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FOURIER TRANSFORM INVERSION IN THE ALEXIEWICZ NORM

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Abstract. If $f\in L^1(\mathbb{R})$ it is proved that $\lim_{S\to\infty}\|f-f*D_S\|=0$, where $D_S(x)=\sin(Sx)/(\pi x)$ is the Dirichlet kernel and $\|f\|=\sup_{\alpha<\beta}|\int_\alpha^\beta f(x)\,dx|$ is the Alexiewicz norm. This gives a symmetric inversion of the Fourier transform on the real line. An asymmetric inversion is also proved. The results also hold for a measure given by dF where F is a continuous function of bounded variation. Such measures need not be absolutely continuous with respect to Lebesgue measure. An example shows there is $f\in L^1(\mathbb{R})$ such that $\lim_{S\to\infty}\|f-f*D_S\|_1\neq 0$.

1. Introduction

If $f:\mathbb{R}\to\mathbb{R}$ then its Fourier transform is $\hat{f}(s)=\int_{-\infty}^{\infty}e^{-ist}f(t)\,dt$. Under the condition $f\in L^1(\mathbb{R})$ it is known that $\hat{f}$ is uniformly continuous on $\mathbb{R}$. The Riemann-Lebesgue lemma says that $\hat{f}$ vanishes at infinity. The inversion formula is $f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{isx}\hat{f}(s)\,ds$. Further assumptions, such as $\hat{f}\in L^1(\mathbb{R})$, are needed for this formula to hold. See [2] and [7] for such background information.

In this paper we prove the Fourier inversion theorem in the Alexiewicz norm, $\lim_{S\to\infty}\|f-f*D_S\|=0$. The Alexiewicz norm is $\|f\|=\sup_{\alpha<\beta}|\int_\alpha^\beta f(x)\,dx|$. It is useful for conditionally convergent integrals [1], [4]. Note that $\|f\|\leqslant\|f\|_1$ with equality if and only if f is of one sign almost everywhere. For $S>0$, the family of functions, $D_S(x)=\sin(Sx)/(\pi x)$, is known as the Dirichlet kernel. Notice that $\int_{-\infty}^{\infty}D_S(x)\,dx=1$.

The Dirichlet kernel arises from a symmetric inversion integral. Let $S>0$. By the Fubini–Tonelli theorem,

$$\begin{aligned}\frac{1}{2\pi}\int_{-S}^{S}e^{isx}\hat{f}(s)\,ds&=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(t)\int_{-S}^{S}e^{-i(t-x)s}\,ds\,dt\\&=\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\frac{\sin[(t-x)S]}{t-x}\,dt\\&=f*D_S(x).\end{aligned}$$

The symmetric inversion formula is then written as convolution with the Dirichlet kernel,

$$f(x)=\lim_{S\to\infty}\frac{1}{2\pi}\int_{-S}^{S}e^{isx}\hat{f}(s)\,ds=\lim_{S\to\infty}f*D_S(x).$$

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For this formula to hold pointwise, further conditions must be imposed on f or $\hat{f}$. For example, it holds at points of differentiability of f [3] or everywhere if f is absolutely continuous and $f' \in L^1(\mathbb{R})$ [6].

We prove in Theorem 1 that the inversion holds in the Alexiewicz norm, $\lim_{S\to\infty}\|f - f*D_S\| = 0$ for all $f \in L^1(\mathbb{R})$. The proof is elementary and does not use any machinery from Fourier analysis. For example, it does not use the Riemann–Lebesgue lemma. It should be noted that the same result is false for the L^1 norm, i.e., there is $f \in L^1(\mathbb{R})$ such that $\lim_{S\to\infty}\|f - f*D_S\|_1 \neq 0$. See Example 2, where $f \in L^1(\mathbb{R})$ is given so that $\|f*D_S\|_1$ exists for no $S > 0$.

In Section 2 we prove that if F is a continuous function of bounded variation and $f = dF$ is the associated signed Lebesgue–Stieltjes measure then the inversion formula holds for dF. Such measures need not be absolutely continuous with respect to Lebesgue measure. An example shows that inversion can fail when F is not continuous.

In Section 3 we prove the inversion still holds with an asymmetric inversion integral.

A similar result holds for Fourier series [5].

