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ON SOME INEQUALITIES CONCERNING GENERALIZED (α,β) RELATIVE ORDER AND GENERALIZED (α,β) RELATIVE TYPE OF ENTIRE FUNCTION WITH RESPECT TO AN ENTIRE FUNCTION

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Abstract. In this paper, we intend to find out some inequalities relating to generalized (α,β) relative order, generalized (α,β) relative type and generalized (α,β) relative weak type of an entire function f with respect to an entire function g when generalized (γ,β) relative order, generalized (γ,β) relative type and generalized (γ,β) relative weak type of f with respect to another entire function h and generalized (γ,α) relative order, generalized (γ,α) relative type and generalized (γ,α) relative weak type of g with respect to h are given, where α, β and γ are continuous non-negative slowly increasing functions defined on $(-\infty,+\infty)$.

1. Introduction, definitions and notations

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be an entire function defined on $\mathbb{C}$. The maximum modulus function $M_f(r)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined as $M_f(r) = \max_{|z|=r} |f(z)|$. Moreover, if f is non-constant entire then $M_f(r)$ is also strictly increasing and continuous functions of r. Therefore its inverse M_f^{-1} : $(M_f(0),\infty) \to (0,\infty)$ exists and is such that $\lim_{s\to\infty} M_f^{-1}(s) = \infty$. We use the standard notations and definitions of the theory of entire functions which are available in [9] and [10], and therefore we do not explain those in details.

Now let L be a class of continuous non-negative functions α defined on $(-\infty,+\infty)$ such that $\alpha(x) = \alpha(x_0) \geqslant 0$ for $x \leqslant x_0$ with $\alpha(x) \uparrow +\infty$ as $x \to +\infty$ and $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x \to +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha(cx) = (1+o(1))\alpha(x)$ as $x_0 \leqslant x \to +\infty$ for each $c \in (0,+\infty)$, i.e., α is slowly increasing function. Clearly $L^0 \subset L$.

Further we assume that throughout the present paper α, β and γ always denote the functions belonging to L^0.

Considering this, the value

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log M_f(r))}{\beta(\log r)} \ (\alpha \in L, \beta \in L)$$

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is called [8] generalized (α,β) order of an entire function f. For details about generalized (α,β) order one may see [8]. During the past decades, several authors made close investigations on the properties of entire functions related to generalized (α,β) order in some different direction. For the purpose of further applications, Biswas et al. [5] rewrote the definition of the generalized (α,β) order of entire function in the following way after giving a minor modification to the original definition (e.g. see, [8]) which considerably extend the definition of φ-order of entire function introduced by Chyzhykov et al. [7]:

DEFINITION 1. [5] The generalized (α,β) order and generalized (α,β) lower order denoted by $\rho_{(\alpha,\beta)}[f]$ and $\lambda_{(\alpha,\beta)}[f]$ respectively of an entire function f are defined as:

$$\rho_{(\alpha,\beta)}[f]=\limsup_{r\to+\infty}\frac{\alpha(M_f(r))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f]=\liminf_{r\to+\infty}\frac{\alpha(M_f(r))}{\beta(r)}.$$

The function f is said to be of regular generalized (α,β) growth when generalized (α,β) order and generalized (α,β) lower order of f are the same. Functions which are not of regular generalized (α,β) growth are said to be of irregular generalized (α,β) growth.

In order to refine the growth scale namely the generalized (α,β) order of an entire function, Biswas et al. [3] have introduced the definitions of other growth indicators, called generalized (α,β) type and generalized (α,β) lower type respectively of an entire function which are as follows:

DEFINITION 2. [3] The generalized (α,β) type and generalized (α,β) lower type denoted by $\sigma_{(\alpha,\beta)}[f]$ and $\overline{\sigma}_{(\alpha,\beta)}[f]$ respectively of an entire function f having finite positive generalized (α,β) order $(0<\rho_{(\alpha,\beta)}[f]<\infty)$ are defined as:

$$\sigma_{(\alpha,\beta)}[f]=\limsup_{r\to+\infty}\frac{\exp(\alpha(M_f(r)))}{(\exp(\beta(r))^{\rho_{(\alpha,\beta)}[f]}} \text{ and } \overline{\sigma}_{(\alpha,\beta)}[f]=\liminf_{r\to+\infty}\frac{\exp(\alpha(M_f(r)))}{(\exp(\beta(r))^{\rho_{(\alpha,\beta)}[f]}}.$$

It is obvious that $0\leqslant\overline{\sigma}_{(\alpha,\beta)}[f]\leqslant\sigma_{(\alpha,\beta)}[f]\leqslant\infty$.

Analogously, to determine the relative growth of two entire functions having same non zero finite generalized (α,β) lower order, Biswas et al. [3] have introduced the definitions of generalized (α,β) weak type and generalized (α,β) upper weak type of an entire function f of finite positive generalized (α,β) lower order in the following way:

DEFINITION 3. [3] The generalized (α,β) upper weak type and generalized (α,β) weak type denoted by $\tau_{(\alpha,\beta)}[f]$ and $\overline{\tau}_{(\alpha,\beta)}[f]$ respectively of an entire function f having finite positive generalized (α,β) lower order $(0<\lambda_{(\alpha,\beta)}[f]<\infty)$ are defined as:

$$\tau_{(\alpha,\beta)}[f]=\limsup_{r\to+\infty}\frac{\exp(\alpha(M_f(r)))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]}} \text{ and } \overline{\tau}_{(\alpha,\beta)}[f]=\liminf_{r\to+\infty}\frac{\exp(\alpha(M_f(r)))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]}}.$$

It is obvious that $0\leqslant\overline{\tau}_{(\alpha,\beta)}[f]\leqslant\tau_{(\alpha,\beta)}[f]\leqslant\infty$.

