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ON THE LOCATION OF ZEROS OF QUASI–ORTHOGONAL POLYNOMIALS WITH APPLICATIONS TO SOME REAL SELF–RECIPROCAL POLYNOMIALS

VANESSA BOTTA* AND MIJAEL HANCCO SUNI

Abstract. In this paper, we present new results on the location of zeros of some classes of quasi-orthogonal polynomials. From the Chebyshev polynomials, we obtain some classes of real self-reciprocal polynomials, and investigate the location and monotonicity of their zeros.

1. Introduction

Since its introduction, the concept of orthogonal polynomials has been an important tool in the analysis of a large variety of problems in mathematics and engineering, like moment problems and numerical quadrature, for example. Afterwards, many other concepts related to orthogonal polynomials were proposed, as quasi-orthogonal polynomials [12, 18, 32, 33], exceptional orthogonal polynomials [21, 23], para-orthogonal polynomials [10, 14, 20, 24], among others.

In this paper, our focus is on the location of zeros of quasi-orthogonal polynomials. Recent contributions we can see in references [6, 9, 16, 17, 25]. We present new results on the location and monotonicity of zeros of some quasi-orthogonal polynomials. These results were used to study the behaviour of zeros of some classes of real self-reciprocal polynomials.

One of the motivations for the study of the location of zeros of quasi-orthogonal polynomials of degree n is related to the positive quadrature formula with n nodes, which is exact for polynomials of degree $2n-r-1$, $0 \leqslant r \leqslant n$. More details and further information about this connection can be found in [37].

This paper is organized as follows. The basic background is introduced in Section 2. In Section 3, we present new results on the location of zeros of some classes of quasi-orthogonal polynomials (Theorems 3, 4 and 5), generalizing [9]. Numerical examples are given to illustrate them. In Section 4, we construct classes of real self-reciprocal polynomials from the family of Chebyshev quasi-orthogonal polynomials of order one and order two, respectively. A complete study on the location and monotonicity behaviour of zeros of these real self-reciprocal polynomials is also given, including numerical examples to illuminate the present work. Finally, we draw conclusions and open issues in Section 5.

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* Corresponding author.

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2. Preliminary results

2.1. Sequence of orthogonal polynomials

Let us consider $P = \{P_n(x)\}_{n\geqslant 0}$ a sequence of polynomials such that $P_n \in P$ has degree n and the following orthogonality relation is satisfied

$$\int_a^b P_m(x)P_n(x)d\mu(x) = K_m\delta_{m,n}, \tag{1}$$

where μ is a positive Borel measure supported in an infinite set of the real line, $\delta_{m,n}$ is the Kronecker delta function and K_m are positive numbers. In particular, if μ has support within some closed interval $[a,b]$, we will say that P_n belongs to the family of orthogonal polynomials on $[a,b]$ with respect to μ. If $\mu(x)$ is absolutely continuous, then $d\mu(x) = w(x)dx$, where $w(x)$ is a non-negative function measurable in Lebesgue's sense for which $\int_a^b w(x)dx > 0$. We shall call $w(x)$ a weight function on $[a,b]$.

There are several important results which follow from the orthogonality properties of these polynomials [11, 34]. We will present some of them in what follows.

1. Polynomials in the orthogonal sequence $\{P_n(x)\}_{n\geqslant 0}$ satisfy a three-term recurrence relation of the form

$$P_{-1}(x) = 0,\ P_0(x) = 1,\ P_n(x) = (\gamma_n x - \beta_n)P_{n-1}(x) - \alpha_n P_{n-2}(x),\ n \geqslant 1, \tag{2}$$

   where γ_n, β_n and α_n are real numbers with $\gamma_n > 0$ and $\alpha_n > 0$ for each $n \geqslant 1$. We prefer to use the non-monic form of $P_n(x)$ to facilitate the construction of real self-reciprocal polynomials from the family of Chebyshev quasi-orthogonal polynomials of order one and two, respectively, in Section 4.

2. The zeros of any polynomial $P_n(x)$, $n > 0$, lie all inside the orthogonality interval $(a,b) \subset \mathbb{R}$ and they are all simple. Furthermore, $P_n(x)$ and $P_{n+1}(x)$ cannot have common zeros.

3. The zeros of $P_{n+1}(x)$ and $P_n(x)$, $n \geqslant 1$, interlace. In other words, between any two consecutive zeros of $P_{n+1}(x)$ there is one, and exactly one, zero of $P_n(x)$.

If $w(x)$ is a positive and an even function on the interval $[-c,c]$, the sequence $P = \{P_n(x)\}_{n\geqslant 0}$ that the orthogonality condition (1) is satisfied is called sequence of symmetric orthogonal polynomials. For example, Hermite and Gegenbauer polynomials are symmetric orthogonal polynomials. In this paper, we will use the following properties of them:

1. $P_n(-x) = (-1)^n P_n(x)$;

2. In the corresponding recurrence formula (2), $\beta_n = 0$;

3. If we consider $x_{k,n}$, $k = 1,\dots,n$, the zeros of $P_n(x)$ such that $x_{n,n} < x_{n-1,n} < \dots < x_{2,n} < x_{1,n}$, then $x_{k,n} = -x_{n-k+1,n}$, $k = 1,\dots,\left\lfloor \frac{n}{2} \right\rfloor$. For n odd, $x_{\left\lfloor \frac{n}{2} \right\rfloor +1,n} = 0$.

2.2. Sequence of quasi-orthogonal polynomials

An important concept related to a sequence of orthogonal polynomials is the quasi-orthogonality [12].

DEFINITION 1. Let R_n be a polynomial of exact degree $n \geqslant r$. If R_n satisfies the conditions

$$\int_a^b x^k R_n(x) w(x) dx \begin{cases} = 0, & \text{for } k = 0, \dots, n-1-r \\ \neq 0, & \text{for } k = n-r \end{cases}, \tag{3}$$

where w is a positive weight function on $[a,b]$, then R_n is quasi–orthogonal of order r on $[a,b]$ with respect to w.