2. Alexiewicz norm inversion theorem

THEOREM 1. *Let* $f \in L^1(\mathbb{R})$. *Then* $\lim_{S\to\infty}\|f - f*D_S\| = 0$, *where* D_S *is the Dirichlet kernel.*

Proof. Let $-\infty < \alpha < \beta < \infty$ and let $S > 0$. Let $F(x) = \int_{-\infty}^x f(t)\,dt$. By the Fubini–Tonelli theorem,

$$\int_\alpha^\beta [f(x) - f*D_S(x)]\,dx = \int_\alpha^\beta \int_{-\infty}^\infty [f(x)D_1(t) - f(x-t)D_S(t)]\,dt\,dx$$
$$= \frac{1}{\pi}\int_{-\infty}^\infty \frac{\sin(t)}{t}\int_\alpha^\beta [f(x) - f(x-t/S)]\,dx\,dt$$
$$= \frac{1}{\pi}\int_{-\infty}^\infty \frac{\sin(t)}{t}[F(\beta) - F(\beta - t/S) - F(\alpha) + F(\alpha - t/S)]\,dt.$$

Let $T > 0$. We split this last integral into integration over the intervals $(-T,T)$, (T,∞) and $(-\infty,-T)$. The supremum over $\alpha < \beta$ of the absolute value of each such integral is then shown to have limit 0 as $S \to \infty$.

First note that

$$\left|\int_{-T}^T \frac{\sin(t)}{t}[F(\beta) - F(\beta - t/S) - F(\alpha) + F(\alpha - t/S)]\,dt\right|$$
$$\leqslant 2T\left(\sup_{\substack{\beta\in\mathbb{R}\\|t|\leqslant T}}|F(\beta) - F(\beta - t/S)| + \sup_{\substack{\alpha\in\mathbb{R}\\|t|\leqslant T}}|F(\alpha) - F(\alpha - t/S)|\right)$$
$\to 0$ as $S \to \infty$ since F is uniformly continuous on $\mathbb{R}$.

Let $\varepsilon > 0$. Let $g(t) = \sin(t)/t$. Note that $\lim_{t\to\infty} F(\beta - t/S) = 0$. Note that $\int_T^\infty |f(\beta - t/S)|\,dt/S = \int_{-\infty}^{\beta - T/S} |f(u)|\,du \leqslant \|f\|_1$ and $\lim_{T\to\infty} \int_T^\infty g(t)\,dt = 0$. Now integrate by parts to get

$$\left|\int_T^\infty g(t)\,[F(\beta) - F(\beta - t/S) - F(\alpha) + F(\alpha - t/S)]\,dt\right|$$
$$= \left|[F(\beta) - F(\alpha)]\int_T^\infty g(t)\,dt - \int_T^\infty \int_T^t g(u)\,du[f(\beta - t/S) - f(\alpha - t/S)]\,dt/S\right|$$
$$\leqslant 3\|f\|_1 \|\chi_{(T,\infty)} g\| < \varepsilon \text{ for large enough } T.$$

Similarly for integration over the interval $(-\infty, -T)$. □

The essential parts of the proof of the theorem use the Fubini–Tonelli theorem and the fact that F is uniformly continuous and has finite variation. Thus, the theorem extends to measures that arise from continuous functions of bounded variation. If μ is a signed measure then its Alexiewicz norm is

$$\|\mu\| = \sup_{\alpha<\beta}\left|\int_{(\alpha,\beta)} d\mu\right| = \sup_{\alpha<\beta}|\mu((\alpha,\beta))|.$$

COROLLARY 1. *Let $F \in C(\mathbb{R})$ such that F is of bounded variation. Define a signed measure $\mu_F = dF$. Then $\lim_{S\to\infty}\|\mu_F - \mu_F * D_S\| = 0$.*

If F is of bounded variation then $\lim_{x\to\infty} F(x)$ and $\lim_{x\to-\infty} F(x)$ exist so that if $F(\infty)$ and $F(-\infty)$ are defined with these respective limits then F is continuous on the extended real line $[-\infty, \infty]$.

The continuity condition in Corollary 1 cannot be dropped.

EXAMPLE 1. The Dirac measure, δ, is generated by the Heaviside step function, $H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x > 0$. We have $\hat{\delta} = 1$. This gives

$$\frac{1}{2\pi}\int_{-S}^S e^{isx}\hat{\delta}(s)\,ds = \frac{1}{2\pi}\int_{-S}^S e^{isx}\,ds = D_S(x).$$

Let $0 < \alpha < \beta$. Then

$$\left|\int_\alpha^\beta d\delta(x) - \frac{1}{\pi}\int_\alpha^\beta \frac{\sin(Sx)}{x}\,dx\right| = \frac{1}{\pi}\left|\int_\alpha^\beta \frac{\sin(Sx)}{x}\,dx\right| = \frac{1}{\pi}\left|\int_{\alpha S}^{\beta S} \frac{\sin(x)}{x}\,dx\right|.$$

If we let $\alpha = 1/S^2$ and $\beta = 1$ then we see that $\limsup_{S\to\infty}\|\delta - \delta * D_S\| \geqslant 1/2$.