Mainly the growth investigation of entire functions has usually been done through their maximum moduli function in comparison with those of exponential function. But if one is paying attention to evaluate the growth rates of any entire function with respect to a new entire function, the notions of relative growth indicators (see e.g. [1, 2]) will come. Now in order to make some progress in the study of relative order, Biswas et al. [4] have introduced the definitions of generalized (α,β) relative order and generalized (α,β) relative lower order of an entire function with respect to another entire function in the following way:

DEFINITION 4. [4] The generalized (α,β) relative order and generalized (α,β) relative lower order denoted by $\rho_{(\alpha,\beta)}[f]_g$ and $\lambda_{(\alpha,\beta)}[f]_g$ respectively of an entire function f with respect to an entire function g are defined as:

$$\rho_{(\alpha,\beta)}[f]_g = \limsup_{r\to+\infty} \frac{\alpha(M_g^{-1}(M_f(r)))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f]_g = \liminf_{r\to+\infty} \frac{\alpha(M_g^{-1}(M_f(r)))}{\beta(r)}.$$

Further if generalized (α,β) relative order and the generalized (α,β) relative lower order of an entire function f with respect to an entire function g are the same, then f is called a function of regular generalized (α,β) relative growth with respect to g. Otherwise, f is said to be irregular generalized (α,β) relative growth.with respect to g.

Now in order to refine the above growth scale, Biswas et al. [4] have introduced the definitions of other growth indicators, such as generalized (α,β) relative type and generalized (α,β) relative lower type of entire function with respect to an entire function which are as follows:

DEFINITION 5. [4] The generalized (α,β) relative type denoted by $\sigma_{(\alpha,\beta)}[f]_g$ and generalized (α,β) relative lower type denoted by $\overline{\sigma}_{(\alpha,\beta)}[f]_g$ of an entire function f with respect to an entire function g having non-zero finite generalized (α,β) relative order are defined as:

$$\sigma_{(\alpha,\beta)}[f]_g = \limsup_{r\to+\infty} \frac{\exp(\alpha(M_g^{-1}M_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]_g}}$$
$$\text{and } \overline{\sigma}_{(\alpha,\beta)}[f]_g = \liminf_{r\to+\infty} \frac{\exp(\alpha(M_g^{-1}M_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]_g}}.$$

Analogously, to determine the relative growth of an entire function f having same non zero finite generalized (α,β) relative lower order with respect to an entire function g, Biswas et al. [4] have introduced the definitions of generalized (α,β) relative upper weak type denoted by $\tau_{(\alpha,\beta)}[f]_g$ and generalized (α,β) relative weak type denoted by $\overline{\tau}_{(\alpha,\beta)}[f]_g$ of f with respect to g of finite positive generalized (α,β) relative lower order in the following way:

DEFINITION 6. [4] The generalized (α,β) relative upper weak type and generalized (α,β) relative weak type of an entire function f with respect to an entire function g having non-zero finite generalized (α,β) relative lower order are defined as:

$$\tau_{(\alpha,\beta)}[f]_g = \limsup_{r\to+\infty} \frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]_g}}$$
$$\text{and } \overline{\tau}_{(\alpha,\beta)}[f]_g = \liminf_{r\to+\infty} \frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]_g}}.$$

In this paper, we intend to find out some inequalities relating to generalized (α,β) relative order, generalized (α,β) relative type and generalized (α,β) relative weak type of an entire function f with respect to an entire function g when generalized (γ,β) relative order, generalized (γ,β) relative type and generalized (γ,β) relative weak type of f with respect to another entire function h and generalized (γ,α) relative order, generalized (γ,α) relative type and generalized (γ,α) relative weak type of g with respect to h are given, where α, β and γ are continuous non-negative slowly increasing functions on $(-\infty,+\infty)$. In fact, the results presented in this paper have been improved and extended some earlier results (see, e.g., [6]). We assume that all the growth indicators are non-zero finite.

2. Main results

In this section we present the main results of the paper.

THEOREM 1. *Let f, g and h be any three entire functions such that* $0<\lambda_{(\gamma,\beta)}[f]_h \leqslant \rho_{(\gamma,\beta)}[f]_h<\infty$ *and* $0<\lambda_{(\gamma,\alpha)}[g]_h \leqslant \rho_{(\gamma,\alpha)}[g]_h<\infty$. *Then*

$$\frac{\lambda_{(\gamma,\beta)}[f]_h}{\rho_{(\gamma,\alpha)}[g]_h} \leqslant \lambda_{(\alpha,\beta)}[f]_g \leqslant \min\left\{\frac{\lambda_{(\gamma,\beta)}[f]_h}{\lambda_{(\gamma,\alpha)}[g]_h}, \frac{\rho_{(\gamma,\beta)}[f]_h}{\rho_{(\gamma,\alpha)}[g]_h}\right\}$$
$$\leqslant \max\left\{\frac{\lambda_{(\gamma,\beta)}[f]_h}{\lambda_{(\gamma,\alpha)}[g]_h}, \frac{\rho_{(\gamma,\beta)}[f]_h}{\rho_{(\gamma,\alpha)}[g]_h}\right\} \leqslant \rho_{(\alpha,\beta)}[f]_g \leqslant \frac{\rho_{(\gamma,\beta)}[f]_h}{\lambda_{(\gamma,\alpha)}[g]_h}.$$