Clearly, x^k can be replaced by $R_k(x)$ in (3):

$$\int_a^b R_k(x) R_n(x) w(x) dx \begin{cases} = 0, & \text{for } k = 0, \dots, n-1-r \\ \neq 0, & \text{for } k = n-r \end{cases}.$$

We can cite the reference [12] for a more general definition of quasi-orthogonality, where the following result is considered:

THEOREM 1. *Let $\{Q_n(x)\}_{n \geqslant 0}$ be the sequence of orthogonal polynomials on $[a,b]$ with respect to a positive weight function $w(x)$. A necessary and sufficient condition for a polynomial R_n of degree n to be quasi-orthogonal of order r on $[a,b]$ with respect to w is that*

$$R_n(x) = c_{n,0} Q_n(x) + c_{n,1} Q_{n-1}(x) + \cdots + c_{n,r} Q_{n-r}(x),$$

where $c_{n,i}$, $i = 0, \dots, r$, are numbers which can depend on n and $c_{n,0} c_{n,r} \neq 0$.

For $r \geqslant 1$, in the following we have a well-known result related to the location of zeros of the quasi-orthogonal polynomial R_n.

THEOREM 2. *If R_n is a quasi-orthogonal polynomial of order r on $[a,b]$ with respect to a positive weight function w, then at least $n-r$ distinct zeros of R_n lie in the interval (a,b).*

It is important to mention that the concept of quasi-orthogonality seems to have been introduced by Riesz [32], for $r = 1$. Féjer [18] considered the case $r = 2$ and the general case was first studied by Shohat [33]. We can cite the references [11, 12] for more information on the properties of the zeros of quasi-orthogonal polynomials. Furthermore, there are many problems involving linear combination of orthogonal polynomials, as mentioned in [22], and this topic was studied by many authors as [1, 2, 3, 9], for example.

2.3. Self-reciprocal polynomials

Let $P(z) = p_0 + p_1 z + \ldots + p_n z^n$ be a polynomial of degree n with complex coefficients. Define the reciprocal polynomial $P^*(z)$ by $P^*(z) = z^n \overline{P(1/\overline{z})}$. If $z_1, z_2, \ldots, z_n$ are the zeros of $P(z)$, it is clear that the zeros of $P^*(z)$ are $1/\overline{z_1}, 1/\overline{z_2}, \ldots, 1/\overline{z_n}$.

If there exists a complex number ω such that $|\omega| = 1$ and $P(z) = \omega P^*(z)$, we say that $P(z)$ is a self-reciprocal polynomial. Observe that if $P(z)$ is a self-reciprocal polynomial, then $p_i = \omega \overline{p_{n-i}}$, for $i = 0, 1, \ldots, n$. Furthermore, their zeros are symmetric with respect to the unit circle. Further details and applications of self-reciprocal polynomials can be found in [26, 31, 35].

In this paper, we consider real self-reciprocal polynomials (i.e., $p_i \in \mathbb{R}$) such that $\omega = 1$, and we present the location and monotonicity of zeros of some classes of these polynomials.

In particular, real self-reciprocal polynomials with all zeros on the unit circle constitute an interesting topic to study and has many applications in some areas of mathematics [19, 27, 35]. Some results in this area can be found in [26, 28, 29, 30]. We would like to highlight two important classes of real self-reciprocal polynomials with all zeros on the unit circle: the classes of real para-orthogonal polynomials described in [15] (referred as the first and second kind *singular predictor polynomials*), which satisfy three-term recurrence formulas and have many interesting properties, as we can see in [7, 15]. These classes of para-orthogonal polynomials are important from the point of view of connecting real orthogonal polynomials on the unit circle to symmetric orthogonal polynomials on the interval $[-1, 1]$. The importance of this connection was explored in [39] and in reference [8] it was used in the problem of frequency analysis.

The goal of this work is the construction of real self-reciprocal polynomials from the linear combination of Chebyshev polynomials – more precisely, from Chebyshev quasi-orthogonal polynomials of order one and order two, respectively. In fact, considering $P(z) = p_0 + p_1 z + \ldots + p_{2n} z^{2n}$ a real self-reciprocal polynomial of degree $2n$ such that $\omega = 1$, it is easy to verify that

$$P(z) = \sum_{j=0}^{2n} p_j z^j = 2z^n \left[\frac{p_{2n}}{2} \left(z^n + \frac{1}{z^n} \right) + \ldots + \frac{p_{n+1}}{2} \left(z + \frac{1}{z} \right) + \frac{p_n}{2} \right]. \quad (4)$$

In the case that a real self-reciprocal polynomial has odd degree (for example, $S(z)$ has degree $2n+1$), it is obvious that it is possible to write $S(z)$ as $S(z) = s_{2n+1}(z+1)Q(z)$, where $s_{2n+1} \in \mathbb{R} - \{0\}$ and $Q(z)$ is a real self-reciprocal polynomial of degree $2n$.

We denote the unit circle by $\mathscr{C} = \{z | z = e^{i\theta}, 0 \leqslant \theta \leqslant 2\pi\}$. For $z = e^{i\theta}$ with $0 \leqslant \theta \leqslant 2\pi$, we consider the transformation

$$x = x(z) = \frac{1}{2} \left(z + \frac{1}{z} \right) = \cos\theta. \quad (5)$$

From basic manipulations it follows that $\frac{1}{2} \left(z^j + \frac{1}{z^j} \right) = T_j(x), \ j = 0, 1, 2, \ldots,$ where $\{T_j(x)\}$ is a sequence of Chebyshev polynomials of the first kind. So, from (4),

$$P(z) = 2z^n \left[p_{2n} T_n(x) + p_{2n-1} T_{n-1}(x) + \ldots + p_{n+1} T_1(x) + \frac{p_n}{2} T_0(x) \right] = 2z^n C_n(x). \quad (6)$$

From relation (6) or, equivalently,

$$C_n(x) = \frac{1}{2} z^{-n} P(z), \tag{7}$$

it is evident that the zeros of $C_n(x)$ are connected with the zeros of $P(z)$ by transformation (5). In fact, if we denote by x_j a zero of $C_n(x)$ and z_j a zero of $P(z)$, from (5) it follows that

$$z_j = x_j \pm \sqrt{x_j^2 - 1}. \tag{8}$$

This connection is an important tool for the analysis of the location and monotonicity of the zeros of $P(z)$.

