality holds in (1) and (2) if and only if $P(z)$ is a non-zero multiple of z^n. The above inequalities have been the starting point of a considerable literature in Approximation theory. Several papers and research monographs have been written on polynomial approximations (see, for example Marden [7], Rahman and Schmeisser [9], or Milovanović et al. [8]). For a polynomial $P(z)$ of degree n, not vanishing in the interior of the unit circle, we have the following version of (1):

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{3}$$

The result is best possible and the equality in (3) holds for any polynomial which has all its zeros on $|z| = 1$. As is well known, inequality (3) was conjectured by Erdös and

Mathematics subject classification (2020): 30A10, 30C10, 30D15.
Keywords and phrases: Polar derivative of a polynomial, maximum modulus principle, Bernstein-inequality.
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Paper JCA-18-04

later proved by Lax [6]. In 1985, Frappier et al. [3] proved that if $P(z)$ is a polynomial of degree n, then

$$\max_{|z|=1} |P'(z)| \leqslant n \max_{1\leqslant l\leqslant 2n} |P(e^{\frac{il\pi}{n}})|. \tag{4}$$

Clearly (4) is a refinement of (1), since the maximum of $|P(z)|$ on $|z|=1$ may be larger than maximum of $|P(z)|$ taken over $2n^{th}$ roots of unity as one can see by taking a simple example $P(z)=z^n+ib$, $b>0$.

Following the approach of Frappier et al. [3], Aziz [1] showed that the bound in (4) can be considerably improved. In fact, Aziz proved that if $P(z)$ is a polynomial of degree n, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{2}\left(M_\alpha + M_{\alpha+\pi}\right) \tag{5}$$

and for $R>1$,

$$\max_{|z|=1} |P(Rz)-P(z)| \leqslant \frac{R^n-1}{2}\left(M_\alpha + M_{\alpha+\pi}\right), \tag{6}$$

where

$$M_\alpha = \max_{1\leqslant l\leqslant n}\left|P\left(e^{i\frac{(\alpha+2l\pi)}{n}}\right)\right| \tag{7}$$

for all real α.

By restricting the zeros of $P(z)$, inequalities (5) and (6) were also improved by Aziz [1] by establishing that if $P(z)$ is a polynomial of degree n such that $P(z)\neq 0$ in $|z|<1$, then for every real α,

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{2}\left(M_\alpha^2 + M_{\alpha+\pi}^2\right)^{\frac{1}{2}} \tag{8}$$

and for every $R>1$,

$$\max_{|z|=1} |P(Rz)-P(z)| \leqslant \frac{R^n-1}{2}\left(M_\alpha^2 + M_{\alpha+\pi}^2\right)^{\frac{1}{2}}, \tag{9}$$

where M_α is defined in (7).

In 2012, Rather and Shah [10] extended (8) by proving that if $P(z)$ is a polynomial of degree n such that $P(z)\neq 0$ in $|z|<k$, $k\geqslant 1$, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{\sqrt{2(1+k^2)}}\left(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m_k^2\right)^{\frac{1}{2}} \tag{10}$$

and also extended (9) for every real α and $R>1$,

$$\max_{|z|=1} |P(Rz)-P(z)| \leqslant \frac{R^n-1}{\sqrt{2(1+k^2)}}\left(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m_k^2\right)^{\frac{1}{2}}, \tag{11}$$

where M_α is defined in (7) and $m_k=\min_{|z|=k}|P(z)|$.

As it is easy to see that inequalities (10) and (11) for $k=1$ provides refinement to inequalities (8) and (9). Recently Hans et al. [5] extended inequalities (10) and (11) to the class of lacunary-type polynomials $P(z)=a_0+\sum\limits_{v=\mu}^{n} a_v z^v$, $1\leqslant\mu\leqslant n$ not vanishing in $|z|<k$, $k\geqslant 1$ and obtained the following results.

THEOREM A. *If* $P(z)=a_0+\sum_{v=\mu}^{n}a_vz^v$, $1\leqslant\mu\leqslant n$ *is a polynomial of degree* n *such that* $P(z)\neq 0$ *in* $|z|<k$, $k\geqslant 1$, *then for every real* α,

$$\max_{|z|=1}|P'(z)|\leqslant\frac{n}{\sqrt{2(1+k^{2\mu})}}\left(M_\alpha^2+M_{\alpha+\pi}^2-2m_k^2\right)^{\frac{1}{2}}, \tag{12}$$

where M_α *is defined in* (7) *and* $m_k=\min_{|z|=k}|P(z)|$.