Proof. From the definitions of $\rho_{(\gamma,\beta)}[f]_h$ and $\lambda_{(\gamma,\beta)}[f]_h$ for all sufficiently large values of r, we have

$$M_f(r) \leqslant M_h(\gamma^{-1}((\rho_{(\gamma,\beta)}[f]_h+\varepsilon)\beta(r))), \tag{1}$$
$$M_f(r) \geqslant M_h(\gamma^{-1}((\lambda_{(\gamma,\beta)}[f]_h-\varepsilon)\beta(r))). \tag{2}$$

For a sequence of values of r tending to infinity, we get

$$M_f(r) \geqslant M_h(\gamma^{-1}((\rho_{(\gamma,\beta)}[f]_h-\varepsilon)\beta(r))). \tag{3}$$

Again for a sequence of values of r tending to infinity, we get

$$M_f(r) \leqslant M_h(\gamma^{-1}((\lambda_{(\gamma,\beta)}[f]_h+\varepsilon)\beta(r))). \tag{4}$$

Further from the definitions of $\rho_{(\gamma,\alpha)}[g]_h$ and $\lambda_{(\gamma,\alpha)}[g]_h$ for all sufficiently large values of $r,$ it follows that

$$M_g(r) \leqslant M_h(\gamma^{-1}((\rho_{(\gamma,\alpha)}[g]_h+\varepsilon)\alpha(r)))$$

$$i.e.,\ M_h(r) \geqslant M_g\Big(\alpha^{-1}\Big(\frac{\gamma(r)}{\rho_{(\gamma,\alpha)}[g]_h+\varepsilon}\Big)\Big) \tag{5}$$

$$\text{and } M_h(r) \leqslant M_g\Big(\alpha^{-1}\Big(\frac{\gamma(r)}{\lambda_{(\gamma,\alpha)}[g]_h-\varepsilon}\Big)\Big). \tag{6}$$

From the definition of $\rho_{(\gamma,\alpha)}[g]_h$, for a sequence of values of r tending to infinity, we obtain

$$M_h(r) \leqslant M_g\Big(\alpha^{-1}\Big(\frac{\gamma(r)}{(\rho_{(\gamma,\alpha)}[g]_h-\varepsilon)}\Big)\Big). \tag{7}$$

Also from the definition of $\lambda_{(\gamma,\alpha)}[g]_h,$ for a sequence of values of r tending to infinity, we have

$$M_h(r) \geqslant M_g\Big(\alpha^{-1}\Big(\frac{\gamma(r)}{\lambda_{(\gamma,\alpha)}[g]_h+\varepsilon}\Big)\Big). \tag{8}$$

Now from (3) and in view of (5), for a sequence of values of r tending to infinity, we get

$$\alpha(M_g^{-1}(M_f(r))) \geqslant \alpha(M_g^{-1}(M_h(\gamma^{-1}((\rho_{(\gamma,\beta)}[f]_h-\varepsilon)\beta(r))))$$

$$\alpha(M_g^{-1}(M_f(r))) \geqslant \alpha\Big(M_g^{-1}\Big(M_g\Big(\alpha^{-1}\Big(\frac{(\rho_{(\gamma,\beta)}[f]_h-\varepsilon)\beta(r)}{\rho_{(\gamma,\alpha)}[g]_h+\varepsilon}\Big)\Big)\Big)\Big).$$

$$i.e.,\ \alpha(M_g^{-1}(M_f(r))) \geqslant \frac{(\rho_{(\gamma,\beta)}[f]_h-\varepsilon)\beta(r)}{(\rho_{(\gamma,\alpha)}[g]_h+\varepsilon)}$$

$$i.e.,\ \frac{\alpha(M_g^{-1}(M_f(r)))}{\beta(r)} \geqslant \frac{\rho_{(\gamma,\beta)}[f]_h-\varepsilon}{\rho_{(\gamma,\alpha)}[g]_h+\varepsilon}.$$

As $\varepsilon(>0)$ is arbitrary, it follows that

$$\rho_{(\alpha,\beta)}[f]_g \geqslant \frac{\rho_{(\gamma,\beta)}[f]_h}{\rho_{(\gamma,\alpha)}[g]_h}. \tag{9}$$

Analogously from (2) and in view of (8), it follows that

$$\rho_{(\alpha,\beta)}[f]_g \geqslant \frac{\lambda_{(\gamma,\beta)}[f]_h}{\lambda_{(\gamma,\alpha)}[g]_h}. \tag{10}$$

Again from (2) and in view of (5), we obtain

$$\lambda_{(\alpha,\beta)}[f]_g \geqslant \frac{\lambda_{(\gamma,\beta)}[f]_h}{\rho_{(\gamma,\alpha)}[g]_h}. \tag{11}$$