Mappings from the unit circle to the interval $[-1,1]$ are used in the literature to analyze asymptotic properties of orthogonal polynomials. For example, in the context of orthogonal polynomials, the transformation (5) was used by Szegö to show that real orthogonal polynomials on the unit circle can be mapped to orthogonal polynomials on the interval $[-1,1]$ [13]. Delsarte and Genin [15], from the transformation $2x(z) = z^{1/2} + z^{-1/2}$, showed that real orthogonal polynomials on the unit circle can be mapped to symmetric orthogonal polynomials on the interval $[-1,1]$. Furthermore, from this transformation it is possible to show a connection between para-orthogonal polynomials on the unit circle and orthogonal polynomials in $[-1,1]$. The importance of this connection was explored in [39]. In a recent paper [36], the authors analyzed the action of the Möbius transformations in a sequence of orthogonal polynomials on the real line. Indeed, the Möbius transformation was used by [38] in the context of biorthogonal rational functions.

3. Properties of the zeros of $R_n(x)$

Let us consider $\{Q_n(x)\}_{n\geqslant 0}$ an orthogonal polynomial sequence on $[a,b]$ with respect to a given positive weight function $w(x)$. If we denote by $x_{k,n}$, $k = 1,\ldots,n$, the zeros of $Q_n(x)$, then

$$a < x_{n+1,n+1} < x_{n,n} < x_{n,n+1} < x_{n-1,n} < \ldots < x_{2,n+1} < x_{1,n} < x_{1,n+1} < b,$$

i.e., the zeros of $Q_n(x)$ and $Q_{n+1}(x)$ interlace.

Let us begin our analysis considering the polynomial

$$R_n(x) = Q_n(x) + a_n Q_{n-1}(x) + b_n Q_{n-2}(x), \; a_n, b_n \in \mathbb{R}, \tag{9}$$

whose zeros we denote by $y_{i,n}$, $i = 1,\ldots,n$.

By Theorems 1 and 2 notice that, if $b_n = 0$ and $a_n \neq 0$, $R_n(x)$ is a quasi-orthogonal polynomial of order one on $[a,b]$ with respect to $w(x)$ and it has at least $n-1$ zeros in (a,b). If $b_n \neq 0$, $R_n(x)$ is a quasi-orthogonal polynomial of order two on $[a,b]$ with respect to $w(x)$ and it has at least $n-2$ zeros in (a,b).

Now, let us consider the function $h_n(x) = a_n + b_n\left(\frac{\gamma_n x - \beta_n}{\alpha_n}\right)$, where $a_n, b_n \in \mathbb{R}$, and γ_n, β_n and α_n are the coefficients of the three-term recurrence relation given in (2). Notice that $\gamma_n, \beta_n, \alpha_n \in \mathbb{R}$, $\gamma_n > 0$ and $\alpha_n > 0$, for each $n \geqslant 1$.

From (2) and (9) it follows that

$$R_n(x_{i,n}) = Q_{n-1}(x_{i,n})h_n(x_{i,n}),\ i = 1,\ldots,n, \text{ and} \tag{10}$$

$$R_n(x_{i,n-1}) = Q_{n-2}(x_{i,n-1})(b_n - \alpha_n),\ i = 1,\ldots,n-1. \tag{11}$$

In the following we shall present the main results of this work, where the location of zeros of the polynomial $R_n(x)$ represented by (9) is established. The proofs can be easier derived from the sign analysis of $R_n(x_{i,n})$, $i = 1,\ldots,n$, and $R_n(x_{i,n-1})$, $i = 1,\ldots,n-1$, and some manipulations. We suppose, without loss of generality, that the leading coefficients of all polynomials $Q_n(x)$ are positive.

THEOREM 3. *If $b_n < \alpha_n$, $R_n(x)$ has n distinct and real zeros $y_{n,n} < y_{n-1,n} < \ldots < y_{1,n}$ such that*

$$y_{n,n} < x_{n-1,n-1},\ x_{i,n-1} < y_{i,n} < x_{i-1,n-1} \textit{ for } i = 2,\ldots,n-1, \textit{ and } y_{1,n} > x_{1,n-1}. \tag{12}$$

Furthermore,

1. *If $h_n(x_{1,n}) < 0$, then $y_{1,n} > x_{1,n}$. In addition, for a finite number b, if $R_n(b) < 0$, then $y_{1,n} \in (b,\infty)$ and if $R_n(b) \geqslant 0$, $y_{1,n} \in (x_{1,n}, b]$.*
2. *If $h_n(x_{1,n}) > 0$, $x_{1,n-1} < y_{1,n} < x_{1,n}$.*
3. *If $h_n(x_{n,n}) < 0$, $x_{n,n} < y_{n,n} < x_{n-1,n-1}$.*
4. *If $h_n(x_{n,n}) > 0$, $y_{n,n} < x_{n,n}$. Further, for a finite number a, if $R_n(a) < 0$, $y_{n,n} \in (a, x_{n,n})$ (n odd) and $y_{n,n} \in (-\infty, a)$ (n even). If $R_n(a) \geqslant 0$, $y_{n,n} \in (-\infty, a]$ (for n odd) and $y_{n,n} \in [a, x_{n,n})$ (n even).*

Moreover, if $h_n(x_{i,n}) = 0$ for a fixed i ($i = 1,\ldots,n$), then $y_{i,n} = x_{i,n}$ and

- *For a finite number a, if $R_n(a) < 0$, $y_{n,n} \in (-\infty, a)$ (for n even) and $y_{n,n} \in (a, x_{n-1,n-1})$ (for n odd). If $R_n(a) \geqslant 0$, $y_{n,n} \in [a, x_{n-1,n-1})$ (for n even) and $y_{n,n} \in (-\infty, a]$ (for n odd).*
- *For a finite number b, if $R_n(b) < 0$, $y_{1,n} \in (b,\infty)$. If $R_n(b) \geqslant 0$, then $y_{1,n} \in (x_{1,n-1}, b]$.*

Proof. From (10) and (11), if $h_n(x_{i,n}) \neq 0$ for $i = 1,\ldots,n$, and $b_n - \alpha_n \neq 0$,

$$sign(R_n(x_{i,n})) = (-1)^{i+1} sign(h_n(x_{i,n})),\ i = 1,\ldots,n,$$
$$sign(R_n(x_{i,n-1})) = (-1)^{i+1} sign(b_n - \alpha_n),\ i = 1,\ldots,n-1.$$

For $b_n < \alpha_n$, it follows that $sign(R_n(x_{i,n-1})) = (-1)^i$, $i = 1,\ldots,n-1$, from which we conclude that there exist $n-2$ distinct and real zeros $y_{2,n}, y_{3,n}, \ldots, y_{n-1,n}$ of $R_n(x)$ satisfying $x_{i,n-1} < y_{i,n} < x_{i-1,n-1}$ for $i = 2,\ldots,n-1$. Furthermore,

1. If $h_n(x_{1,n}) < 0$, $sign(R_n(x_{1,n})) = -1$. Notice that $\lim_{x\to+\infty} R_n(x) > 0$. So, it follows that there is a sign change of $R_n(x)$ in $(x_{1,n},\infty)$, showing that $y_{1,n} > x_{1,n}$ and, of course, $y_{1,n} > x_{1,n-1}$.