EXAMPLE 2. Let $a > 0$ and take $f = \chi_{(0,a)}$. For $x > 0$, integrate by parts to get

$$\pi f * D_S(x) = \int_{(x-a)S}^{xS} \sin(t)\,\frac{dt}{t} = \frac{\cos[(x-a)S]}{(x-a)S} - \frac{\cos(xS)}{xS} - \int_{(x-a)S}^{xS} \cos(t)\,\frac{dt}{t^2}.$$

Note that $1/(x-a) = 1/x + a/x^2 + O(1/x^3)$ as $x \to \infty$. Then

$$\left| \int_{(x-a)S}^{xS} \cos(t) \frac{dt}{t^2} \right| \leqslant \left| \int_{(x-a)S}^{xS} \frac{dt}{t^2} \right| = \frac{1}{(x-a)S} - \frac{1}{xS} = \frac{a}{x^2 S} + O(1/x^3) \text{ as } x \to \infty.$$

This gives

$$\pi f * D_S(x) = \frac{\sin(aS)\sin(xS) - [1-\cos(aS)]\cos(xS)}{xS} + O\left(\frac{1}{x^2}\right) \text{ as } x \to \infty.$$

Hence, if $S \neq 2n\pi/a$ for some $n \in \mathbb{N}$ then $f * D_S \notin L^1(\mathbb{R})$.

If we write $f_a = \chi_{(0,a)}$ then

$$\begin{aligned} &\pi(f_a + f_b) * D_S(x) \\ &= \frac{[\sin(aS) + \sin(bS)]\sin(xS) - [2 - \cos(aS) - \cos(bS)]\cos(xS)}{xS} + O\left(\frac{1}{x^2}\right). \end{aligned}$$

It follows that $(f_a + f_b) * D_S \in L^1(\mathbb{R})$ if and only if $\sin(aS) = \sin(bS) = 0$ and $\cos(aS) = \cos(bS) = 1$. Taking $a = 2\pi$ and $b = 2$ then gives an example of a function $f \in L^1(\mathbb{R})$ such that $\|f - f * D_S\|_1$ fails to exist for each $S > 0$.

3. Asymmetric kernel

Let $S_1, S_2 > 0$. Consider the asymmetric inversion,

$$\begin{aligned} \frac{1}{2\pi} \int_{-S_1}^{S_2} e^{isx} \hat{f}(s)\, ds &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \int_{-S_1}^{S_2} e^{-i(t-x)s}\, ds\, dt \\ &= \frac{i}{2\pi} \int_{-\infty}^{\infty} f(t) \left[e^{-i(t-x)S_2} - e^{i(t-x)S_1} \right] \frac{dt}{t-x} \\ &= f * A_{S_1,S_2}(x), \end{aligned}$$

where the asymmetric kernel is

$$A_{S_1,S_2}(x) = \frac{e^{ixS_2} - e^{-ixS_1}}{2\pi i x} = \frac{D_{S_1}(x)}{2} + \frac{D_{S_2}(x)}{2} + B_{S_1,S_2}(x),$$

where

$$B_{S_1,S_2}(x) = \frac{1}{\pi i x} \sin\left[\left(\frac{S_1 + S_2}{2} \right) x \right] \sin\left[\left(\frac{S_1 - S_2}{2} \right) x \right].$$

Notice that $\int_{-\infty}^{\infty} A_{S_1,S_2}(x)\, dx = 1$. As with Theorem 1 there is inversion in the Alexiewicz norm, provided the ratios S_1/S_2 and S_2/S_1 remain bounded.

THEOREM 2. *Let $q \geqslant p > 0$. Let $f \in L^1(\mathbb{R})$. Let $S_1, S_2 \to \infty$ such that $p \leqslant S_1/S_2 \leqslant q$. Then $\|f - f * A_{S_1,S_2}\| \to 0$.*

Proof. The proof is similar to that of Theorem 1. Let $-\infty<\alpha<\beta<\infty$ and let $S_1,S_2>0$. Let $F(x)=\int_{-\infty}^x f(t)\,dt$. Write

$$f-f*A_{S_1,S_2}=\frac{1}{2}(f-f*D_{S_1})+\frac{1}{2}(f-f*D_{S_1})-f*B_{S_1,S_2}.$$

