Journal of
Classical
Analysis
Volume 19, Number 1 (2022), 21–37
doi:10.7153/jca-2022-19-03

PROJECTIONS OF MUTUAL MULTIFRACTAL FUNCTIONS

ZIED DOUZI AND BILEL SELMI*

Abstract. The aim of this article is to study the behavior of the relative multifractal spectrum under projections. First of all, we depict a relationship between the mutual multifractal spectra of a couple of measures (μ, ν) and its orthogonal projections in Euclidean space. As an application, we improve Svetova's results in [46] and study the mutual multifractal analysis of the projections of measures.

1. Introduction

In the previous years, there has been great interest in understanding the fractal dimensions of projections of sets and measures. The first significant work in this area was the result of Marstrand [25] showed a well-known theorem according to which the Hausdorff dimension of a planar set is preserved under orthogonal projections. In [22], Kaufman had employed potential theoretic methods in order to prove Marstrand result, which has been generalized later by Mattila in [26]. Let us mention that Falconer et al [19, 20] have proved that the packing dimension of the projected set or measure will be the same for almost all projections. Other works were carried out in this sense for classes of similar measures in euclidean and symbolic spaces [7, 21]. However, despite these substantial advances for fractal sets, only very little is known about the multifractal structure of projections of measures [2, 6, 15, 16, 17, 32, 34, 35, 38, 39, 40, 41, 44, 45].

Recently, mixed (mutual and relative) multifractal spectra have generated an enormous interest in the mathematical literature. Many authors were interested in mixed multifractal spectra and their applications [1, 4, 5, 10, 12, 13, 14, 23, 24, 28, 29, 31, 42, 46, 47, 48]. Previously, only the scaling behavior

$$\lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$$

of a single measure μ has been investigated (see for example [8, 30]). However, the mixed multifractal analysis of measures on $\mathbb{R}^n$ investigates the simultaneous scaling behavior

$$\lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}, \quad \lim_{r\to 0} \frac{\log \nu(B(x,r))}{\log r}$$

Mathematics subject classification (2020): 28A20, 28A75, 28A78, 28A80, 49Q15.
Keywords and phrases: Hausdorff dimension, packing dimension, projection, multifractal analysis.
* Corresponding author.

Paper JCA-19-03

of finitely many measures μ and ν. It combines local characteristics which depend simultaneously on various different aspects of the underlying dynamical system and provides the basis for a significantly better understanding of the underlying dynamics. Olsen [31] conjectured a mixed multifractal formalism which links the mixed spectrum to the Legendre transform of mixed Rényi dimensions. Olsen obtained a general upper bound and proved that this bound is equality if both measures are self-similar with same contracting similarities. We note also that Peyrière [33] has also guessed a general vectorial multifractal formalism that is valid under some Frostman assumptions.

But the natural fractal-like objects that one wants to understand do not come always from simultaneous functions but from simultaneous measures. This is why, in [28, 46, 47], a mixed multifractal formalism associated with the mixed multifractal generalizations of Hausdorff and packing measures and dimensions is proved, in some cases, based on a generalization of the well-known large deviation formalism. Furthermore, a mixed multifractal formalism has been proved for the Gibbs-like measures. In general, one needs some degree of similarity to prove the existence of Gibbs-like measures. For example, in dynamic contexts, the existence of such measures is often natural. More specifically, given two compactly supported Borel probability measures μ and ν on $\mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$, Svetova estimated the size of the iso-Hölder set

$$E_{\mu,\nu}(\alpha,\beta) = \Big\{x \in \operatorname{supp}\mu \cap \operatorname{supp}\nu;\ \ \alpha_\mu(x) = \alpha \quad \text{and} \quad \alpha_\nu(x) = \beta\Big\},$$

where $\alpha_\mu(x) = \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$ and $B(x,r)$ is the closed ball of center x and radius r. The mutual multifractal analysis of measures allows relating the Hausdorff and packing dimensions of this level set to the Legendre transforms of some multifractal functions. We write for $\gamma \geqslant 0$,

$$\mathscr{B}_{\mu,\nu}(\gamma) = \left\{x \in \operatorname{supp}\mu \cap \operatorname{supp}\nu;\ \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log \nu(B(x,r))} = \gamma\right\}.$$

It is clear that

$$\bigcup_{\substack{(\alpha,\beta)\in\mathbb{R}_+\times\mathbb{R}_+^*,\\ \frac{\alpha}{\beta}=\gamma}} E_{\mu,\nu}\big(\alpha,\beta\big) \subseteq \mathscr{B}_{\mu,\nu}(\gamma).$$

The latter union is composed of an uncountable number of pairwise disjoint nonempty sets. Then, the Hausdorff and packing dimension of $\mathscr{B}_{\mu,\nu}(\gamma)$ is fully carried by some subset $E_{\mu,\nu}\big(\alpha,\beta\big)$. Also, Selmi et al. investigated the projection properties of the ν-Hausdorff, and the ν-packing dimensions of $\mathscr{B}_{\mu,\nu}(\gamma)$ in [15, 17]. In this article, they derived global bounds on the relative multifractal dimensions of a projection of measures in terms of its original relative multifractal dimensions. It is more difficult to obtain a lower and upper bound for the dimension of the set $\mathscr{B}_{\mu_V,\nu_V}(\gamma)$, where V is a m-dimensional linear subspace of $\mathbb{R}^n$.

The purpose of this paper is to improve Svetova's results and to propose a sufficient condition that gives the lower bound for the Hausdorff and the packing dimensions of $\mathscr{B}_{\mu_V,\nu_V}(\gamma)$. Our first aim is to study the behavior of the mutual Hausdorff, packing, and pre-packing dimensions under projections. The second aim is to investigate a relationship between the mutual multifractal spectra and their projections onto a lower-dimensional linear subspace.

2. Preliminaries

Let us recall the multifractal formalism introduced by Svetova in [46]. Let μ and ν be two compactly supported Borel probability measures on $\mathbb{R}^n$. We denote by supp μ the topological support of μ.

DEFINITION 1. For $q,t,s \in \mathbb{R}$, $E \subseteq \mathbb{R}^n$ and $\delta > 0$, we define

$$\overline{\mathscr{P}}^{q,t,s}_{\mu,\nu,\delta}(E) = \sup \sum_i \mu(B(x_i,r_i))^q \nu(B(x_i,r_i))^t (2r_i)^s,$$

where the supremum is taken over all centered δ-packings of E,

$$\overline{\mathscr{P}}^{q,t,s}_{\mu,\nu}(E) = \inf_{\delta>0} \overline{\mathscr{P}}^{q,t,s}_{\mu,\nu,\delta}(E),$$

and we introduce the mutual packing measure relatively to μ and ν

$$\mathscr{P}^{q,t,s}_{\mu,\nu}(E) = \inf_{E\subseteq \bigcup_i E_i} \sum_i \overline{\mathscr{P}}^{q,t,s}_{\mu,\nu}(E_i).$$

In a similar way we define the mutual Hausdorff measure relatively to μ and ν by

$$\overline{\mathscr{H}}^{q,t,s}_{\mu,\nu,\delta}(E) = \inf \sum_i \mu(B(x_i,r_i))^q \nu(B(x_i,r_i))^t (2r_i)^s,$$

where the infinimum is taken over all centered δ-coverings of E,

$$\overline{\mathscr{H}}^{q,t,s}_{\mu,\nu}(E) = \sup_{\delta>0} \overline{\mathscr{H}}^{q,t,s}_{\mu,\nu,\delta}(E),$$

and we introduce the mutual Hausdorff measure relatively to μ and ν

$$\mathscr{H}^{q,t,s}_{\mu,\nu}(E) = \sup_{F\subseteq E} \overline{\mathscr{H}}^{q,t,s}_{\mu,\nu}(F),$$

with the conventions $0^q = \infty$ for $q \leqslant 0$ and $0^q = 0$ for $q > 0$.

REMARK 1.