   For a finite number b, if we consider $R_n(b) < 0$, then there is a sign change of $R_n(x)$ in (b,∞) and, of course, $R_n(x)$ has one real zero in (b,∞), i.e., $y_{1,n} \in (b,\infty)$.

   In the case that $R_n(b) > 0$, there is a sign change of $R_n(x)$ in $(x_{1,n},b)$. Consequently, $R_n(x)$ has one real zero in $(x_{1,n},b)$ and $y_{1,n} \in (x_{1,n},b)$.

   If $R_n(b) = 0$, $y_{1,n} = b$.

2. If $h_n(x_{1,n}) > 0$, the proof follows from the fact that $sign(R_n(x_{1,n})) = 1$ and $sign(R_n(x_{1,n-1})) = -1$. That is, there is a sign change of $R_n(x)$ in $(x_{1,n-1},x_{1,n})$ and then $R_n(x)$ has one real zero in $(x_{1,n-1},x_{1,n})$. So, it follows that $x_{1,n-1} < y_{1,n} < x_{1,n}$.

3. If $h_n(x_{n,n}) < 0$, then $sign(R_n(x_{n,n})) = (-1)^n$ and $sign(R_n(x_{n-1,n-1})) = (-1)^{n-1}$. Consequently, using the same arguments as above, $x_{n,n} < y_{n,n} < x_{n-1,n-1}$.

4. If $h_n(x_{n,n}) > 0$, $sign(R_n(x_{n,n})) = (-1)^{n+1}$. Furthermore, for n even, $\lim_{x\to-\infty} R_n(x) > 0$ and, for n odd, $\lim_{x\to-\infty} R_n(x) < 0$. So, there is a sign change of $R_n(x)$ in $(-\infty,x_{n,n})$, i.e., $y_{n,n} < x_{n,n}$ and, of course, $y_{n,n} < x_{n-1,n-1}$.

   For a finite number a, in the case that $R_n(a) < 0$, for n odd, there is a sign change of $R_n(x)$ in $(a,x_{n,n})$ and, consequently, $R_n(x)$ has one real zero in $(a,x_{n,n})$, i.e., $y_{n,n} \in (a,x_{n,n})$. For n even, there is a sign change of $R_n(x)$ in $(-\infty,a)$ and $R_n(x)$ has one real zero in $(-\infty,a)$, i.e., $y_{n,n} \in (-\infty,a)$.

   If $R_n(a) > 0$, for n odd, there is a sign change of $R_n(x)$ in $(-\infty,a)$ and, consequently, $R_n(x)$ has one real zero in $(-\infty,a)$, i.e., $y_{n,n} \in (-\infty,a)$. For n even, there is a sign change of $R_n(x)$ in $(a,x_{n,n})$ and $R_n(x)$ has one real zero in $(a,x_{n,n})$, i.e., $y_{n,n} \in (a,x_{n,n})$.

   If $R_n(a) = 0$, $y_{n,n} = a$.

If $h_n(x_{i,n}) = 0$ for a fixed i ($i = 1,\ldots,n$), then $R_n(x_{i,n}) = 0$ and, consequently, $y_{i,n} = x_{i,n}$. The result follows using the same idea of the previous items, from the analysis of sign change of $R_n(x)$ in the intervals $(-\infty,a)$ and $(a,x_{n-1,n-1})$ (for a finite number a), (b,∞) and $(x_{1,n-1},b)$ (for a finite number b). □

THEOREM 4. *If $b_n = \alpha_n$, $R_n(x)$ has n real zeros. Furthermore, $n-1$ zeros of $R_n(x)$ coincide with the zeros of $Q_{n-1}(x)$ and*

1. *If $h_n(x_{1,n}) < 0$, then $y_{1,n} \in (x_{1,n},\infty)$. In addition, for a finite number b, if $R_n(b) < 0$, $y_{1,n} \in (b,\infty)$ and if $R_n(b) \geqslant 0$, $y_{1,n} \in (x_{1,n},b]$.*

2. *If $h_n(x_{n,n}) > 0$, then $y_{n,n} \in (-\infty,x_{n,n})$. Furthermore, for a finite number a, if $R_n(a) < 0$, $y_{n,n} \in (-\infty,a)$ (for n even) and $y_{n,n} \in (a,x_{n,n})$ (n odd). If $R_n(a) \geqslant 0$, $y_{n,n} \in (-\infty,a]$ (for n odd) and $y_{n,n} \in [a,x_{n,n})$ (n even).*

Moreover, if $h_n(x_{i,n}) = 0$ *for a fixed* i, $i = 1,\dots,n$, *then* $x_{i,n}$ *is zero of* $R_n(x)$. *Furthermore, if* $a_n = \beta_n - \gamma_n x_{i,n-1}$ *for a fixed* i, $i = 1,\dots,n-1$, $x_{i,n-1}$ *is a double zero of* $R_n(x)$.

Proof. From (11), if $b_n = \alpha_n$, it follows that $R_n(x_{i,n-1}) = 0$ for all $i = 1,\dots,n-1$, and then $x_{i,n-1}$ is zero of $R_n(x)$, $i = 1,\dots,n-1$. Furthermore,

1. If $h_n(x_{1,n}) < 0$, from (10) it follows that $sign(R_n(x_{1,n})) = -1$. As $\lim_{x\to+\infty} R_n(x) > 0$, there is a sign change of $R_n(x)$ in $(x_{1,n},\infty)$. Using the same arguments as the proof of item 1 of Theorem 3, the result follows.