Now in view of (6) we have from (1) for all sufficiently large values of r that

$$\alpha(M_g^{-1}(M_f(r))) \leqslant \alpha(M_g^{-1}(M_h(\gamma^{-1}((\rho_{(\gamma,\beta)}[f]_h+\varepsilon)\beta(r)))))$$

$$\alpha(M_g^{-1}(M_f(r))) \leqslant \alpha\Big(M_g^{-1}\Big(M_g\Big(\alpha^{-1}\Big(\frac{\rho_{(\gamma,\beta)}[f]_h+\varepsilon)\beta(r)}{\lambda_{(\gamma,\alpha)}[g]_h-\varepsilon}\Big)\Big)\Big)\Big)$$

$$i.e.,\ \alpha(M_g^{-1}(M_f(r))) \leqslant \frac{(\rho_{(\gamma,\beta)}[f]_h+\varepsilon)\beta(r)}{(\lambda_{(\gamma,\alpha)}[g]_h-\varepsilon)}$$

$$i.e.,\ \frac{\alpha(M_g^{-1}(M_f(r)))}{\beta(r)} \leqslant \frac{\rho_{(\gamma,\beta)}[f]_h+\varepsilon}{\lambda_{(\gamma,\alpha)}[g]_h-\varepsilon}.$$

Since $\varepsilon(>0)$ is arbitrary, we obtain that

$$\rho_{(\alpha,\beta)}[f]_g \leqslant \frac{\rho_{(\gamma,\beta)}[f]_h}{\lambda_{(\gamma,\alpha)}[g]_h}. \tag{12}$$

Similarly in view of (7), we get from (1) that

$$\lambda_{(\alpha,\beta)}[f]_g \leqslant \frac{\rho_{(\gamma,\beta)}[f]_h}{\rho_{(\gamma,\alpha)}[g]_h}. \tag{13}$$

Again from (4) and in view of (6) it follows that

$$\lambda_{(\alpha,\beta)}[f]_g \leqslant \frac{\lambda_{(\gamma,\beta)}[f]_h}{\lambda_{(\gamma,\alpha)}[g]_h}. \tag{14}$$

The theorem follows from (9), (10), (11), (12), (13) and (14). $\square$

REMARK 1. From the conclusion of the above result, one may write $\rho_{(\alpha,\beta)}[f]_g = \frac{\rho_{(\gamma,\beta)}[f]_h}{\rho_{(\gamma,\alpha)}[g]_h}$ and $\lambda_{(\alpha,\beta)}[f]_g = \frac{\lambda_{(\gamma,\beta)}[f]_h}{\lambda_{(\gamma,\alpha)}[g]_h}$ when $\lambda_{(\gamma,\alpha)}[g]_h = \rho_{(\gamma,\alpha)}[g]_h$. Similarly $\rho_{(\alpha,\beta)}[f]_g = \frac{\lambda_{(\gamma,\beta)}[f]_h}{\lambda_{(\gamma,\alpha)}[g]_h}$ and $\lambda_{(\alpha,\beta)}[f]_g = \frac{\rho_{(\gamma,\beta)}[f]_h}{\rho_{(\gamma,\alpha)}[g]_h}$ when $\lambda_{(\gamma,\beta)}[f]_h = \rho_{(\gamma,\beta)}[f]_h$.

THEOREM 2. *Let f, g and h be any three entire functions such that $0<\rho_{(\gamma,\beta)}[f]_h < \infty$ and $0<\lambda_{(\gamma,\alpha)}[g]_h \leqslant \rho_{(\gamma,\alpha)}[g]_h < \infty$. Then*

$$\max\Big\{\Big(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h}{\overline{\tau}_{(\gamma,\alpha)}[g]_h}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}, \Big(\frac{\sigma_{(\gamma,\beta)}[f]_h}{\tau_{(\gamma,\alpha)}[g]_h}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}\Big\}$$
$$\leqslant \sigma_{(\alpha,\beta)}[f]_g \leqslant \Big(\frac{\sigma_{(\gamma,\beta)}[f]_h}{\overline{\sigma}_{(\gamma,\alpha)}[g]_h}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}.$$

Proof. Let us consider that $\varepsilon(>0)$ is arbitrary number. Now from the definitions of $\sigma_{(\gamma,\beta)}[f]_h$ and $\overline{\sigma}_{(\gamma,\beta)}[f]_h$, for all sufficiently large values of r, we have

$$M_f(r) \leqslant M_h(\gamma^{-1}(\log[(\sigma_{(\gamma,\beta)}[f]_h+\varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]_h}])), \tag{15}$$

$$M_f(r) \geqslant M_h(\gamma^{-1}(\log[(\overline{\sigma}_{(\gamma,\beta)}[f]_h-\varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]_h}])). \tag{16}$$

Also for a sequence of values of r tending to infinity, we get

$$M_f(r) \geqslant M_h(\gamma^{-1}(\log[(\sigma_{(\gamma,\beta)}[f]_h-\varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]_h}])). \tag{17}$$

Again for a sequence of values of r tending to infinity, we get

$$M_f(r) \leqslant M_h(\gamma^{-1}(\log[(\overline{\sigma}_{(\gamma,\beta)}[f]_h+\varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]_h}])). \tag{18}$$

Similarly from the definitions of $\sigma_{(\gamma,\alpha)}[g]_h$ and $\overline{\sigma}_{(\gamma,\alpha)}[g]_h$, it follows for all sufficiently large values of r that

$$M_g(r) \leqslant M_h(\gamma^{-1}(\log[(\sigma_{(\gamma,\alpha)}[g]_h+\varepsilon)(\exp(\alpha(r)))^{\rho_{(\gamma,\alpha)}[g]_h}]))$$

$$i.e., M_h(r) \geqslant M_g\left(\alpha^{-1}\left(\log\left(\frac{\exp(\gamma(r))}{(\sigma_{(\gamma,\alpha)}[g]_h+\varepsilon)}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}\right)\right) \text{ and} \tag{19}$$

$$M_h(r) \leqslant M_g\left(\alpha^{-1}\left(\log\left(\frac{\exp(\gamma(r))}{(\overline{\sigma}_{(\gamma,\alpha)}[g]_h-\varepsilon)}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}\right)\right). \tag{20}$$