1. The functions $\mathscr{H}^{q,t,s}_{\mu,\nu}$ and $\mathscr{P}^{q,t,s}_{\mu,\nu}$ are metric outer measures and thus measures on the Borel family of subsets of $\mathbb{R}^n$. An important feature of the Hausdorff and packing measures is that $\mathscr{P}^{q,t,s}_{\mu,\nu} \leqslant \overline{\mathscr{P}}^{q,t,s}_{\mu,\nu}$ and that there exists an integer $\xi \in \mathbb{N}$, such that $\mathscr{H}^{q,t,s}_{\mu,\nu} \leqslant \xi \mathscr{P}^{q,t,s}_{\mu,\nu}$ (see [48]).
2. In the special case where $q=0$ or $t=0$, the mutual multifractal spectra is strictly related to Olsen's multifractal formalism [30].

3. The mutual multifractal spectra represent the relative multifractal analysis introduced by Cole [9] in the case where $s=0$. Other works were carried out in this sense in probability and symbolic spaces [3, 10, 36, 37, 43].

The functions $\overline{\mathscr{P}}^{q,t,s}_{\mu,\nu}$, $\mathscr{P}^{q,t,s}_{\mu,\nu}$ and $\mathscr{H}^{q,t,s}_{\mu,\nu}$ assign in the usual way a dimension to each subset E of $\mathbb{R}^n$. They are respectively denoted by $\Lambda^{q,t}_{\mu,\nu}(E)$, $B^{q,t}_{\mu,\nu}(E)$ and $b^{q,t}_{\mu,\nu}(E)$ and satisfy

$$b^{q,t}_{\mu,\nu}(E) = \inf\Big\{s\in\mathbb{R};\ \mathscr{H}^{q,t,s}_{\mu,\nu}(E)=0\Big\},\quad B^{q,t}_{\mu,\nu}(E) = \inf\Big\{s\in\mathbb{R};\ \mathscr{P}^{q,t,s}_{\mu,\nu}(E)=0\Big\},$$

$$\Lambda^{q,t}_{\mu,\nu}(E) = \inf\Big\{s\in\mathbb{R};\ \overline{\mathscr{P}}^{q,t,s}_{\mu,\nu}(E)=0\Big\}.$$

PROPOSITION 1. ([46, 48])

1. There exists a unique number $b^{q,t}_{\mu,\nu}(E)\in[-\infty,+\infty]$ such that

$$\mathscr{H}^{q,t,s}_{\mu,\nu}(E) = \begin{cases} \infty \text{ if } s < b^{q,t}_{\mu,\nu}(E), \\ 0 \text{ if } b^{q,t}_{\mu,\nu}(E) < s. \end{cases}$$

2. There exists a unique number $B^{q,t}_{\mu,\nu}(E)\in[-\infty,+\infty]$ such that

$$\mathscr{P}^{q,t,s}_{\mu,\nu}(E) = \begin{cases} \infty \text{ if } s < B^{q,t}_{\mu,\nu}(E), \\ 0 \text{ if } B^{q,t}_{\mu,\nu}(E) < s. \end{cases}$$

3. There exists a unique number $\Lambda^{q,t}_{\mu,\nu}(E)\in[-\infty,+\infty]$ such that

$$\overline{\mathscr{P}}^{q,t,s}_{\mu,\nu}(E) = \begin{cases} \infty \text{ if } s < \Lambda^{q,t}_{\mu,\nu}(E), \\ 0 \text{ if } \Lambda^{q,t}_{\mu,\nu}(E) < s. \end{cases}$$

Let $E\subseteq\mathbb{R}^n$ and $q,t\in\mathbb{R}$. We can remark that

$$b^{q,t}_{\mu,\nu}(E) \leqslant B^{q,t}_{\mu,\nu}(E) \leqslant \Lambda^{q,t}_{\mu,\nu}(E).$$

Then we are able to define the multifractal dimension functions $b_{\mu,\nu}$, $B_{\mu,\nu}$ and $\Lambda_{\mu,\nu}$: $\mathbb{R}^2\to[-\infty,+\infty]$ by

$$b_{\mu,\nu}(q,t) = b^{q,t}_{\mu,\nu}(\operatorname{supp}\mu\cap\operatorname{supp}\nu),\quad B_{\mu,\nu}(q,t) = B^{q,t}_{\mu,\nu}(\operatorname{supp}\mu\cap\operatorname{supp}\nu)$$

$$\text{and}\quad \Lambda_{\mu,\nu}(q,t) = \Lambda^{q,t}_{\mu,\nu}(\operatorname{supp}\mu\cap\operatorname{supp}\nu).$$

It is well known that the functions $b_{\mu,\nu}$, $B_{\mu,\nu}$ and $\Lambda_{\mu,\nu}$ are decreasing and $B_{\mu,\nu}$, $\Lambda_{\mu,\nu}$ are convex (see [48]).

3. Projection results

Let m be an integer with $0 < m < n$ and $G_{n,m}$ stand for the Grassmannian manifold of all m-dimensional linear subspaces of $\mathbb{R}^n$. We denote by $\gamma_{n,m}$ the invariant Haar measure on $G_{n,m}$ such that $\gamma_{n,m}(G_{n,m}) = 1$. For $V \in G_{n,m}$, we define the projection map, $\pi_V : \mathbb{R}^n \longrightarrow V$ as the usual orthogonal projection onto V. Now, for a Borel probability measure μ on $\mathbb{R}^n$, supported on the compact set supp μ and for $V \in G_{n,m}$, we define μ_V, the projection of μ onto V, by

$$\mu_V(A) = \mu(\pi_V^{-1}(A)), \quad \forall A \subseteq V.$$

Since μ has a compact support, then $\operatorname{supp}\mu_V = \pi_V(\operatorname{supp}\mu)$ for all $V \in G_{n,m}$.

In the following, we are interested in the behavior of mutual Hausdorff, packing, and pre-packing dimensions under projections. Throughout this paper, we suppose that $K := \operatorname{supp}\mu = \operatorname{supp}\nu$. We are based on the ideas of Selmi et al in [15, 17], to show the following results.

THEOREM 1. *Let μ and ν be two compactly supported Borel probability measures on $\mathbb{R}^n$. Then, for $(q,t) \in \big(]-\infty,0]^2\big) \cup \big(]-\infty,0] \times [0,1]\big) \cup \big([0,1]\times]-\infty,0]\big)$, $E \subseteq K$ and for all $V \in G_{n,m}$, we have*

$$\Lambda^{q,t}_{\mu_V,\nu_V}(E) \leqslant \Lambda^{q,t}_{\mu,\nu}(E).$$

Proof. Let $s \in \mathbb{R}$ such that $\Lambda^{q,t}_{\mu,\nu}(E) < s$. Consider $V \in G_{n,m}$ and fix $\delta > 0$. Let $\Big(B_i = B(x_i,r_i)\Big)_i$ be a δ-centered packing of $\pi_V(E)$. Then there exists an integer K_m depending on m only such that we can divide up the balls $B(x_i,2r_i)$ into $K \leqslant K_m$ families of disjoint balls $\mathscr{B}_1,\ldots,\mathscr{B}_K$. Let $1 \leqslant l \leqslant K$. For each $B(x_i,r_i) \in \mathscr{B}_l$, denote $E_i = E \cap \pi_V^{-1}\big(B(x_i,r_i)\big)$. We have $E_i \subset \bigcup_{y\in E_i} B(y,r_i)$, so Besicovitch's covering theorem [27] provides a positive integer K_n as well as $K_i \leqslant K_n$ families of pairwise disjoint balls $\mathscr{B}_{i,k} = \Big\{B_j^{(i,k)} = B(y_j^{(i,k)}, r_{ijk});\, r_{ijk} = \frac{r_i}{2}\Big\}$, $1 \leqslant k \leqslant K_i$, extracted from $\Big\{B(y,r_i)\Big\}_{y\in E_i}$ such that

$$E_i \subseteq \bigcup_{k=1}^{K_i}\bigcup_j B_j^{(i,k)}.$$

Case 1: For $q \leqslant 0$ and $t \leqslant 0$, we have

$$\begin{aligned}\sum_i \mu_V(B_i)^q \nu_V(B_i)^t (2r_i)^s &\leqslant 2^s \sum_i \mu\big(B_j^{(i,k)}\big)^q \nu\big(B_j^{(i,k)}\big)^t (2r_{ijk})^s\\ &\leqslant 2^s \sum_{i,j}\sum_{k=1}^{K_i} \mu\big(B_j^{(i,k)}\big)^q \nu\big(B_j^{(i,k)}\big)^t (2r_{ijk})^s.\end{aligned}$$