2. If $h_n(x_{n,n}) > 0$, from (10) it follows that $sign(R_n(x_{n,n})) = (-1)^{n+1}$. Furthermore, for n even, $\lim_{x\to-\infty} R_n(x) > 0$ and, for n odd, $\lim_{x\to-\infty} R_n(x) < 0$. From the same arguments as the proof of item 4 of Theorem 3, the result follows.

If $h_n(x_{i,n}) = 0$ for a fixed i, $i = 1,\dots,n$, it follows that $R_n(x_{i,n}) = 0$. So, we may consider $y_{1,n} = x_{1,n-1},\dots,\ y_{i-1,n} = x_{i-1,n-1},\ y_{i,n} = x_{i,n},\ y_{i+1,n} = x_{i,n-1},\dots,\ y_{n,n} = x_{n-1,n-1}$. From the known properties of the zeros $x_{i,n-1}$, $i = 1,\dots,n-1$, of $Q_{n-1}(x)$, it is clear that

$$a < y_{n,n} < y_{n-1,n} < \dots < y_{i+1,n} < y_{i,n} < y_{i-1,n} < \dots < y_{1,n} < b.$$

The conclusions on the multiplicity of $x_{i,n-1}$ for a fixed i, $i = 1,\dots,n-1$, are easily obtained from the following expression

$$R'_n(x_{i,n-1}) = Q'_{n-1}(x_{i,n-1})(\gamma_n x_{i,n-1} - \beta_n + a_n),\ i = 1,\dots,n-1.$$

Notice that, as $Q'_{n-1}(x_{i,n-1}) \neq 0$ for all $i = 1,\dots,n-1$, we may conclude that, for a fixed i, $R'_n(x_{i,n-1}) = 0$ if and only if $a_n = \beta_n - \gamma_n x_{i,n-1}$. So, if $a_n = \beta_n - \gamma_n x_{i,n-1}$, as $R''_n(x_{i,n-1}) = 2\gamma_n Q'_{n-1}(x_{i,n-1}) \neq 0$ for all $i = 1,\dots,n-1$, it follows that $x_{i,n-1}$ is a double zero of $R_n(x)$. □

THEOREM 5. *If* $b_n > \alpha_n$, $R_n(x)$ *has* $n-2$ *distinct and real zeros such that between two zeros of* $Q_{n-1}(x)$ *there is exactly one zero of* $R_n(x)$ *(interlacing property). Furthermore, if the following conditions are satisfied, then we may conclude that the two remaining zeros of* $R_n(x)$ *are real:*

1. $h_n(x_{1,n}) < 0$, *where the remaining zeros are located one in the interval* $(x_{1,n-1},x_{1,n})$ *and the other in* $(x_{1,n},\infty)$. *For a finite number* b, *if* $R_n(b) < 0$, $y_{1,n} \in (b,\infty)$, *and if* $R_n(b) \geqslant 0$, $y_{1,n} \in (x_{1,n},b]$.

2. $h_n(x_{1,n}) > 0$ *and* $R_n(b) \leqslant 0$ *(for a finite number* b*), where the remaining zeros are located one in the interval* $(x_{1,n},b]$ *and the other in* (b,∞).

3. $h_n(x_{n,n}) < 0$ *and, for a finite number* a, $R_n(a) \leqslant 0$ *(for* n *even) and* $R_n(a) \geqslant 0$ *(for* n *odd), where the remaining zeros are located one in the interval* $[a,x_{n,n})$ *and the other in* $(-\infty,a)$.

4. *$h_n(x_{n,n}) > 0$, where the remaining zeros are located one in the interval $(x_{n,n}, x_{n-1,n-1})$ and the other in $(-\infty, x_{n,n})$. For a finite number a, if $R_n(a) < 0$, $y_{n,n} \in (a, x_{n,n})$ (n odd) and $y_{n,n} \in (-\infty, a)$ (n even). If $R_n(a) \geqslant 0$, $y_{n,n} \in [a, x_{n,n})$ (n even) and $y_{n,n} \in (-\infty, a]$ (n odd).*

Moreover, if $h_n(x_{i,n}) = 0$, then for a fixed $i = 1, \ldots, n$, $x_{i,n}$ is zero of $R_n(x)$.

Proof. According to (11), if $b_n > \alpha_n$, $sign(R_n(x_{i,n-1})) = (-1)^{i+1}$, $i = 1, \ldots, n-1$. So, there are $n-2$ sign changes of $R_n(x)$ in $(x_{n-1,n-1}, x_{1,n-1})$, and then we have the existence of $n-2$ real zeros satisfying the interlacing property.

Furthermore,

1. From (10) and (11), we deduce that $sign(R_n(x_{1,n})) = -1$ and $sign(R_n(x_{1,n-1})) = 1$, respectively. So, $R_n(x)$ has one zero in $(x_{1,n-1}, x_{1,n})$. As $\lim_{x \to +\infty} R_n(x) > 0$, from $sign(R_n(x_{1,n})) = -1$ it follows that there is a sign change of $R_n(x)$ in $(x_{1,n}, \infty)$, i.e., the other zero of $R_n(x)$ is located in $(x_{1,n}, \infty)$.

   For a finite number b, if we consider $R_n(b) < 0$, from $\lim_{x \to +\infty} R_n(x) > 0$ it follows that $y_{1,n} \in (b, \infty)$. If $R_n(b) > 0$, $y_{1,n} \in (x_{1,n}, b)$. If $R_n(b) = 0$, $y_{1,n} = b$.

2. From (10) notice that $sign(R_n(x_{1,n})) = 1$.

   For a finite number b, if $R_n(b) < 0$, there is a sign change of $R_n(x)$ in $(x_{1,n}, b)$, proving the existence of one zero of $R_n(x)$ in $(x_{1,n}, b)$. From the fact that $\lim_{x \to +\infty} R_n(x) > 0$, the existence of other zero of $R_n(x)$ in (b, ∞) follows.

   The case $R_n(b) = 0$ is obvious.