For a sequence of values of r tending to infinity, we obtain

$$M_h(r) \leqslant M_g\left(\alpha^{-1}\left(\log\left(\frac{\exp(\gamma(r))}{(\sigma_{(\gamma,\alpha)}[g]_h-\varepsilon)}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}\right)\right) \tag{21}$$

and for a sequence of values of r tending to infinity, we get

$$M_h(r) \geqslant M_g\left(\alpha^{-1}\left(\log\left(\frac{\exp(\gamma(r))}{(\overline{\sigma}_{(\gamma,\alpha)}[g]_h+\varepsilon)}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}\right)\right). \tag{22}$$

Further from the definitions of $\tau_{(\gamma,\beta)}[f]_h$ and $\overline{\tau}_{(\gamma,\beta)}[f]_h$, for all sufficiently large values of r, we have

$$M_f(r) \leqslant M_h(\gamma^{-1}(\log((\tau_{(\gamma,\beta)}[f]_h+\varepsilon)(\exp(\beta(r)))^{\lambda_{(\gamma,\beta)}[f]_h}))), \tag{23}$$

$$M_f(r) \geqslant M_h(\gamma^{-1}(\log((\overline{\tau}_{(\gamma,\beta)}[f]_h-\varepsilon)(\exp(\beta(r)))^{\lambda_{(\gamma,\beta)}[f]_h}))). \tag{24}$$

Also for a sequence of values of r tending to infinity, we get

$$M_f(r) \geqslant M_h(\gamma^{-1}(\log((\tau_{(\gamma,\beta)}[f]_h-\varepsilon)(\exp(\beta(r)))^{\lambda_{(\gamma,\beta)}[f]_h}))) \tag{25}$$

and for a sequence of values of r tending to infinity, we get

$$M_f(r) \leqslant M_h(\gamma^{-1}(\log((\overline{\tau}_{(\gamma,\beta)}[f]_h+\varepsilon)(\exp(\beta(r)))^{\lambda_{(\gamma,\beta)}[f]_h}))). \tag{26}$$

Similarly from the definitions of $\tau_{(\gamma,\alpha)}[g]_h$ and $\overline{\tau}_{(\gamma,\alpha)}[g]_h$, it follows for all sufficiently large values of r that

$$M_h(r) \geqslant M_g\Big(\alpha^{-1}\Big(\log\Big(\frac{\exp(\gamma(r))}{(\tau_{(\gamma,\alpha)}[g]_h+\varepsilon)}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}\Big)\Big) \text{ and } \tag{27}$$

$$M_h(r) \leqslant M_g\Big(\alpha^{-1}\Big(\log\Big(\frac{\exp(\gamma(r))}{(\overline{\tau}_{(\gamma,\alpha)}[g]_h-\varepsilon)}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}\Big)\Big) \tag{28}$$

Also for a sequence of values of r tending to infinity, we obtain

$$M_h(r) \leqslant M_g\Big(\alpha^{-1}\Big(\log\Big(\frac{\exp(\gamma(r))}{(\tau_{(\gamma,\alpha)}[g]_h-\varepsilon)}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}\Big)\Big) \text{ and } \tag{29}$$

$$M_h(r) \geqslant M_g\Big(\alpha^{-1}\Big(\log\Big(\frac{\exp(\gamma(r))}{(\overline{\tau}_{(\gamma,\alpha)}[g]_h+\varepsilon)}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}\Big)\Big). \tag{30}$$

Now from (17) and in view of (27), for a sequence of values of r tending to infinity, we get

$$\exp(\alpha(M_g^{-1}(M_f(r))))$$
$$\geqslant \exp(\alpha(M_g^{-1}(M_h(\gamma^{-1}(\log((\sigma_{(\gamma,\beta)}[f]_h-\varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]_h}))))))$$

$$\exp(\alpha(M_g^{-1}(M_f(r)))) \geqslant \Big(\frac{(\sigma_{(\gamma,\beta)}[f]_h-\varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]_h}}{(\tau_{(\gamma,\alpha)}[g]_h+\varepsilon)}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}$$

$$i.e., \ \frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\frac{\rho_{(\gamma,\beta)}[f]_h}{\lambda_{(\gamma,\alpha)}[g]_h}}} \geqslant \Big(\frac{\sigma_{(\gamma,\beta)}[f]_h-\varepsilon}{\tau_{(\gamma,\alpha)}[g]_h+\varepsilon}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}.$$

Since in view of Theorem 1, $\frac{\rho_{(\gamma,\beta)}[f]_h}{\lambda_{(\gamma,\alpha)}[g]_h} \geqslant \rho_{(\alpha,\beta)}[f]_g$, and as $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty}\frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]_g}} \geqslant \Big(\frac{\sigma_{(\gamma,\beta)}[f]_h}{\tau_{(\gamma,\alpha)}[g]_h}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}$$

$$i.e., \ \sigma_{(\alpha,\beta)}[f]_g \geqslant \Big(\frac{\sigma_{(\gamma,\beta)}[f]_h}{\tau_{(\gamma,\alpha)}[g]_h}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}. \tag{31}$$