Case 2: For $q \leqslant 0$ and $0 \leqslant t \leqslant 1$, we have

$$\sum_i \mu_V(B_i)^q \nu_V(B_i)^t (2r_i)^s \leqslant 2^s \sum_i \mu\left(B_j^{(i,k)}\right)^q \nu\left(\bigcup_{k=1}^{K_i}\bigcup_j B_j^{(i,k)}\right)^t (2r_{ijk})^s$$
$$\leqslant 2^s \sum_{i,j}\sum_{k=1}^{K_i} \mu\left(B_j^{(i,k)}\right)^q \nu\left(B_j^{(i,k)}\right)^t (2r_{ijk})^s.$$

Case 3: For $0 \leqslant q \leqslant 1$ and $t \leqslant 0$. The proof of *Case* 3 is identical to the proof of *Case* 2 and is therefore omitted.

In all these cases and by construction, since the balls $B(x_i, 2r_i) \in \mathscr{B}_l$ are pairwise disjoint, if $B(y,r) \in \mathscr{B}_{i,k}$ and $B(y',r') \in \mathscr{B}_{i',k'}$ with $i \neq i'$, then $B(y,r) \cap B(y',r') = \emptyset$. Consequently, we can collect the balls $B(y,r)$ invoked in the above sum into at most K_n centered packing of E. This holds for all $1 \leqslant l \leqslant K$, which implies that

$$\sum_i \mu_V(B_i)^q \nu_V(B_i)^t (2r_i)^s \leqslant 2^s K_m K_n \sup\left\{\sum_j \mu(B(y_j,r_j))^q \nu(B(y_j,r_j))^t (2r_j)^s\right\},$$

where the supremum is taken over all centered δ-packings of E by closed balls of radius r_j. Now, we can deduce that

$$\overline{\mathscr{P}}^{q,t,s}_{\mu_V,\nu_V,\delta}(\pi_V(E)) \leqslant 2^s K_n K_m \overline{\mathscr{P}}^{q,t,s}_{\mu,\nu,\delta}(E).$$

Letting $\delta \downarrow 0$, we give

$$\overline{\mathscr{P}}^{q,t,s}_{\mu_V,\nu_V}(\pi_V(E)) \leqslant 2^s K_n K_m \overline{\mathscr{P}}^{q,t,s}_{\mu,\nu}(E), \tag{1}$$

and the result yields. □

COROLLARY 1. *Let μ and ν be two compactly supported Borel probability measures on $\mathbb{R}^n$. Then for $(q,t) \in \left(]-\infty,0]^2\right) \cup \left(]-\infty,0] \times [0,1]\right) \cup \left([0,1] \times]-\infty,0]\right)$ and for all $V \in G_{n,m}$, we have*

$$\Lambda_{\mu_V,\nu_V}(q,t) \leqslant \Lambda_{\mu,\nu}(q,t).$$

Proof. It follows immediately from Theorem 1. □

THEOREM 2. *Let μ and ν be two compactly supported Borel probability measures on $\mathbb{R}^n$. Then for $(q,t) \in \left(]-\infty,0]^2\right) \cup \left(]-\infty,0] \times [0,1]\right) \cup \left([0,1] \times]-\infty,0]\right)$ and for all $V \in G_{n,m}$, we have*

$$B_{\mu_V,\nu_V}(q,t) \leqslant B_{\mu,\nu}(q,t).$$

Proof. Let $s \in \mathbb{R}$ such that $B_{\mu,\nu}(q,t) < s$. Consider $F \subseteq \mathbb{R}^n$ and $V \in G_{n,m}$. Due to inequality (1), we have

$$\overline{\mathscr{P}}^{q,t,s}_{\mu_V,\nu_V}(\pi_V(F)) \leqslant 2^s K_n K_m \overline{\mathscr{P}}^{q,t,s}_{\mu,\nu}(F).$$

Since $\mathscr{P}^{q,t,s}_{\mu,\nu}(K)=0$, there exists $(E_i)_i$ a covering of K such that

$$\sum_i \overline{\mathscr{P}}^{q,t,s}_{\mu,\nu}(E_i) < 1.$$

It follows that $\pi_V(K) \subseteq \bigcup_i \pi_V(E_i)$ and we have

$$\begin{aligned}\mathscr{P}^{q,t,s}_{\mu_V,\nu_V}(\pi_V(K)) &\leqslant \sum_i \overline{\mathscr{P}}^{q,t,s}_{\mu_V,\nu_V}(\pi_V(E_i))\\ &\leqslant 2^s K_n K_m \sum_i \overline{\mathscr{P}}^{q,t,s}_{\mu,\nu}(E_i) < \infty.\end{aligned}$$

Which implies that $B_{\mu_V,\nu_V}(q,t) \leqslant s$. □

THEOREM 3. *Let μ, ν be two compactly supported Borel probability measures on $\mathbb{R}^n$. Then for $(q,t) \in \big(]-\infty,0[^2\big) \cup \big(]-\infty,0[\times]0,1]\big) \cup \big(]0,1]\times]-\infty,0[\big)$ and for all $V \in G_{n,m}$, we have*

$$b_{\mu_V,\nu_V}(q,t) = b_{\mu,\nu}(q,t).$$

Proof. Let's prove that $b_{\mu,\nu}(q,t) \leqslant b_{\mu_V,\nu_V}(q,t)$. Fix $s\in\mathbb{R}$ such that $s < b_{\mu,\nu}(q,t)$ and choose $F \subseteq K$ and $V \in G_{n,m}$. Let $\delta > 0$ and $\Big(B_i = B(x_i,r_i)\Big)_i$ be a δ-centered covering of F. Let E_i such that $\pi_V^{-1}(E_i) = F \cap B(x_i,r_i)$. We have $E_i \subset \bigcup_{y\in E_i} B(y,r_i)$, so Besicovitch's covering theorem provides a positive integer K_n as well as $K_i \leqslant K_n$ families of pairwise disjoint balls $\mathscr{B}_{i,k} = \Big\{B^{(i,k)}_j = B(y^{(i,k)}_j, r_{ijk});\, r_{ijk} = \frac{r_i}{2}\Big\}$, $1 \leqslant k \leqslant K_i$, extracted from $\Big\{B(y,r_i)\Big\}_{y\in E_i}$ and such that

$$E_i \subseteq \bigcup_{k=1}^{K_i}\bigcup_j B^{(i,k)}_j.$$

Case 1: For $q<0$ and $t<0$, we have

$$\begin{aligned}\sum_i \mu(B_i)^q \nu(B_i)^t (2r_i)^s &\leqslant 2^s \sum_i \mu_V\big(B^{(i,k)}_j\big)^q \nu_V\big(B^{(i,k)}_j\big)^t (2r_{ijk})^s\\ &\leqslant 2^s \sum_{i,j}\sum_{k=1}^{K_i} \mu_V(B^{(i,k)}_j)^q \nu_V(B^{(i,k)}_j)^t (2r_{ijk})^s.\end{aligned}$$

Case 2: For $q<0$ and $0<t\leqslant 1$, we have

$$\begin{aligned}\sum_i \mu(B_i)^q \nu(B_i)^t (2r_i)^s &\leqslant 2^s \sum_i \mu_V(B^{(i,k)}_j)^q \nu_V\left(\bigcup_{k=1}^{K_i}\bigcup_j B^{(i,k)}_j\right)^t (2r_{ijk})^s\\ &\leqslant 2^s \sum_{i,j}\sum_{k=1}^{K_i} \mu_V(B^{(i,k)}_j)^q \nu_V(B^{(i,k)}_j)^t (2r_{ijk})^s.\end{aligned}$$

Case 3: For $0 < q \leqslant 1$ and $t < 0$. The proof of *Case* 3 is identical to the proof of *Case* 2 and is therefore omitted.