THEOREM B. *If* $P(z)=a_0+\sum_{v=\mu}^{n}a_vz^v$, $1\leqslant\mu\leqslant n$ *is a polynomial of degree* n *such that* $P(z)\neq 0$ *in* $|z|<k$, $k\geqslant 1$, *then for every real* α *and* $R\geqslant r\geqslant 1$,

$$\max_{|z|=1}|P(Rz)-P(rz)|\leqslant\frac{R^n-r^n}{\sqrt{2(1+k^{2\mu})}}\left(M_\alpha^2+M_{\alpha+\pi}^2-2m_k^2\right)^{\frac{1}{2}}, \tag{13}$$

where M_α *is defined in* (7) *and* $m_k=\min_{|z|=k}|P(z)|$.

The authors are curious to know how the inequalities in Theorems A and B and other related inequalities mentioned above can be sharpened by using some of the coefficients of $P(z)=a_0+\sum_{v=\mu}^{n}a_vz^v$, $\mu\geqslant 1$. Indeed, this paper is mainly motivated by the desire to establish some more refined bounds than given by (10)–(13).

2. Main results

THEOREM 1. *If* $P(z)=a_0+\sum_{v=\mu}^{n}a_vz^v$, $1\leqslant\mu\leqslant n$ *is a polynomial of degree* n *such that* $P(z)\neq 0$ *in* $|z|<k$, $k\geqslant 1$ *with* $m_k=\min_{|z|=k}|P(z)|$, *then for every real* α *and* $0\leqslant\lambda\leqslant 1$, *we have*

$$\max_{|z|=1}|P'(z)|\leqslant\frac{n}{\sqrt{2\left(1+\psi_\lambda^2(\mu)\right)}}\left(M_\alpha^2+M_{\alpha+\pi}^2-2m_k^2\lambda^2\right)^{\frac{1}{2}}, \tag{14}$$

where M_α *is defined in* (7) *and* $\psi_\lambda(\mu)=k^{\mu+1}\left\{\dfrac{\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-\lambda m_k}k^{\mu-1}+1}{\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-\lambda m_k}k^{\mu+1}+1}\right\}$.

REMARK 1. It is important to mention that bound obtained from Theorem 1 is optimal when $\lambda=1$. However, the parameter λ plays a vital role for making Theorem 1 more general and to get different bounds from it by simply giving different values to it from 0 to 1 and without changing the hypothesis of the Theorem. For example, for $\lambda=0$ (without assuming that $P(z)$ has a zero on $|z|=k$), we obtain the following result.

COROLLARY 1. *If* $P(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leqslant \mu \leqslant n$ *is a polynomial of degree* n *such that* $P(z) \neq 0$ *in* $|z| < k$, $k \geqslant 1$, *then for every real* α,

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{\sqrt{2\left(1+\psi_0^2(\mu)\right)}} \left(M_\alpha^2 + M_{\alpha+\pi}^2\right)^{\frac{1}{2}},$$

where M_α *is defined in* (7) *and* $\psi_0(\mu) = k^{\mu+1}\left\{\frac{\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|}k^{\mu-1}+1}{\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|}k^{\mu+1}+1}\right\}$.

Setting $\lambda = 1$ in (14), we obtain the following result which provides a refinement to Theorem A.

COROLLARY 2. *If* $P(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leqslant \mu \leqslant n$ *is a polynomial of degree* n *such that* $P(z) \neq 0$ *in* $|z| < k$, $k \geqslant 1$ *with* $m_k = \min_{|z|=k} |P(z)|$, *then for every real* α,

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{\sqrt{2\left(1+\psi_1^2(\mu)\right)}} \left(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m_k^2\right)^{\frac{1}{2}}, \tag{15}$$

where M_α *is defined in* (7) *and* $\psi_1(\mu) = k^{\mu+1}\left\{\frac{\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-m}k^{\mu-1}+1}{\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}+1}\right\}$.