Analogously from (16) and (30), we get

$$\sigma_{(\alpha,\beta)}[f]_g \geqslant \Big(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h}{\overline{\tau}_{(\gamma,\alpha)}[g]_h}\Big)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}. \tag{32}$$

Again in view of (20), we have from (15) for all sufficiently large values of r that

$$\exp(\alpha(M_g^{-1}(M_f(r)))) \leqslant \Big(\frac{(\sigma_{(\gamma,\beta)}[f]_h+\varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]_h}}{(\overline{\sigma}_{(\gamma,\alpha)}[g]_h-\varepsilon)}\Big)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}$$

$$i.e., \frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\frac{\rho_{(\gamma,\beta)}[f]_h}{\rho_{(\gamma,\alpha)}[g]_h}}} \leqslant \left(\frac{\sigma_{(\gamma,\beta)}[f]_h+\varepsilon}{\overline{\sigma}_{(\gamma,\alpha)}[g]_h-\varepsilon}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}.$$

Since in view of Theorem 1, it follows that $\frac{\rho_{(\gamma,\beta)}[f]_h}{\rho_{(\gamma,\alpha)}[g]_h} \leqslant \rho_{(\alpha,\beta)}[f]_g$ and $\varepsilon(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to\infty} \frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\frac{\rho_{(\gamma,\beta)}[f]_h}{\rho_{(\gamma,\alpha)}[g]_h}}} \leqslant \left(\frac{\sigma_{(\gamma,\beta)}[f]_h}{\overline{\sigma}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}$$

$$i.e., \ \sigma_{(\alpha,\beta)}[f]_g \leqslant \left(\frac{\sigma_{(\gamma,\beta)}[f]_h}{\overline{\sigma}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}. \tag{33}$$

Thus the theorem follows from (31), (32) and (33). □

The proof of the following theorem can be carried out from (20) and (23); (23) and (28) respectively after applying the same technique of Theorem 2 and with the help of Theorem 1. Therefore its proof is omitted.

THEOREM 3. *Let f, g and h be any three entire functions such that* $0<\lambda_{(\gamma,\beta)}[f]_h<\infty$ *and* $0<\lambda_{(\gamma,\alpha)}[g]_h \leqslant \rho_{(\gamma,\alpha)}[g]_h<\infty$. *Then*

$$\sigma_{(\alpha,\beta)}[f]_g \leqslant \min\left\{\left(\frac{\tau_{(\gamma,\beta)}[f]_h}{\overline{\tau}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}, \left(\frac{\tau_{(\gamma,\beta)}[f]_h}{\overline{\sigma}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}\right\}.$$

Similarly in the line of Theorem 2 and with the help of Theorem 1, one may easily carry out the proof of following theorem from pairwise inequalities numbers (24) and (27); (21) and (23); (20) and (26) respectively and therefore its proofs is omitted:

THEOREM 4. *Let f, g and h be any three entire functions such that* $0<\lambda_{(\gamma,\beta)}[f]_h \leqslant \rho_{(\gamma,\beta)}[f]_h<\infty$ *and* $0<\lambda_{(\gamma,\alpha)}[g]_h \leqslant \rho_{(\gamma,\alpha)}[g]_h<\infty$. *Then*

$$\left(\frac{\overline{\tau}_{(\gamma,\beta)}[f]_h}{\tau_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}} \leqslant \overline{\tau}_{(\alpha,\beta)}[f]_g \leqslant \min\left\{\left(\frac{\overline{\tau}_{(\gamma,\beta)}[f]_h}{\overline{\sigma}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}, \left(\frac{\tau_{(\gamma,\beta)}[f]_h}{\sigma_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}\right\}.$$

THEOREM 5. *Let f, g and h be any three entire functions such that* $0<\rho_{(\gamma,\beta)}[f]_h<\infty$ *and* $0<\lambda_{(\gamma,\alpha)}[g]_h \leqslant \rho_{(\gamma,\alpha)}[g]_h<\infty$. *Then*

$$\overline{\tau}_{(\alpha,\beta)}[f]_g \geqslant \max\left\{\left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h}{\sigma_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}, \left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h}{\tau_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}\right\}.$$

With the help of Theorem 1, the proof of the above theorem can be carried out from (16), (19) and (16), (27) respectively after applying the same technique of Theorem 2 and therefore its proof is omitted.

THEOREM 6. *Let f, g and h be any three entire functions such that* $0<\rho_{(\gamma,\beta)}[f]_h<\infty$ *and* $0<\lambda_{(\gamma,\alpha)}[g]_h\leqslant\rho_{(\gamma,\alpha)}[g]_h<\infty$. *Then*

$$\left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h}{\tau_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}\leqslant\overline{\sigma}_{(\alpha,\beta)}[f]_g$$
$$\leqslant\min\left\{\left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h}{\overline{\sigma}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}},\left(\frac{\sigma_{(\gamma,\beta)}[f]_h}{\sigma_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}\right\}.$$