This implies that

$$\overline{\mathscr{H}}^{q,t,s}_{\mu,\nu,\delta}(F) \leqslant 2^s \overline{\mathscr{H}}^{q,t,s}_{\mu_V,\nu_V,\delta}(\pi_V(F)).$$

Letting $\delta \downarrow 0$, we obtain

$$\overline{\mathscr{H}}^{q,t,s}_{\mu,\nu}(F) \leqslant 2^s \overline{\mathscr{H}}^{q,t,s}_{\mu_V,\nu_V}(\pi_V(F)).$$

We can deduce that

$$\begin{aligned} \overline{\mathscr{H}}^{q,t,s}_{\mu,\nu}(F) &\leqslant 2^s \overline{\mathscr{H}}^{q,t,s}_{\mu_V,\nu_V}(\pi_V(F)) \\ &\leqslant 2^s \mathscr{H}^{q,t,s}_{\mu_V,\nu_V}(\pi_V(F)) \\ &\leqslant 2^s \mathscr{H}^{q,t,s}_{\mu_V,\nu_V}(\pi_V(K)). \end{aligned}$$

The arbitrary on F implies that

$$\mathscr{H}^{q,t,s}_{\mu,\nu}(K) \leqslant 2^s \mathscr{H}^{q,t,s}_{\mu_V,\nu_V}(\pi_V(K)) \tag{2}$$

and the result holds.

In order to prove the other inequality, let $E \subseteq \mathbb{R}^n$ and $s \in \mathbb{R}$ such that $b^{q,t}_{\mu,\nu}(E) < s$. Fix $V \in G_{n,m}$, $\delta > 0$ and suppose that $\Big(B(x_i, r_i)\Big)_i$ is a centered δ-cover of $\pi_V(E)$. Denote $E_i = E \bigcap \pi_V^{-1}\big(B(x_i, r_i)\big)$ which implies that $E_i = \bigcup_{y \in E_i \cap \pi_V^{-1}(\{x_i\})} B(y, \frac{r_i}{n})$. By applying Besicovitch covering theorem, we can find an integer K_n, depending only on n as well as $K_i \leqslant K_n$ families of pairwise disjoint balls $\mathscr{B}_{i,k} = \Big\{B^{(i,k)}_j = B(y^{(i,k)}_j, \frac{r_{ijk}}{n});\ r_{ijk} = \frac{r_i}{2}\Big\}$, $1 \leqslant k \leqslant K_i$ such that

$$E \cap \pi_V^{-1}\big(B(x_i, r_i)\big) \subseteq \bigcup_{k=1}^{K_i} \bigcup_j B^{(i,k)}_j.$$

Case 1: For $q < 0$ and $t < 0$, we have

$$\begin{aligned} \sum_i \mu_V(B_i)^q \nu_V(B_i)^t (2r_i)^s &\leqslant 2^s \sum_i \mu\big(B^{(i,k)}_j\big)^q \nu\big(B^{(i,k)}_j\big)^t (2r_{ijk})^s \\ &\leqslant 2^s \sum_{i,j} \sum_{k=1}^{K_i} \mu(B^{(i,k)}_j)^q \nu(B^{(i,k)}_j)^t (2r_{ijk})^s. \end{aligned}$$

Case 2: For $q < 0$ and $0 < t \leqslant 1$, we have

$$\begin{aligned} \sum_i \mu_V(B_i)^q \nu_V(B_i)^t (2r_i)^s &\leqslant 2^s \sum_i \mu(B^{(i,k)}_j)^q \nu\left(\bigcup_{k=1}^{K_i} \bigcup_j B^{(i,k)}_j\right)^t (2r_{ijk})^s \\ &\leqslant 2^s \sum_{i,j} \sum_{k=1}^{K_i} \mu(B^{(i,k)}_j)^q \nu(B^{(i,k)}_j)^t (2r_{ijk})^s. \end{aligned}$$

Case 3: For $0 < q \leqslant 1$ and $t < 0$. The proof of *Case* 3 is identical to the proof of *Case* 2 and is therefore omitted.

Then

$$\overline{\mathscr{H}}^{q,t,s}_{\mu_V,\nu_V,\delta}(\pi_V(E)) \leqslant 2^s \overline{\mathscr{H}}^{q,t,s}_{\mu,\nu,\delta}(E).$$

Letting $\delta \downarrow 0$, we obtain

$$\overline{\mathscr{H}}^{q,t,s}_{\mu_V,\nu_V}(\pi_V(E)) \leqslant 2^s \overline{\mathscr{H}}^{q,t,s}_{\mu,\nu}(E).$$

Thus, given a subset E of K, $\pi_V(E) \subseteq \pi_V(K)$ and

$$\begin{aligned}\overline{\mathscr{H}}^{q,t,s}_{\mu_V,\nu_V}(\pi_V(E)) &\leqslant 2^s \overline{\mathscr{H}}^{q,t,s}_{\mu,\nu}(E)\\ &\leqslant 2^s \mathscr{H}^{q,t,s}_{\mu,\nu}(K).\end{aligned}$$

The arbitrary on E implies that

$$\mathscr{H}^{q,t,s}_{\mu_V,\nu_V}(\pi_V(K)) \leqslant 2^s \mathscr{H}^{q,t,s}_{\mu,\nu}(K)$$

and the result holds. This achieves the proof of Theorem 3. □

THEOREM 4. *Let μ and ν be two compactly supported Borel probability measures on $\mathbb{R}^n$. Then for $q,t \geqslant 1$ and all $V \in G_{n,m}$, we have*

$$b_{\mu_V,\nu_V}(q,t) \geqslant b_{\mu,\nu}(q,t).$$

Proof. Fix $V \in G_{n,m}$ and $\delta > 0$ and suppose that $\big(B_i = B(x_i,r_i)\big)_i$ is a δ-cover of $\pi_V(E)$ where $E \subseteq K$. For each i, we may use the Besicovitch covering theorem to find a constant ξ, depending only on n, and a family of balls $\big(B_{ij} = B(x_{ij},r_{ij})\big)_{j\in\mathbb{N}}$ with $r_{ij} = \frac{r_i}{2}$ which is a δ-cover of $\pi_V^{-1}(B_i)\bigcap E$ such that

$$\bigcup_j B(x_{ij},r_{ij}) \subseteq \pi_V^{-1}\Big(B(x_i,2r_i)) \cap V\Big).$$

Note that $\widetilde{B}_i = B(x_i,2r_i)$. It follows that

$$\begin{aligned}\sum_i \mu_V\big(\widetilde{B}_i\big)^q \nu_V\big(\widetilde{B}_i\big)^t (4r_i)^s &\geqslant \xi^{-(q+t)} \sum_i (4r_i)^s \left(\sum_j \mu(B_{ij})\right)^q \left(\sum_j \nu(B_{ij})\right)^t\\ &\geqslant \xi^{-(q+t)} \sum_{i,j} (4r_i)^s \mu(B_{ij})^q \nu(B_{ij})^t\\ &\geqslant 4^s \xi^{-(q+t)} \sum_{i,j} \mu(B_{ij})^q \nu(B_{ij})^t (2r_{ij})^s.\end{aligned}$$

Consequently, as $(B_i)_i$ way any centered δ-cover of $\pi_V(E)$, we conclude that

$$\overline{\mathscr{H}}^{q,t,s}_{\mu,\nu,\delta}(E) \leqslant 4^{-s} \xi^{(q+t)} \overline{\mathscr{H}}^{q,t,s}_{\mu_V,\nu_V,2\delta}(\pi_V(E)).$$