REMARK 2. It is easy to verify that the function

$$x \longrightarrow \frac{n}{\sqrt{2(1+x^2)}} \left(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m_k^2\right)^{\frac{1}{2}},$$

is a non-increasing function of x. If we combine this fact with inequality (21) (Lemma 1, for $\lambda = 1$) according to which $\psi_1(\mu) \geqslant k^\mu$, $\mu \geqslant 1$, we observe that right hand side of (15) does not exceed the right hand side of (12). This shows that Corollary 2 sharpens the bound in Theorem A.

REMARK 3. By the same argument as in remark 2, if we choose $\mu = \lambda = 1$ in (14), then right hand side of (14) does not exceed the right hand side of (10). Hence inequality (14) for $\mu = \lambda = 1$ sharpens the bound in (10).

Setting $\mu = k = 1$ in (14), we obtain the following result which generalizes as well as refines (8).

COROLLARY 3. *If* $P(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n *such that* $P(z) \neq 0$ *in* $|z| < 1$ *with* $m = \min_{|z|=1} |P(z)|$, *then for every real* α *and* $0 \leqslant \lambda \leqslant 1$,

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{2} \left(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2\lambda^2\right)^{\frac{1}{2}}, \tag{16}$$

where M_α *is defined in* (7).

REMARK 4. For $\lambda = 1$, inequality (16) provides a refinement to (8) and for $\lambda = 0$, it reduces to (8).

Next, we prove the following strengthening of (13).

THEOREM 2. *If* $P(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leqslant \mu \leqslant n$ *is a polynomial of degree* n *such that* $P(z) \neq 0$ *in* $|z| < k$, $k \geqslant 1$ *with* $m_k = \min_{|z|=k} |P(z)|$, *then for every real* α, $R \geqslant r \geqslant 1$ *and* $0 \leqslant \lambda \leqslant 1$,

$$\max_{|z|=1} |P(Rz) - P(rz)| \leqslant \frac{R^n - r^n}{\sqrt{2\left(1 + \psi_\lambda^2(\mu)\right)}} \left(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m_k^2\lambda^2\right)^{\frac{1}{2}}, \tag{17}$$

where M_α *is defined in* (7) *and* $\psi_\lambda(\mu)$ *is defined in Theorem* 1.

If in (17), we take $\lambda = 1$, we obtain the following result which provides a refinement to inequality (13).

COROLLARY 4. *If* $P(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leqslant \mu \leqslant n$ *is a polynomial of degree* n *such that* $P(z) \neq 0$ *in* $|z| < k$, $k \geqslant 1$ *with* $m_k = \min_{|z|=k} |P(z)|$, *then for every real* α *and for* $R \geqslant r \geqslant 1$,

$$\max_{|z|=1} |P(Rz) - P(rz)| \leqslant \frac{R^n - r^n}{\sqrt{2\left(1 + \psi_1^2(\mu)\right)}} \left(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m_k^2\right)^{\frac{1}{2}},$$

where M_α *is defined in* (7) *and* $\psi_1(\mu)$ *is defined in Corollary* 2.

REMARK 5. For $\lambda = \mu = r = 1$, inequality (17) provides a refinement to (11).

Setting $\mu = k = 1$ in (17), we obtain the following result which generalizes as well as refines (9).

COROLLARY 5. *If* $P(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n *such that* $P(z) \neq 0$ *in* $|z| < 1$ *with* $m = \min_{|z|=1} |P(z)|$, *then for every real* α, $R \geqslant r \geqslant 1$ *and for* $0 \leqslant \lambda \leqslant 1$,

$$\max_{|z|=1} |P(Rz) - P(rz)| \leqslant \frac{R^n - r^n}{2} \left(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2\lambda^2\right)^{\frac{1}{2}}, \tag{18}$$

where M_α *is defined in* (7).

REMARK 6. For $\lambda = r = 1$, inequality (18) provides a refinement to (9) and for $\lambda = 0$ and $r = 1$, it reduces to (9).

DEFINITION 1. For a polynomial $P(z)$ of degree n, now we define the so-called polar derivative of $P(z)$ with respect to the point β as

$$D_\beta P(z) := nP(z) + (\beta - z)P'(z).$$

The polynomial $D_\beta P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\beta\to\infty}\left\{\frac{D_\beta P(z)}{\beta}\right\} = P'(z),$$

uniformly with respect to z for $|z| \leqslant R$, $R > 0$.