Proof. From (16) and in view of (27), for all sufficiently large values of r, we get

$$\exp(\alpha(M_g^{-1}(M_f(r))))$$
$$\geqslant\exp(\alpha(M_g^{-1}(M_h(\gamma^{-1}(\log((\overline{\sigma}_{(\gamma,\beta)}[f]_h-\varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]_h}))))))$$

$$i.e., \exp(\alpha(M_g^{-1}(M_f(r))))\geqslant\left(\frac{(\overline{\sigma}_{(\gamma,\beta)}[f]_h-\varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]_h}}{(\tau_{(\gamma,\alpha)}[g]_h+\varepsilon)}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}$$

$$i.e., \ \frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\frac{\rho_{(\gamma,\beta)}[f]_h}{\lambda_{(\gamma,\alpha)}[g]_h}}}\geqslant\left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h-\varepsilon}{\tau_{(\gamma,\alpha)}[g]_h+\varepsilon}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}.$$

Since in view of Theorem 1, $\frac{\rho_{(\gamma,\beta)}[f]_h}{\lambda_{(\gamma,\alpha)}[g]_h}\geqslant\rho_{(\alpha,\beta)}[f]_g$, and $\varepsilon(>0)$ is arbitrary, we get from above that

$$\liminf_{r\to\infty}\frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]_g}}\geqslant\left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h}{\tau_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}$$

$$i.e., \ \overline{\sigma}_{(\alpha,\beta)}[f]_g\geqslant\left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h}{\overline{\tau}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}. \tag{34}$$

Further in view of (21), we get from (15) for a sequence of values of r tending to infinity that

$$\exp(\alpha(M_g^{-1}(M_f(r))))\leqslant\left(\frac{(\sigma_{(\gamma,\beta)}[f]_h+\varepsilon)(\exp(\beta(r)))^{\rho_{(\gamma,\beta)}[f]_h}}{(\sigma_{(\gamma,\alpha)}[g]_h-\varepsilon)}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}$$

$$i.e., \ \frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\frac{\rho_{(\gamma,\beta)}[f]_h}{\rho_{(\gamma,\alpha)}[g]_h}}}\leqslant\left(\frac{\sigma_{(\gamma,\beta)}[f]_h+\varepsilon}{\sigma_{(\gamma,\alpha)}[g]_h-\varepsilon}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}.$$

Again as in view of Theorem 1, $\frac{\rho_{(\gamma,\beta)}[f]_h}{\rho_{(\gamma,\alpha)}[g]_h}\leqslant\rho_{(\alpha,\beta)}[f]_g$ and $\varepsilon(>0)$ is arbitrary, therefore we get from above that

$$\liminf_{r\to\infty}\frac{\exp(\alpha(M_g^{-1}(M_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]_g}}\leqslant\left(\frac{\sigma_{(\gamma,\beta)}[f]_h}{\sigma_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}$$

$$i.e., \ \overline{\sigma}_{(\alpha,\beta)}[f]_g \leqslant \left(\frac{\sigma_{(\gamma,\beta)}[f]_h}{\sigma_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}. \tag{35}$$

Similarly from (18) and (20), we get

$$i.e., \ \overline{\sigma}_{(\alpha,\beta)}[f]_g \leqslant \left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h}{\overline{\sigma}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}. \tag{36}$$

Thus the theorem follows from (34), (35) and (36). □

THEOREM 7. *Let f, g and h be any three entire functions such that $0<\lambda_{(\gamma,\beta)}[f]_h<\infty$ and $0<\lambda_{(\gamma,\alpha)}[g]_h \leqslant \rho_{(\gamma,\alpha)}[g]_h<\infty$. Then*

$$\overline{\sigma}_{(\alpha,\beta)}[f]_g \leqslant \min\left\{\left(\frac{\overline{\tau}_{(\gamma,\beta)}[f]_h}{\overline{\tau}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}, \left(\frac{\tau_{(\gamma,\beta)}[f]_h}{\tau_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}, \left(\frac{\tau_{(\gamma,\beta)}[f]_h}{\sigma_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}, \left(\frac{\overline{\tau}_{(\gamma,\beta)}[f]_h}{\overline{\sigma}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}\right\}.$$

The proof of the above theorem can be carried out from pairwise inequalities numbered (20) and (26); (21) and (23); (26) and (28); (23) and (29) respectively after applying the same technique of Theorem 6 and with the help of Theorem 1. Therefore its proof is omitted.

Similarly in the line of Theorem 2 and with the help of Theorem 1, one may easily carry out the following theorem from pairwise inequalities numbered (25) and (27); (24) and (30); (20) and (23) respectively and therefore its proof is omitted:

THEOREM 8. *Let f, g and h be any three entire functions such that $0<\lambda_{(\gamma,\beta)}[f]_h<\infty$ and $0<\lambda_{(\gamma,\alpha)}[g]_h \leqslant \rho_{(\gamma,\alpha)}[g]_h<\infty$. Then*

$$\max\left\{\left(\frac{\tau_{(\gamma,\beta)}[f]_h}{\tau_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}, \left(\frac{\overline{\tau}_{(\gamma,\beta)}[f]_h}{\overline{\tau}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}\right\} \leqslant \tau_{(\alpha,\beta)}[f]_g \leqslant \left(\frac{\tau_{(\gamma,\beta)}[f]_h}{\overline{\sigma}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}.$$

THEOREM 9. *Let f, g and h be any three entire functions such that $0<\lambda_{(\gamma,\beta)}[f]_h \leqslant \rho_{(\gamma,\beta)}[f]_h<\infty$ and $0<\lambda_{(\gamma,\alpha)}[g]_h \leqslant \rho_{(\gamma,\alpha)}[g]_h<\infty$. Then*

$$\tau_{(\alpha,\beta)}[f]_g \geqslant \max\left\{\left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h}{\overline{\sigma}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}, \left(\frac{\sigma_{(\gamma,\beta)}[f]_h}{\sigma_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}, \left(\frac{\sigma_{(\gamma,\beta)}[f]_h}{\tau_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}, \left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h}{\overline{\tau}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}\right\}.$$

The proof of the above theorem can be carried out from pairwise inequalities numbered (17) and (19); (16) and (22); (17) and (27); (16) and (30) respectively after applying the same technique of Theorem 6 and with the help of Theorem 1. Therefore its proof is omitted.