Letting $\delta \downarrow 0$, gives that

$$\overline{\mathscr{H}}^{q,t,s}_{\mu,\nu}(E) \leqslant 4^{-s}\xi^{(q+t)}\overline{\mathscr{H}}^{q,t,s}_{\mu_V,\nu_V}(\pi_V(E)).$$

Which implies that

$$\begin{aligned}\overline{\mathscr{H}}^{q,t,s}_{\mu,\nu}(E) &\leqslant 4^{-s}\xi^{(q+t)}\overline{\mathscr{H}}^{q,t,s}_{\mu_V,\nu_V}(\pi_V(E))\\ &\leqslant 4^{-s}\xi^{(q+t)}\mathscr{H}^{q,t,s}_{\mu_V,\nu_V}(\pi_V(E))\\ &\leqslant 4^{-s}\xi^{(q+t)}\mathscr{H}^{q,t,s}_{\mu_V,\nu_V}(\pi_V(K)).\end{aligned}$$

The arbitrary on E implies that

$$\mathscr{H}^{q,t,s}_{\mu,\nu}(K) \leqslant 4^{-s}\xi^{(q+t)}\mathscr{H}^{q,t,s}_{\mu_V,\nu_V}(\pi_V(K))$$

and the result yields. $\square$

REMARK 2. Notice that in the case where $t=0$ or $q=0$, the preceding results were treated by O′Neil in [32]. Also, when $s=0$, Selmi et al. investigated the projection properties of the mutual Hausdorff, packing, and pre-packing measures in [15, 17]. They derived global bounds on the relative multifractal dimensions of a projection of measures in terms of its original relative multifractal dimensions.

4. Application

This section is devoted to studying the behavior of projections of measures obeying the mutual multifractal formalism. More precisely, we prove that for

$$(q,t) \in \Big\{ \big(]-\infty,0[^2\big) \cup \big(]-\infty,0[\times]0,1]\big) \cup \big(]0,1]\times]-\infty,0[\big) \Big\},$$

if the mutual multifractal formalism holds for the couple (μ,ν) at $\alpha = -\frac{\partial B_{\mu,\nu}(q,t)}{\partial q}$ and $\beta = -\frac{\partial B_{\mu,\nu}(q,t)}{\partial t}$, it holds for (μ_V,ν_V) for all $V \in G_{n,m}$. Before detailing our results let us recall the mutual multifractal formalism introduced by Svetova [46]. For $\alpha,\beta \geqslant 0$, let

$$E_{\mu,\nu}(\alpha,\beta) = \left\{ x\in K;\ \lim_{r\to 0}\frac{\log\big(\mu(B(x,r))\big)}{\log r} = \alpha \text{ and } \lim_{r\to 0}\frac{\log\big(\nu(B(x,r))\big)}{\log r} = \beta \right\}.$$

We are interested in the estimation of the Hausdorff and packing dimension of $E_{\mu,\nu}(\alpha,\beta)$. Let us mention that in the last decay there has been a great interest for the multifractal analysis and positive results have been written in various situations (see for example [8, 9, 15, 30, 42]). Our purpose in the following theorem is to prove the result of Theorem 3 in [46] under less restrictive hypotheses.

THEOREM 5. *Let μ,ν be two compactly supported Borel probability measures on $\mathbb{R}^n$. Suppose that $B_{\mu,\nu}$ is differentiable at (q,t), we set $\alpha = -\frac{\partial B_{\mu,\nu}(q,t)}{\partial q}$ and $\beta = -\frac{\partial B_{\mu,\nu}(q,t)}{\partial t}$, and assume that $\mathscr{H}^{q,t,B_{\mu,\nu}(q,t)}_{\mu,\nu}\big(\mathrm{supp}\,\mu \cap \mathrm{supp}\,\nu\big) > 0$. Then, we have*

$$\dim_H E_{\mu,\nu}(\alpha,\beta) = \dim_P E_{\mu,\nu}(\alpha,\beta) = B^*_{\mu,\nu}(\alpha,\beta) = b^*_{\mu,\nu}(\alpha,\beta),$$

where $f^*(\alpha,\beta)=\inf_{q,t}\big(\alpha q+\beta t+f(\alpha,\beta)\big)$ *denotes the Legendre transform of the function* f. *Here* $\dim_H$ *and* $\dim_P$ *denote the Hausdorff and packing dimensions (see* [18] *for the definitions) and in this case we say that the mutual multifractal formalism is valid.*

Proof. It is known (for instance, see [46]) that, for all reals α and β, one has

$$\dim_P E_{\mu,\nu}(\alpha,\beta)\leqslant \alpha q+\beta t+B_{\mu,\nu}(q,t).$$

Then Theorem 5 is a immediately consequence from the following lemmas.

LEMMA 1. *Let* $\eta_1,\eta_2>0$ *and we set* $\alpha=-\frac{\partial B_{\mu,\nu}(q,t)}{\partial q}$ *and* $\beta=-\frac{\partial B_{\mu,\nu}(q,t)}{\partial t}$. *Then*

$$\mathscr{H}^{\alpha q+\beta t+B_{\mu,\nu}(q,t)-\eta_1-\eta_2}\big(E_{\mu,\nu}(\alpha,\beta)\big)\geqslant 2^{\alpha q+\beta t-\eta_1-\eta_2}\mathscr{H}^{q,t,B_{\mu,\nu}(q,t)}_{\mu,\nu}\big(E_{\mu,\nu}(\alpha,\beta)\big).$$

Proof. We treat the case $q\leqslant 0$ and $t\leqslant 0$. The other cases are proved similarly. The result is true for $q=t=0$, so we may assume that $q<0$ and $t<0$. For $m\in\mathbb{N}^*$, write

$$E_m:=\Big\{x\in E_{\mu,\nu}(\alpha,\beta)\,;\ \frac{\log\big(\mu(B(x,r))\big)}{\log r}\leqslant\alpha-\frac{\eta_1}{q}\\ \text{and}\ \frac{\log\big(\nu(B(x,r))\big)}{\log r}\leqslant\beta-\frac{\eta_2}{t}\ \text{for}\ 0<r<\frac{1}{m}\Big\}.$$

Given $F\subseteq E_m$, $0<\delta<\dfrac{1}{m}$ and $\Big(B(x_i,r_i)\Big)_i$ a centered δ-covering of F, we have

$$\frac{\log\mu(B(x_i,r_i))}{\log r_i}\leqslant\alpha-\frac{\eta_1}{q}\ \text{and}\ \frac{\log\nu(B(x_i,r_i))}{\log r_i}\leqslant\beta-\frac{\eta_2}{q}.$$

Which implies that

$$\mu(B(x_i,r_i))^q\leqslant {r_i}^{\alpha q-\eta_1}\ \text{and}\ \nu(B(x_i,r_i))^t\leqslant {r_i}^{\beta t-\eta_2}.$$

Now, we can deduce that

$$\mu(B(x_i,r_i))^q\nu(B(x_i,r_i))^t(2r_i)^{B_{\mu,\nu}(q,t)}\leqslant 2^{B_{\mu,\nu}(q,t)}{r_i}^{\alpha q+\beta t+B_{\mu,\nu}(q,t)-\eta_1-\eta_2}.$$

So

$$\begin{aligned}\overline{\mathscr{H}}^{q,t,B_{\mu,\nu}(q,t)}_{\mu,\nu,\delta}(F)&\leqslant\sum_i\mu(B(x_i,r_i))^q\nu(B(x_i,r_i))^t(2r_i)^{B_{\mu,\nu}(q)}\\&\leqslant 2^{-\alpha q-\beta t+\eta_1+\eta_2}\sum_i(2r_i)^{\alpha q+\beta t+B_{\mu,\nu}(q,t)-\eta_1-\eta_2}.\end{aligned}$$