3. If $h_n(x_{n,n}) < 0$, from (10) it follows that $sign(R_n(x_{n,n})) = (-1)^n$.

   For a finite number a, considering $R_n(a) < 0$ (for n even) and $R_n(a) > 0$ (for n odd), there is a sign change of $R_n(x)$ in $(a, x_{n,n})$ and, consequently, $R_n(x)$ has one zero in the interval $(a, x_{n,n})$. Furthermore, for n even, $\lim_{x \to -\infty} R_n(x) > 0$ and, for n odd, $\lim_{x \to -\infty} R_n(x) < 0$. So, there is a sign change of $R_n(x)$ in $(-\infty, a)$, showing the existence of a zero of $R_n(x)$ in $(-\infty, a)$.

   The case $R_n(a) = 0$ is obvious.

4. From (10) and (11) it follows that $sign(R_n(x_{n,n})) = (-1)^{n+1}$ and $sign(R_n(x_{n-1,n-1})) = (-1)^n$, respectively.

   Consequently, using the same arguments as above, $R_n(x)$ has one zero in $(x_{n,n}, x_{n-1,n-1})$ and the other in $(-\infty, x_{n,n})$.

   For a finite number a, if $R_n(a) > 0$ (for n even) and $R_n(a) < 0$ (for n odd), there is a sign change of $R_n(x)$ in $(a, x_{n,n})$ and, consequently, $R_n(x)$ has one zero in the interval $(a, x_{n,n})$. Furthermore, for n even, $\lim_{x \to -\infty} R_n(x) > 0$ and, for n odd, $\lim_{x \to -\infty} R_n(x) < 0$. So, there is a sign change of $R_n(x)$ in $(-\infty, a)$, proving the existence of a zero of $R_n(x)$ in $(-\infty, a)$.

   The case $R_n(a) = 0$ is obvious.

From $R_n(x_{i,n}) = Q_{n-1}(x_{i,n})h_n(x_{i,n})$, $i = 1,\ldots,n$, if $h_n(x_{i,n}) = 0$ for a fixed i, then $x_{i,n}$ is zero of $R_n(x)$. □

For simplicity, the sign of $R_n(a)$ and $R_n(b)$ we may calculate from the following expressions. If $\gamma_n a - \beta_n + a_n + (b_n - \alpha_n)f_{n-2}(a) \neq 0$ and $\gamma_n b - \beta_n + a_n + (b_n - \alpha_n)f_{n-2}(b) \neq 0$, for finite numbers a and b, we have

$$sign(R_n(a)) = (-1)^{n-1} sign(\gamma_n a - \beta_n + a_n + (b_n - \alpha_n)f_{n-2}(a)) \text{ and}$$
$$sign(R_n(b)) = sign(\gamma_n b - \beta_n + a_n + (b_n - \alpha_n)f_{n-2}(b)),$$

where $f_{n-2}(x) = \dfrac{Q_{n-2}(x)}{Q_{n-1}(x)}$.

These expressions are easily obtained from relations (2) and (9):

$$R_n(x) = Q_{n-1}(x)\left[\gamma_n x - \beta_n + a_n + (b_n - \alpha_n)f_{n-2}(x)\right].$$

THEOREM 6. *If $b_n < \alpha_n$, each real zero $y_{i,n}$, $i = 1,\ldots,n$, is a decreasing function of a_n.*

Proof. We consider $\varepsilon \geqslant 0$ and

$$R_{n_\varepsilon}(x) = Q_n(x) + (a_n + \varepsilon)Q_{n-1}(x) + b_n Q_{n-2}(x)$$

with n real zeros

$$y_{1,n_\varepsilon} > y_{2,n_\varepsilon} > \ldots > y_{n,n_\varepsilon}.$$

It is clear that $y_{i,n} = y_{i,n_0}$, $i = 1,\ldots,n$, and

$$R_{n_\varepsilon}(x) = R_n(x) + \varepsilon Q_{n-1}(x).$$

Consequently, $R_{n_\varepsilon}(y_{i,n}) = \varepsilon Q_{n-1}(y_{i,n})$, $i = 1,\ldots,n$.

So, for $\varepsilon > 0$ and from (12),

$$sgn(R_{n_\varepsilon}(y_{i,n})) = (-1)^{i+1},\ i = 1,\ldots,n. \tag{13}$$

Since the zeros of $R_{n_\varepsilon}(x)$ are real, from (13) it follows that $y_{i,n} > y_{i,n_\varepsilon}$, completing the proof. □

3.1. Special case: sequence of symmetric orthogonal polynomials

Considering $\{Q_n(x)\}_{n\geqslant 0}$ be a sequence of symmetric orthogonal polynomials on the finite interval $[-c,c]$ with respect to a positive and even weight function $w(x)$, it follows immediately that $\beta_n = 0$ in (2). Furthermore, the zeros of $Q_n(x)$ are symmetric with respect to the origin and then $0 < x_{1,n} = -x_{n,n}$. So, as a consequence of Theorems 3, 4 and 5, we present the following results.

COROLLARY 1. *The zeros of $R_n(x)$ are located in $[-c,c]$ when*

1. $b_n < \alpha_n$, $a_n < -\frac{|b_n|\gamma_n x_{1,n}}{\alpha_n}$ *and* $R_n(c) \geqslant 0$;

2. $b_n < \alpha_n$, $a_n > \frac{|b_n|\gamma_n x_{1,n}}{\alpha_n}$, $R_n(-c) \leqslant 0$ *(n odd) and* $R_n(-c) \geqslant 0$ *(n even);*

3. $b_n < 0$, $\frac{b_n\gamma_n x_{1,n}}{\alpha_n} < a_n < -\frac{b_n\gamma_n x_{1,n}}{\alpha_n}$, $R_n(c) \geqslant 0$, $R_n(-c) \leqslant 0$ *(n odd) and* $R_n(-c) \geqslant 0$ *(n even);*

4. $0 < b_n < \alpha_n$ *and* $-\frac{b_n\gamma_n x_{1,n}}{\alpha_n} < a_n < \frac{b_n\gamma_n x_{1,n}}{\alpha_n}$;

5. $b_n \geqslant \alpha_n$, $a_n < -\frac{b_n\gamma_n x_{1,n}}{\alpha_n}$ *and* $R_n(c) \geqslant 0$;

6. $b_n \geqslant \alpha_n$, $a_n > \frac{b_n\gamma_n x_{1,n}}{\alpha_n}$, $R_n(-c) \leqslant 0$ *(n odd) and* $R_n(-c) \geqslant 0$ *(n even).*

REMARK 1. If $b_n = \alpha_n$ and $a_n = -\gamma_n x_{i,n-1}$ for a fixed i $(i = 1, \ldots, n-1)$, $x_{i,n-1}$ is a double zero of $R_n(x)$. In this case, all the zeros of $R_n(x)$ are located in $[-c, c]$.