Due to Theorem 1 we just need to consider the last term above. Since B_{S_1,S_2} is an odd function we can write

$$\begin{aligned}
&-\pi i\int_\alpha^\beta f*B_{S_1,S_2}(x)\,dx\\
&=\pi i\int_\alpha^\beta f(x)\,dx\int_{-\infty}^\infty B_{S_1,S_2}(t)\,dt-\pi i\int_\alpha^\beta\int_{-\infty}^\infty f(x-t)B_{S_1,S_2}(t)\,dt\,dx\\
&=\int_\alpha^\beta\int_{-\infty}^\infty [f(x)-f(x-2t/(S_1+S_2))]\sin(t)\sin\left[\left(\frac{S_1-S_2}{S_1+S_2}\right)t\right]\frac{dt}{t}\,dx,
\end{aligned}$$

after the change of variables $t\mapsto 2t/(S_1+S_2)$.

Let $T>0$. Then

$$\begin{aligned}
&\left|\int_\alpha^\beta\int_{-T}^T [f(x)-f(x-2t/(S_1+S_2))]\sin(t)\sin\left[\left(\frac{S_1-S_2}{S_1+S_2}\right)t\right]\frac{dt}{t}\,dx\right|\\
&\leqslant 2T\left(\sup_{\substack{\beta\in\mathbb{R}\\|t|\leqslant T}}|F(\beta)-F(\beta-2t/(S_1+S_2))|+\sup_{\substack{\alpha\in\mathbb{R}\\|t|\leqslant T}}|F(\alpha)-F(\alpha-2t/(S_1+S_2))|\right)
\end{aligned}$$

$\to 0$ as $S_1,S_2\to\infty$ since F is uniformly continuous on $\mathbb{R}$.

And,

$$\begin{aligned}
&\int_\alpha^\beta\int_T^\infty [f(x)-f(x-2t/(S_1+S_2))]\sin(t)\sin\left[\left(\frac{S_1-S_2}{S_1+S_2}\right)t\right]\frac{dt}{t}\,dx\\
&=\frac{1}{2}\int_T^\infty [F(\beta)-F(\beta-2t/(S_1+S_2))]\left[\cos\left(\frac{2S_2t}{S_1+S_2}\right)-\cos\left(\frac{2S_1t}{S_1+S_2}\right)\right]\frac{dt}{t}\\
&\quad+\text{ similar terms involving }\alpha.
\end{aligned}$$

Let $\varepsilon>0$ be given. Let $h(t)=\cos(t)/t$. Write $A_1=2S_1T/(S_1+S_2)$ and $A_2=2S_2T/(S_1+S_2)$. Use the change of variables $u_2=2S_2t/(S_1+S_2)$ and $u_1=2S_1t/(S_1+S_2)$. Integrate by parts as in the last paragraph of the proof of Theorem 1. Then

$$\begin{aligned}
&\int_T^\infty [F(\beta)-F(\beta-2t/(S_1+S_2))]\left[\cos\left(\frac{2S_2t}{S_1+S_2}\right)-\cos\left(\frac{2S_1t}{S_1+S_2}\right)\right]\frac{dt}{t}\\
&=\int_{A_2}^\infty [F(\beta)-F(\beta-u_2/S_2)]h(u_2)\,du_2-\int_{A_1}^\infty [F(\beta)-F(\beta-u_1/S_1)]h(u_1)\,du_1\\
&=F(\beta)\int_{A_2}^\infty h(u_2)\,du_2-\int_{A_2}^\infty\int_{A_2}^{u_2} h(v)\,dv\,f(\beta-u_2/S_2)\frac{du_2}{S_2}\\
&\quad-F(\beta)\int_{A_1}^\infty h(u_1)\,du_1+\int_{A_1}^\infty\int_{A_1}^{u_1} h(v)\,dv\,f(\beta-u_1/S_1)\frac{du_1}{S_1}.
\end{aligned}$$

Notice that $A_2 = 2S_2T/(S_1+S_2) \geqslant 2T/(q+1)$ and $S_1 = 2S_1T/(S_1+S_2) \geqslant 2T/(1+1/p)$. Hence, if T is large then so are A_2 and A_1. This gives

$$\begin{aligned}
&\left| \int_T^\infty [F(\beta) - F(\beta - 2t/(S_1+S_2))] \left[\cos\left(\frac{2S_2 t}{S_1+S_2} \right) - \cos\left(\frac{2S_1 t}{S_1+S_2} \right) \right] \frac{dt}{t} \right| \\
&\leqslant 2\|f\|_1 \left(\|\chi_{(A_2,\infty)} h\| + \|\chi_{(A_1,\infty)} h\| \right) \\
&< \varepsilon \text{ for large enough } T.
\end{aligned}$$

The same estimates hold for the terms containing α. □

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