We obtain

$$\overline{\mathscr{H}}^{q,t,B_{\mu,\nu}(q,t)}_{\mu,\nu,\delta}(F)\leqslant 2^{-\alpha q-\beta t+\eta_1+\eta_2}\overline{\mathscr{H}}^{\alpha q+\beta t+B_{\mu,\nu}(q,t)-\eta_1-\eta_2}_{\delta}(F).$$

Letting $\delta \searrow 0$ gives that

$$\overline{\mathscr{H}}_{\mu,\nu}^{q,t,B_{\mu,\nu}(q,t)}(F) \leqslant 2^{-\alpha q-\beta t+\eta_1+\eta_2}\overline{\mathscr{H}}^{\alpha q+\beta t+B_{\mu,\nu}(q,t)-\eta_1-\eta_2}(F)$$
$$\leqslant 2^{-\alpha q-\beta t+\eta_1+\eta_2}\mathscr{H}^{\alpha q+\beta t+B_{\mu,\nu}(q,t)-\eta_1-\eta_2}(E_m)$$

for all $F \subseteq E_m$. This implies that

$$\mathscr{H}_{\mu,\nu}^{q,t,B_{\mu,\nu}(q,t)}(E_m) \leqslant 2^{-\alpha q-\beta t+\eta_1+\eta_2}\mathscr{H}^{\alpha q+\beta t+B_{\mu,\nu}(q,t)-\eta_1-\eta_2}(E_m)$$

and the result follows since $E_{\mu,\nu}(\alpha,\beta) = \bigcup_m E_m$. □

LEMMA 2. *We have* $\mathscr{H}_{\mu,\nu}^{q,t,B_{\mu,\nu}(q,t)}\Big(\big(supp\ \mu \cap supp\ \nu\big)\setminus E_{\mu,\nu}(\alpha,\beta)\Big) = 0.$

Proof. Let us introduce, for $\alpha,\beta \in \mathbb{R}$

$$F_{\alpha,\beta} = \left\{x;\ \limsup_{r\to 0}\frac{\log\big(\mu(B(x,r))\big)}{\log r} > \alpha,\ \text{or}\ \limsup_{r\to 0}\frac{\log\big(\nu(B(x,r))\big)}{\log r} > \beta\right\},$$

$$F^1_{\alpha,\beta} = \left\{x;\ \liminf_{r\to 0}\frac{\log\big(\mu(B(x,r))\big)}{\log r} < \alpha,\ \text{or}\ \liminf_{r\to 0}\frac{\log\big(\nu(B(x,r))\big)}{\log r} < \beta\right\},$$

$$F^2_{\alpha,\beta} = \left\{x;\ \limsup_{r\to 0}\frac{\log\big(\mu(B(x,r))\big)}{\log r} > \alpha,\ \text{or}\ \liminf_{r\to 0}\frac{\log\big(\nu(B(x,r))\big)}{\log r} < \beta\right\},$$

$$F^3_{\alpha,\beta} = \left\{x;\ \liminf_{r\to 0}\frac{\log\big(\mu(B(x,r))\big)}{\log r} < \alpha,\ \text{or}\ \limsup_{r\to 0}\frac{\log\big(\nu(B(x,r))\big)}{\log r} > \beta\right\}.$$

We have to prove that

$$\mathscr{H}_{\mu,\nu}^{q,t,B_{\mu,\nu}(q,t)}(F_{\alpha,\beta}) = 0 \text{ for every } \alpha > -\frac{\partial B_{\mu,\nu}(q,t)}{\partial q} \text{ and } \beta > -\frac{\partial B_{\mu,\nu}(q,t)}{\partial t} \tag{3}$$

$$\mathscr{H}_{\mu,\nu}^{q,t,B_{\mu,\nu}(q,t)}(F^1_{\alpha,\beta}) = 0 \text{ for every } \alpha < -\frac{\partial B_{\mu,\nu}(q,t)}{\partial q} \text{ and } \beta < -\frac{\partial B_{\mu,\nu}(q,t)}{\partial t} \tag{4}$$

$$\mathscr{H}_{\mu,\nu}^{q,t,B_{\mu,\nu}(q,t)}(F^2_{\alpha,\beta}) = 0 \text{ for every } \alpha > -\frac{\partial B_{\mu,\nu}(q,t)}{\partial q} \text{ and } \beta < -\frac{\partial B_{\mu,\nu}(q,t)}{\partial t} \tag{5}$$

and

$$\mathscr{H}_{\mu,\nu}^{q,t,B_{\mu,\nu}(q,t)}(F^3_{\alpha,\beta}) = 0 \text{ for every } \alpha < -\frac{\partial B_{\mu,\nu}(q,t)}{\partial q} \text{ and } \beta > -\frac{\partial B_{\mu,\nu}(q,t)}{\partial t}. \tag{6}$$

Let us sketch the proof of assertion (3). Given

$$\alpha > -\frac{\partial B_{\mu,\nu}(q,t)}{\partial q} \text{ and } \beta > -\frac{\partial B_{\mu,\nu}(q,t)}{\partial t},$$

then we can choose $h > 0$ such that

$$B_{\mu,\nu}(q-h,t) < B_{\mu,\nu}(q,t) + \alpha h \text{ and } B_{\mu,\nu}(q,t-h) < B_{\mu,\nu}(q,t) + \beta h.$$

Which implies that

$$\mathscr{P}_{\mu,\nu}^{q-h,t,B_{\mu,\nu}(q,t)+\alpha h}\big(\operatorname{supp}\mu \cap \operatorname{supp}\nu\big) = 0$$

and

$$\mathscr{P}_{\mu,\nu}^{q,t-h,B_{\mu,\nu}(q,t)+\beta h}\big(\operatorname{supp}\mu \cap \operatorname{supp}\nu\big) = 0.$$

Let $\delta > 0$. For each $x \in F_{\alpha,\beta}$, there exists $0 < r_x < \delta$ such that

$$\mu(B(x,r_x)) \leqslant r_x^{\alpha} \qquad \text{or} \qquad \nu(B(x,r_x)) \leqslant r_x^{\beta}.$$

The family $\Big(B(x,r_x)\Big)_{x\in F_{\alpha,\beta}}$ is then a centered δ-covering of $F_{\alpha,\beta}$. By using Besicovitch's covering theorem, we can construct ξ finite or countable sub-families

$$\Big(B(x_{1j},r_{1j})\Big)_j,\dots,\Big(B(x_{\xi j},r_{\xi j})\Big)_j$$

such that each $F_{\alpha,\beta} \subseteq \bigcup_{i=1}^{\xi}\bigcup_j B(x_{ij},r_{ij})$ and $\Big(B(x_{ij},r_{ij})\Big)_j$ is a δ-packing of $F_{\alpha,\beta}$. It follows that

$$\begin{aligned}&\mu(B(x_{ij},r_{ij}))^q\nu(B(x_{ij},r_{ij}))^t(2r_{ij})^{B_{\mu,\nu}(q,t)}\\ \leqslant\ &\mu(B(x_{ij},r_{ij}))^{q-h}\nu(B(x_{ij},r_{ij}))^t(2r_{ij})^{B_{\mu,\nu}(q,t)+\alpha h}\end{aligned}$$

or

$$\begin{aligned}&\mu(B(x_{ij},r_{ij}))^q\nu(B(x_{ij},r_{ij}))^t(2r_{ij})^{B_{\mu,\nu}(q,t)}\\ \leqslant\ &\mu(B(x_{ij},r_{ij}))^{q}\nu(B(x_{ij},r_{ij}))^{t-h}(2r_{ij})^{B_{\mu,\nu}(q,t)+\beta h},\end{aligned}$$

which implies that

$$\overline{\mathscr{H}}_{\mu,\nu}^{q,t,B_{\mu,\nu}(q,t)}(F_{\alpha,\beta}) \leqslant \xi\,\overline{\mathscr{P}}_{\mu,\nu}^{q-h,t,B_{\mu,\nu}(q,t)+\alpha h}(F_{\alpha,\beta})$$

or

$$\overline{\mathscr{H}}_{\mu,\nu}^{q,t,B_{\mu,\nu}(q,t)}(F_{\alpha,\beta}) \leqslant \xi\,\overline{\mathscr{P}}_{\mu,\nu}^{q,t-h,B_{\mu,\nu}(q,t)+\beta h}(F_{\alpha,\beta}).$$