COROLLARY 2. *$R_n(x)$ has only one zero outside the interval $[-c, c]$ if*

1. $b_n < \alpha_n$, $a_n < \frac{b_n\gamma_n x_{1,n}}{\alpha_n}$ *and* $R_n(c) < 0$;

2. $b_n < \alpha_n$, $a_n > \frac{|b_n|\gamma_n x_{1,n}}{\alpha_n}$, $R_n(-c) > 0$ *(n odd) and* $R_n(-c) < 0$ *(n even);*

3. $b_n < 0$, $\frac{b_n\gamma_n x_{1,n}}{\alpha_n} < a_n < -\frac{b_n\gamma_n x_{1,n}}{\alpha_n}$, $R_n(c) < 0$, $R_n(-c) \leqslant 0$ *(n odd) and* $R_n(-c) \geqslant 0$ *(n even);*

4. $b_n < 0$, $\frac{b_n\gamma_n x_{1,n}}{\alpha_n} < a_n < -\frac{b_n\gamma_n x_{1,n}}{\alpha_n}$, $R_n(c) \geqslant 0$, $R_n(-c) > 0$ *(n odd) and* $R_n(-c) < 0$ *(n even);*

5. $b_n \geqslant \alpha_n$, $a_n < -\frac{b_n\gamma_n x_{1,n}}{\alpha_n}$ *and* $R_n(c) < 0$;

6. $b_n \geqslant \alpha_n$, $a_n > \frac{b_n\gamma_n x_{1,n}}{\alpha_n}$, $R_n(-c) > 0$ *(n odd) and* $R_n(-c) < 0$ *(n even);*

7. $b_n > \alpha_n$, $a_n > -\frac{b_n\gamma_n x_{1,n}}{\alpha_n}$ *and* $R_n(c) < 0$;

8. $b_n > \alpha_n$, $a_n < \frac{b_n\gamma_n x_{1,n}}{\alpha_n}$, $R_n(-c) > 0$ *(n odd) and* $R_n(-c) < 0$ *(n even).*

COROLLARY 3. *If $b_n < 0$, $\frac{b_n\gamma_n x_{1,n}}{\alpha_n} < a_n < -\frac{b_n\gamma_n x_{1,n}}{\alpha_n}$, $R_n(c) < 0$, $R_n(-c) < 0$ (n even) and $R_n(-c) > 0$ (n odd), $R_n(x)$ has exactly two zeros outside the interval $[-c, c]$.*

3.2. Numerical examples

In this section we illustrate the behaviour of the zeros of polynomial $R_4(x) = Q_4(x) + a_4Q_3(x) + b_4Q_2(x)$, considering two families of orthogonal polynomials: Chebyshev polynomials of the first kind and Laguerre polynomials.

EXAMPLE 1. Let $\{T_n(x)\}_{n\geqslant 0}$ be the sequence of Chebyshev polynomials of the first kind. They are orthogonal with respect to the weight function $w(x) = \dfrac{1}{\sqrt{1-x^2}}$ on $[-1,1]$. From the recurrence formula $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$ ($n \geqslant 2$, $T_0(x) = 1$ and $T_1(x) = x$) it follows that $\gamma_n = 2$, $\beta_n = 0$ and $\alpha_n = 1$ for each $n \geqslant 2$.

The following figures display the behaviour of the zeros of polynomial $R_4(x) = T_4(x) + a_4T_3(x) + b_4T_2(x)$ for certain values of a_4 and b_4.

Figures 1, 2 and 3 illustrate the case $b_4 < \alpha_4 = 1$. Observe that the interlacing property (12) is satisfied by the zeros of $R_4(x)$. In Figure 1 we consider $a_4 = -0.2$, where the conditions 1 and 4 of Theorem 3 are satisfied, i.e., $y_{1,4} > 1$ and $y_{4,4} < -1$, respectively. In Figure 2, where $a_4 = -0.9$, as $h_4(x_{1,4}) < 0$ and $R_4(1) > 0$, then $y_{1,4} \in (x_{1,4}, 1)$ (item 1 of Theorem 3); and $h_4(x_{4,4}) < 0$, wich implies that $x_{4,4} < y_{4,4} < x_{3,3}$ by item 3 of Theorem 3. In Figure 3 we consider $a_4 = -1.53$ and, as $h_4(x_{3,4}) = 0$, $x_{3,4}$ is zero of $R_4(x)$ and $y_{1,4} > 1$.

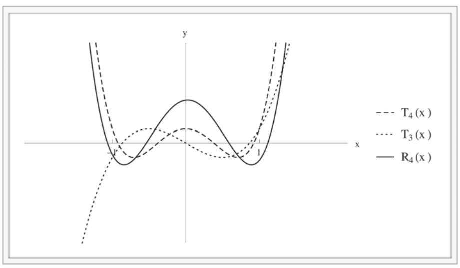

Figure 1: $a_4 = -0.2$ and $b_4 = -2$.

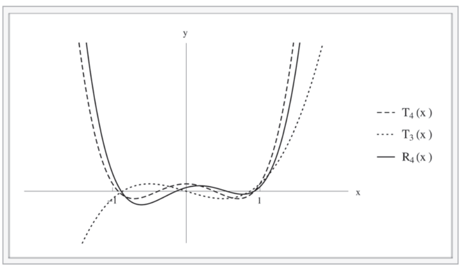

Figure 2: $a_4 = -0.9$ and $b_4 = 0.5$.

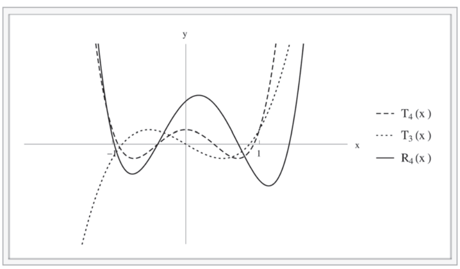

Figure 3: $a_4 = -1.53$ and $b_4 = -2$.