THEOREM 10. *Let f, g and h be any three entire functions such that* $0 < \rho_{(\gamma,\beta)}[f]_h < \infty$ *and* $0 < \rho_{(\gamma,\alpha)}[g]_h (= \lambda_{(\gamma,\alpha)}[g]_h) < \infty$. *Then*

$$\left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h}{\sigma_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}} \leqslant \overline{\sigma}_{(\alpha,\beta)}[f]_g$$
$$\leqslant \min\left\{\left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h}{\overline{\sigma}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}, \left(\frac{\sigma_{(\gamma,\beta)}[f]_h}{\sigma_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}\right\}$$
$$\leqslant \max\left\{\left(\frac{\overline{\sigma}_{(\gamma,\beta)}[f]_h}{\overline{\sigma}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}, \left(\frac{\sigma_{(\gamma,\beta)}[f]_h}{\sigma_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}\right\}$$
$$\leqslant \sigma_{(\alpha,\beta)}[f]_g \leqslant \left(\frac{\sigma_{(\gamma,\beta)}[f]_h}{\overline{\sigma}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\rho_{(\gamma,\alpha)}[g]_h}}.$$

The proof of the above theorem can be carried out from pairwise inequalities numbered (16) and (19); (18) and (20); (15) and (21); (16) and (22); (17) and (19); (15) and (20) respectively after applying the same technique of Theorem 6 and with the help of Theorem 1. Therefore its proof is omitted.

REMARK 2. In Theorem 10, if we replace the conditions "$0 < \rho_{(\gamma,\beta)}[f]_h < \infty$ and $0 < \rho_{(\gamma,\alpha)}[g]_h (= \lambda_{(\gamma,\alpha)}[g]_h) < \infty$" by "$0 < \rho_{(\gamma,\beta)}[f]_h (= \lambda_{(\gamma,\beta)}[f]_h) < \infty$ and $0 < \rho_{(\gamma,\alpha)}[g]_h < \infty$" respectively, then Theorem 10 remains valid with $\overline{\tau}_{(\alpha,\beta)}[f]_g$ and $\tau_{(\alpha,\beta)}[f]_g$ in place of $\overline{\sigma}_{(\alpha,\beta)}[f]_g$ and $\sigma_{(\alpha,\beta)}[f]_g$ respectively.

THEOREM 11. *Let f, g and h be any three entire functions such that* $0 < \rho_{(\gamma,\beta)}[f]_h (= \lambda_{(\gamma,\beta)}[f]_h) < \infty$ *and* $0 < \lambda_{(\gamma,\alpha)}[g]_h < \infty$. *Then*

$$\left(\frac{\overline{\tau}_{(\gamma,\beta)}[f]_h}{\tau_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}} \leqslant \overline{\sigma}_{(\alpha,\beta)}[f]_g$$
$$\leqslant \min\left\{\left(\frac{\tau_{(\gamma,\beta)}[f]_h}{\tau_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}, \left(\frac{\overline{\tau}_{(\gamma,\beta)}[f]_h}{\overline{\tau}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}\right\}$$
$$\leqslant \max\left\{\left(\frac{\tau_{(\gamma,\beta)}[f]_h}{\tau_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}, \left(\frac{\overline{\tau}_{(\gamma,\beta)}[f]_h}{\overline{\tau}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}\right\}$$
$$\leqslant \sigma_{(\alpha,\beta)}[f]_g \leqslant \left(\frac{\tau_{(\gamma,\beta)}[f]_h}{\overline{\tau}_{(\gamma,\alpha)}[g]_h}\right)^{\frac{1}{\lambda_{(\gamma,\alpha)}[g]_h}}.$$

The proof of the above theorem can be carried out from pairwise inequalities numbered (24) and (27); (26) and (28); (23) and (29); (24) and (30); (25) and (27); (23) and (28) respectively after applying the same technique of Theorem 6 and with the help of Theorem 1. Therefore its proof is omitted.

REMARK 3. In Theorem 11, if we replace the conditions "$0 < \rho_{(\gamma,\beta)}[f]_h$ $(= \lambda_{(\gamma,\beta)}[f]_h) < \infty$ and $0 < \lambda_{(\gamma,\alpha)}[g]_h < \infty$" by "$0 < \lambda_{(\gamma,\beta)}[f]_h < \infty$ and $0 < \rho_{(\gamma,\alpha)}[g]_h$ $(= \lambda_{(\gamma,\alpha)}[g]_h) < \infty$" respectively, then Theorem 11 remains valid with $\overline{\tau}_{(\alpha,\beta)}[f]_g$ and $\tau_{(\alpha,\beta)}[f]_g$ in place of $\overline{\sigma}_{(\alpha,\beta)}[f]_g$ and $\sigma_{(\alpha,\beta)}[f]_g$ respectively.

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