Notice that, in the last inequality, we can replace $F_{\alpha,\beta}$ by any arbitrary subset of $F_{\alpha,\beta}$. Then, we can finally conclude that

$$\begin{aligned}\mathscr{H}_{\mu,\nu}^{q,t,B_{\mu,\nu}(q,t)}(F_{\alpha,\beta}) &\leqslant \xi\,\mathscr{P}_{\mu,\nu}^{q-h,t,B_{\mu,\nu}(q,t)+\alpha h}(F_{\alpha,\beta})\\ &\leqslant \xi\,\mathscr{P}_{\mu,\nu}^{q-h,t,B_{\mu,\nu}(q,t)+\alpha h}\big(\operatorname{supp}\mu\cap\operatorname{supp}\nu\big) = 0\end{aligned}$$

or

$$\mathscr{H}_{\mu,\nu}^{q,t,B_{\mu,\nu}(q,t)}(F_{\alpha,\beta}) \leqslant \xi\, \mathscr{P}_{\mu,\nu}^{q,t-h,B_{\mu,\nu}(q,t)+\beta h}(F_{\alpha,\beta})$$
$$\leqslant \xi\, \mathscr{P}_{\mu,\nu}^{q,t-h,B_{\mu,\nu}(q,t)+\beta h}\big(\text{supp } \mu \cap \text{supp } \nu\big) = 0.$$

The proof of (4), (5) and (6) is similar to (3) and is therefore omitted. □

Let us return to the proof of Theorem 5. By using Lemmas 1 and 2, we have for all $\eta_1, \eta_2 > 0$,

$$\mathscr{H}^{\alpha q+\beta t+B_{\mu,\nu}(q,t)-\eta_1-\eta_2}\big(E_{\mu,\nu}(\alpha,\beta)\big) \geqslant 2^{\alpha q+\beta t-\eta_1-\eta_2}\mathscr{H}_{\mu,\nu}^{q,t,B_{\mu,\nu}(q,t)}\big(E_{\mu,\nu}(\alpha,\beta)\big) > 0.$$

Which implies that

$$\dim_H E_{\mu,\nu}(\alpha,\beta) \geqslant \alpha q+\beta t+B_{\mu,\nu}(q,t)-\eta_1-\eta_2.$$

Letting $\eta_1 \to 0$ and $\eta_2 \to 0$ yields

$$\dim_H E_{\mu,\nu}(\alpha,\beta) \geqslant \alpha q+\beta t+B_{\mu,\nu}(q,t).$$

Which achieves the proof of Theorem 5. □

As a consequence of this result, we finish our paper by establishing an important result studying the validity of the multifractal formalism under projections.

THEOREM 6. *Let μ,ν be two compactly supported Borel probability measures on $\mathbb{R}^n$ with supp $\mu =$ supp ν.*
For $(q,t) \in \big(]-\infty,0[^2\big) \cup \big(]-\infty,0[\times]0,1]\big) \cup \big(]0,1]\times]-\infty,0[\big)$, suppose that
H_1) $\mathscr{H}_{\mu,\nu}^{q,t,B_{\mu,\nu}(q,t)}$(supp $\mu \cap$ supp ν) > 0,
H_2) $B_{\mu,\nu}$ is differentiable at (q,t).
Then, for all $V \in G_{n,m}$, we have

$$\dim_H E_{\mu_V,\nu_V}(\alpha,\beta) = \dim_P E_{\mu_V,\nu_V}(\alpha,\beta) = \dim_H E_{\mu,\nu}(\alpha,\beta)$$
$$= \dim_P E_{\mu,\nu}(\alpha,\beta) = B^*_{\mu,\nu}(\alpha,\beta) = b^*_{\mu,\nu}(\alpha,\beta),$$

where $\alpha = -\frac{\partial B_{\mu,\nu}(q,t)}{\partial q}$ and $\beta = -\frac{\partial B_{\mu,\nu}(q,t)}{\partial t}$.

Proof. It follows from Theorems 2 and 3, and (H_1) that

$$b_{\mu,\nu}(q,t) = B_{\mu,\nu}(q,t) = b_{\mu_V,\nu_V}(q,t) = B_{\mu_V,\nu_V}(q,t), \qquad \forall V \in G_{n,m}. \tag{7}$$

(H_1), (2) and (7) ensure that

$$\mathscr{H}_{\mu_V,\nu_V}^{q,t,B_{\mu_V,\nu_V}(q,t)}(\text{supp } \mu_V) \geqslant \mathscr{H}_{\mu,\nu}^{q,t,B_{\mu,\nu}(q,t)}(\text{supp } \mu) > 0, \qquad \forall V \in G_{n,m}.$$

Now, Theorem 5 and the equalities (7) imply that

$$\dim_H E_{\mu_V,\nu_V}(\alpha,\beta) \geqslant \alpha q+\beta t+B_{\mu,\nu}(q,t).$$

The other estimation is satisfied since

$$\begin{aligned}\dim_P E_{\mu_V,\nu_V}\big(\alpha,\beta\big) &\leqslant \alpha q+\beta t+B_{\mu_V,\nu_V}(q,t)\\ &= \alpha q+\beta t+B_{\mu,\nu}(q,t),\end{aligned}$$

which achieves the proof of Theorem 6. □

REMARK 3. Let μ and ν be two compactly supported Borel probability measures on $\mathbb{R}^n$. We write for $\gamma \geqslant 0$,

$$\mathscr{B}_{\mu,\nu}(\gamma)=\left\{x\in \operatorname{supp}\mu\cap\operatorname{supp}\nu;\ \lim_{r\to 0}\frac{\log\big(\mu(B(x,r))\big)}{\log\big(\nu(B(x,r))\big)}=\gamma\right\}.$$

It is clear that

$$\bigcup_{\substack{(\alpha,\beta)\in\mathbb{R}_+\times\mathbb{R}_+^*,\\ \frac{\alpha}{\beta}=\gamma}} E_{\mu,\nu}\big(\alpha,\beta\big)\subseteq \mathscr{B}_{\mu,\nu}(\gamma).$$

The latter union is composed of an uncountable number of pairwise disjoint nonempty sets. Theorem 5 shows that surprisingly the Hausdorff and packing dimension of $\mathscr{B}_{\mu,\nu}(\gamma)$ is fully carried by some subset $E_{\mu,\nu}\big(\alpha,\beta\big)$. Together with Theorem 6, this relationship provides a lower bound to the relative multifractal spectra of the projections of measures introduced in Theorem 4.2 in [15].

Acknowledgements. The authors would like to thank the anonymous referees for their valuable comments and suggestions that led to the improvement of the manuscript.

This work was supported by Analysis, Probability & Fractals Laboratory (No: LR18ES17).