Over the last four decades many different authors produced a large number of results pertaining to the polar derivative of polynomials. More information on this topic can be found in the books of Milovanović et al. [8], Rahman and Schmeisser [9] and Marden [7]. Finally, in this paper, as an application of Theorem 1, we establish the following result for the polar derivative of polynomial.

THEOREM 3. *If* $P(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leqslant \mu \leqslant n$ *is a polynomial of degree* n *such that* $P(z) \neq 0$ *in* $|z| < k$, $k \geqslant 1$ *with* $m_k = \min_{|z|=k}|P(z)|$, *then for every real* α, $0 \leqslant \lambda \leqslant 1$ *and for every complex number* β *with* $|\beta| \geqslant 1$, *we have*

$$\max_{|z|=1}|D_\beta P(z)| \leqslant n\max_{|z|=1}|P(z)| + \frac{n(|\beta|-1)}{\sqrt{2\left(1+\psi_\lambda^2(\mu)\right)}}\left(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m_k^2\lambda^2\right)^{\frac{1}{2}}, \tag{19}$$

where M_α *is defined in* (7) *and* $\psi_\lambda(\mu)$ *is defined in Theorem* 1.

REMARK 7. If we divide both sides of (19) by $|\beta|$ and let $|\beta| \to \infty$, we get (14).

3. Lemmas

In order to prove our main results, we need the following lemmas.

LEMMA 1. [2] *If* $P(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leqslant \mu \leqslant n$ *is a polynomial of degree* n *such that* $P(z) \neq 0$ *in* $|z| < k$, $k \geqslant 1$, $0 \leqslant \lambda \leqslant 1$ *and for* $R \geqslant 1$,

$$\psi_\lambda(\mu)|P(Rz) - P(z)| \leqslant |Q(Rz) - Q(z)| - (R^n - 1)\lambda m_k, \tag{20}$$

$$k^{\mu+1}\left\{\frac{\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-\lambda m_k}k^{\mu-1}+1}{\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-\lambda m_k}k^{\mu+1}+1}\right\} = \psi_\lambda(\mu) \geqslant k^\mu,\ \mu \geqslant 1, \tag{21}$$

where $Q(z) = z^n\overline{P\left(\frac{1}{\bar z}\right)}$ *and* $m_k = \min_{|z|=k}|P(z)|$.

LEMMA 2. [4] *If $P(z)$ is a polynomial of degree n, then for $|z|=1$,*

$$|P'(z)|+|Q'(z)| \leqslant n \max_{|z|=1} |P(z)|,$$

where $Q(z)=z^n\overline{P(\frac{1}{\bar{z}})}$.

LEMMA 3. [1] *If $P(z)$ is a polynomial of degree n, then for $|z|=1$,*

$$|P'(z)|^2+|Q'(z)|^2 \leqslant \frac{n^2}{2}\left(M_\alpha^2+M_{\alpha+\pi}^2\right),$$

where $Q(z)=z^n\overline{P(\frac{1}{\bar{z}})}$ *and* M_α *is defined in* (7).

4. Proofs of main results

Proof of Theorem 1. Since $P(z)=a_0+\sum\limits_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leqslant \mu \leqslant n$ is a polynomial of degree n such that $P(z) \neq 0$ in $|z|<k$, $k \geqslant 1$. Therefore on dividing both sides of inequality (20) of Lemma 1 by $R-1$ and taking $R \to 1$, we have for $|z|=1$,

$$\psi_\lambda(\mu)|P'(z)| \leqslant |Q'(z)|-nm_k\lambda,$$

which implies

$$\left(\psi_\lambda(\mu)|P'(z)|+nm_k\lambda\right)^2 \leqslant |Q'(z)|^2. \tag{22}$$

Also

$$\left(\psi_\lambda(\mu)|P'(z)|+nm_k\lambda\right)^2 \geqslant \psi_\lambda^2(\mu)|P'(z)|^2+n^2m_k^2\lambda^2. \tag{23}$$

From (22) and (23), we have for $|z|=1$,

$$\psi_\lambda^2(\mu)|P'(z)|^2+n^2m_k^2\lambda^2 \leqslant |Q'(z)|^2.$$

By using Lemma 3, we have

$$\psi_\lambda^2(\mu)|P'(z)|^2+|P'(z)|^2+n^2m_k^2\lambda^2 \leqslant \frac{n^2}{2}\left(M_\alpha^2+M_{\alpha+\pi}^2\right),$$

which implies for $|z|=1$,

$$\left(1+\psi_\lambda^2(\mu)\right)|P'(z)|^2 \leqslant \frac{n^2}{2}(M_\alpha^2+M_{\alpha+\pi}^2)-n^2m_k^2\lambda^2.$$