REFERENCES

[1] M. ABID, M. SLIMANE, I. OMRANE AND B. HALOUANI, *Mixed wavelet leaders multifractal formalism in a product of critical besov spaces*, Mediterr. J. Math., **14**, (2017), 1–20.
[2] N. ATTIA AND B. SELMI, *Relative multifractal box-dimensions*, Filomat, **33**, (2019), 2841–2859.
[3] N. ATTIA, B. SELMI AND CH. SOUISSI, *Some density results of relative multifractal analysis*, Chaos, Solitons and Fractals, **103**, (2017), 1–11.
[4] L. BARREIRA AND B. SAUSSOL, *Variational principles and mixed multifractal spectra*, Trans. Amer. Math. Soc., **353**, (2001) 3919–3944.
[5] L. BARREIRA AND P. DOUTOR, *Birkhoff Averages for Hyperbolic Flows: Variational Principles and Applications*, Journal of Statistical Physics, **115**, (2004), 1567–1603.
[6] J. BARRAL AND I. BHOURI, *Multifractal analysis for projections of Gibbs and related measures*, Ergodic Theory and Dynamic systems, **31**, (2011), 673–701.
[7] J. BARRAL AND D. J. FENG, *Projections of planar Mandelbrot random measures*, Advances in Mathematics, **325**, (2018), 640–718.
[8] F. BEN NASR, I. BHOURI AND Y. HEURTEAUX, *The validity of the multifractal formalism: results and examples*, Adv. in Math., **165**, (2002), 264–284.
[9] J. COLE, *Relative multifractal analysis*, Chaos, Solitons and Fractals, **11**, (2000), 2233–2250.
[10] C. DAI AND Y. LI, *A multifractal formalism in a probability space*, Chaos Solitons Fractals, **27**, (2006), 57–73.

[11] M. DAI, *Mixed self-conformal multifractal measures*, Analysis in Theory and Applications, **25**, (2009), 154–165.

[12] M. DAI AND Y. SHI, *Typical behavior of mixed L^q-dimensions*, Nonlinear Analysis: Theory, Methods & Applications, **72**, (2010), 2318–2325.

[13] M. DAI AND W. LI, *The mixed L^q-spectra of self-conformal measures satisfying the weak separation condition*, J. Math. Anal. Appl., **382**, (2011), 140–147.

[14] M. DAI, C. WANG AND H. SUN, *Mixed generalized dimensions of random self-similar measures*, Int. J. Nonlinear. Sci., **13**, (2012), 123–128.

[15] Z. DOUZI AND B. SELMI, *Multifractal variation for projections of measures*, Chaos, Solitons and Fractals, **91**, (2016), 414–420.

[16] Z. DOUZI AND B. SELMI, *On the projections of the mutual multifractal Rényi dimensions*, Anal. Theory Appl., **37**, (2021), 572–592.

[17] Z. DOUZI AND B. SELMI, *A relative multifractal analysis: box-dimensions, densities, and projections*, Quaestiones Mathematicae, (to appear), http://doi.org/10.2989/16073606.2021.1941375.

[18] K. J. FALCONER, *The Geometry of Fractal sets*, Cambridge univ. Press New. York-London., (1985).

[19] K. J. FALCONER AND J. D. HOWROYD, *Packing Dimensions of Projections and Dimensions Profiles*, Math. Proc. Cambridge Philos. Soc., **121**, (1997), 269–286.

[20] K. J. FALCONER AND P. MATTILA, *The Packing Dimensions of Projections and Sections of Measures*, Math. Proc. Cambridge Philos. Soc., **119**, (1996), 695–713.

[21] M. HOCHMAN AND P. SHMERKIN, *Local entropy averages and projections of fractal measures*, Annals Math., **175**, (2012), 1001–1059.

[22] R. KAUFMAN, *On Hausdorff dimension of projections*, Mathematika, **15**, (1968), 153–155.

[23] M. KHELIFI, H. LOTFI, A. SAMTI AND B. SELMI, *A joint multifractal analysis*, Choas, Solitons & Fractals, **140**, (2020), 110091.

[24] Z. LI AND B. SELMI, *On the multifractal analysis of measures in a probability space*, Illinois J. Math., **65**, (2021), 687–718.

[25] J. M. MARSTRAND, *Some fundamental geometrical properties of plane sets of fractional dimensions*, Proceedings of the London Mathematical Society, **4**, (1954), 257–302.

[26] P. MATTILA, *Hausdorff dimension, orthogonal projections and intersections with planes*, Annales Academiae Scientiarum Fennicae. Series A I. Mathematica, **1**, (1975), 227–244.

[27] P. MATTILA, *The Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, Cambrdige, (1995).

[28] M. MENCEUR, A. BEN MABROUK AND K. BETINA, *The multifractal formalism for measures, review and extension to mixed cases*, Anal. Theory Appl., **32**, (2016), 77–106.

[29] J. LI, L. OLSEN AND M. WU, *Bounds for the L^q-spectra of self-similar measures without any separation conditions*, J. Math. Anal. Appl., **387**, (2012), 77–89.

[30] L. OLSEN, *A multifractal formalism*, Advances in Mathematics, **116**, (1995), 82–196.

[31] L. OLSEN, *Mixed generalized dimensions of self-similar measures*, J. Math. Anal. Appl., **306**, (2005), 516–539.

[32] T. C. O'NEIL, *The multifractal spectra of projected measures in Euclidean spaces*, Chaos, Solitons and Fractals, **11**, (2000), 901–921.

[33] J. PEYRIÈRE, *A vectorial multifractal formalism*, Proc. Sympos. Pure Math., **72**, (2004), 217–230.

[34] B. SELMI, *Multifractal dimensions for projections of measures*, Bol. Soc. Paran. Mat., **40**, (2022), 1–15.

[35] B. SELMI, *A note on the effect of projections on both measures and the generalization of q-dimension capacity*, Probl. Anal. Issues Anal., **5**, (2016), 38–51.

[36] B. SELMI, *Measure of relative multifractal exact dimensions*, Advances and Applications in Mathematical Sciences, **17**, (2018), 629–643.

[37] B. SELMI, *On the strong regularity with the multifractal measures in a probability space*, Anal. Math. Phys., **9**, (2019), 1525–1534.

[38] B. SELMI, *On the projections of the multifractal packing dimension for $q > 1$*, Ann. Mat. Pura Appl., **199**, (2020), 1519–1532.

[39] B. SELMI, *On the effect of projections on the Billingsley dimensions*, Asian-Eur. J. Math., **13**, (2020), 2050128.

[40] B. SELMI, *Projection estimates for mutual multifractal dimensions*, J. Pure Appl. Math. Adv. Appl., **22**, (2020), 71–89.

[41] B. SELMI, *Projections of measures with small supports*, Annales Universitatis Paedagogicae Cracoviensis Studia Mathematica, **20**, (2021), 5–15.

[42] B. SELMI, *The relative multifractal analysis, review and examples*, Acta Scientiarum Mathematicarum, **86**, (2020), 635–666.

[43] B. SELMI, *The relative multifractal densities: a review and application*, Journal of Interdisciplinary Mathematics, **24**, (2021), 1627–1644

[44] B. SELMI AND N. YU. SVETOVA, *On the projections of mutual $L^{q,t}$-spectrum*, Probl. Anal. Issues Anal., **6** (2017), 94–108.

[45] B. SELMI AND N. YU. SVETOVA, *Projections and Slices of measures*, Commun. Korean Math. Soc., **36**, (2021), 327–342.

[46] N. YU. SVETOVA, *Mutual multifractal spectra I: Exact spectra*, Tr. Petrozavodsk. Gos. Univ. Ser. Mat., **11**, (2004), 41–46.

[47] N. YU. SVETOVA, *Mutual multifractal spectra II: Legendre and Hentschel-Procaccia spectra, and spectra defined for partitions*, Tr. Petrozavodsk. Gos. Univ. Ser. Mat., **11**, (2004), 47–56.

[48] N. YU. SVETOVA, *The property of convexity of mutual multifractal dimension*, Tr. Petrozavodsk. Gos. Univ. Ser. Mat., **17**, (2010), 15–24.

(Received June 2, 2021)

Zied Douzi
Analysis, Probability & Fractals Laboratory: LR18ES17
Department of Mathematics, Faculty of Sciences of Monastir
University of Monastir, 5000-Monastir-Tunisia
e-mail: zied.douzi@fsm.rnu.tn

Bilel Selmi
Analysis, Probability & Fractals Laboratory: LR18ES17
Department of Mathematics, Faculty of Sciences of Monastir
University of Monastir, 5000-Monastir-Tunisia
e-mail: bilel.selmi@fsm.rnu.tn
bilel.selmi@isetgb.rnu.tn