Equivalently,

$$\max_{|z|=1}|P'(z)| \leqslant \frac{n}{\sqrt{2\left(1+\psi_\lambda^2(\mu)\right)}}\left(M_\alpha^2+M_{\alpha+\pi}^2-2m_k^2\lambda^2\right)^{\frac{1}{2}}, \tag{24}$$

which is the desired result and hence completes the proof of Theorem 1. □

Proof of Theorem 2. On applying inequality (2) to the polynomial $P'(z)$ which is of degree $n-1$ and using (14), we obtain for $\zeta \geqslant 1$ and $0 \leqslant \theta < 2\pi$,

$$\begin{aligned} |P'(\zeta e^{i\theta})| &\leqslant \zeta^{n-1}|P'(e^{i\theta})| \\ &\leqslant \zeta^{n-1}\frac{n}{\sqrt{2\left(1+\psi_\lambda^2(\mu)\right)}}\left(M_\alpha^2+M_{\alpha+\pi}^2-2m_k^2\lambda^2\right)^{\frac{1}{2}}. \end{aligned} \tag{25}$$

Hence for each θ, $0 \leqslant \theta < 2\pi$ and $R \geqslant r \geqslant 1$, we have

$$\begin{aligned} P(Re^{i\theta})-P(re^{i\theta}) &= \int_r^R \frac{d}{d\zeta}P(\zeta e^{i\theta})d\zeta \\ &= \int_r^R P'(\zeta e^{i\theta})e^{i\theta}d\zeta. \end{aligned}$$

This implies,

$$\left|P(Re^{i\theta})-P(re^{i\theta})\right| \leqslant \int_r^R |P'(\zeta e^{i\theta})|d\zeta,$$

which on using (25), gives

$$\left|P(Re^{i\theta})-P(re^{i\theta})\right| \leqslant \frac{n}{\sqrt{2\left(1+\psi_\lambda^2(\mu)\right)}}\left(M_\alpha^2+M_{\alpha+\pi}^2-2m_k^2\lambda^2\right)^{\frac{1}{2}}\int_r^R \zeta^{n-1}d\zeta.$$

Consequently,

$$\max_{|z|=1}|P(Rz)-P(rz)| \leqslant \frac{R^n-r^n}{\sqrt{2\left(1+\psi_\lambda^2(\mu)\right)}}\left(M_\alpha^2+M_{\alpha+\pi}^2-2m_k^2\lambda^2\right)^{\frac{1}{2}}.$$

This completes the proof of Theorem 2. □

Proof of Theorem 3. If $Q(z)=z^n\overline{P(\frac{1}{\bar{z}})}$, then it can be easily seen that

$$|nP(z)-zP'(z)| = |Q'(z)| \quad \text{for } |z|=1.$$

Now for $\beta \in \mathbb{C}$, with $|\beta| \geqslant 1$

$$\begin{aligned} |D_\beta P(z)| &= |nP(z)+(\beta-z)P'(z)| \\ &= |nP(z)+\beta P'(z)-zP'(z)| \\ &\leqslant |nP(z)-zP'(z)|+|\beta||P'(z)|. \end{aligned}$$

Therefore we have for $|z| = 1$,

$$|D_\beta P(z)| \leqslant |Q'(z)| + |\beta||P'(z)|$$

which on using Lemma 2 gives for $|z| = 1$,

$$|D_\beta P(z)| \leqslant n|P(z)| + (|\beta| - 1)|P'(z)|. \tag{26}$$

Combining inequalities (24) and (26), we obtain for $|z| = 1$,

$$|D_\beta P(z)| \leqslant n|P(z)| + \frac{n(|\beta| - 1)}{\sqrt{2\left(1 + \psi_\lambda^2(\mu)\right)}} \left(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m_k^2\lambda^2\right)^{\frac{1}{2}}.$$

This immediately leads to the desired result and hence completes the proof of Theorem 3. □

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(Received May 2, 2021)

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