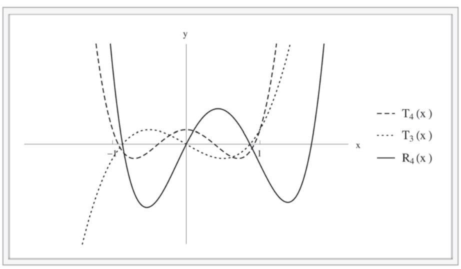

Figure 4: $a_4 = -3.4$ and $b_4 = 1$.

The case $b_4 = \alpha_4 = 1$ is illustrated in Figures 4 and 5. Observe that three zeros of $R_4(x)$ coincide with the zeros of $T_3(x)$, according to Theorem 4. In Figure 4, where $a_4 = -3.4$, we have $h_4(x_{1,4}) < 0$ and $R_4(1) < 0$, which implies that $y_{1,4} > 1$ (item 1 of Theorem 4). In Figure 5, $a_4 = -2x_{1,3} = -1.73$ and, consequently, $x_{1,3}$ is a double zero of $R_4(x)$.

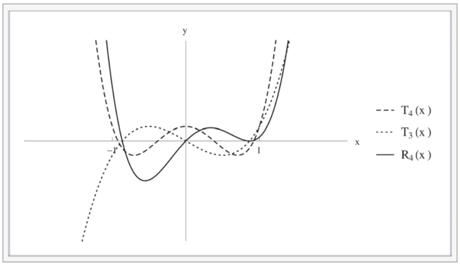

Figure 5: $a_4 = -1.73$ and $b_4 = 1$.

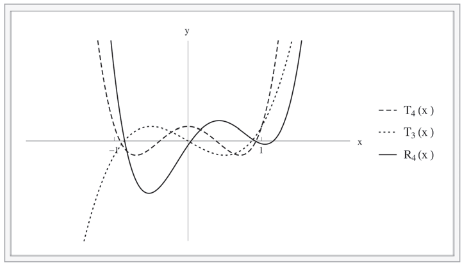

Figure 6: $a_4 = -2.5$ and $b_4 = 1.3$.

Figure 6 illustrates the case $a_4 = -2.5$ and $b_4 = 1.3 > \alpha_4$. Observe that the zeros of $R_4(x)$ satisfy the interlacing property of Theorem 5. As $h_4(x_{1,4}) < 0$ and $R_4(1) < 0$, by item 1 of Theorem 5 it follows that $R_4(x)$ has one real zero in the interval $(x_{1,3}, x_{1,4})$ and the other in $(1, \infty)$.

EXAMPLE 2. Let $\{L_n(x)\}_{n \geqslant 0}$ be the sequence of Laguerre polynomials. They are orthogonal with respect to the weight function $w(x) = e^{-x}$ on $[0, \infty)$.

Laguerre polynomials are generated by the following three-term recurrence relation

$$nL_n(x) = (2n - 1 - x)L_{n-1}(x) - (n-1)L_{n-2}(x), n \geqslant 2,$$

with initial conditions $L_0(x) = 1$ and $L_1(x) = -x + 1$. From this formula, for n odd, the leading coefficient of $L_n(x)$ is a negative number. In the hypothesis of Theorems 3, 4 and 5, we consider that the leading coefficients of all polynomials $Q_n(x)$ are positive. So, it is necessary to consider, for n odd, the polynomial $(-1)L_n(x)$ instead of $L_n(x)$. Then, we obtain $\gamma_n = \frac{1}{n}$, $\beta_n = \frac{2n-1}{n}$ and $\alpha_n = \frac{n-1}{n}$.

In the following pictures we present the behaviour of the zeros of polynomial $R_4(x) = L_4(x) - a_4 L_3(x) + b_4 L_2(x)$ for certain values of a_4 and b_4, illustrating some situations described in Theorems 3, 4 and 5.

In Figures 7 and 8 we consider $b_4 = -1 < \alpha_4$. Observe that the interlacing property (12) is satisfied by the zeros of $R_4(x)$. In Figure 7, $a_4 = 1$ and $x_{1,3} < y_{1,4} < x_{1,4}$ because $h_4(x_{1,4}) > 0$ (item 2 of Theorem 3); from item 4 of Theorem 3, as $h_4(x_{4,4}) > 0$ and $R_n(0) < 0$, it follows that $y_{4,4} \in (-\infty, 0)$. In Figure 8, where $a_4 = 0$, as $h_4(x_{1,4}) < 0$, then $y_{1,4} > x_{1,4}$ (item 1 of Theorem 3); indeed, as $h_4(x_{4,4}) > 0$, by item 4 of Theorem 3 it follows that $y_{4,4} < x_{4,4}$. In this case, $R_4(0) = 0$ and then $y_{4,4} \in [0, x_{4,4})$.

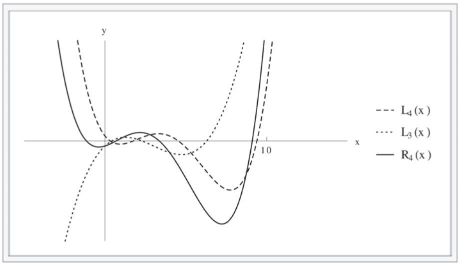

Figure 7: $a_4 = 1$ and $b_4 = -1$.

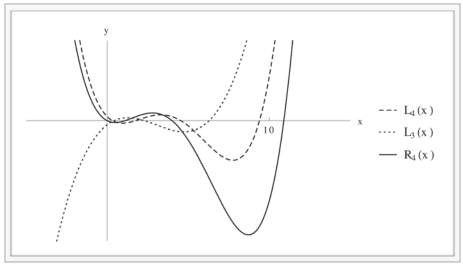

Figure 8: $a_4 = 0$ and $b_4 = -1$.

In Figures 9 and 10, where $b_4 = \alpha_4 = 0.75$, observe that the zeros of $R_4(x)$ coincide with the zeros of $L_3(x)$, according to Theorem 4; in Figure 9, $a_4 = 2.2$ and $h_4(x_{4,4}) > 0$, which implies that $y_{4,4} \in (-\infty, x_{4,4})$ (item 2 of Theorem 4); in Figure 10, $a_4 = 1.176$ and $x_{2,3}$ is a double zero of $R_4(x)$.

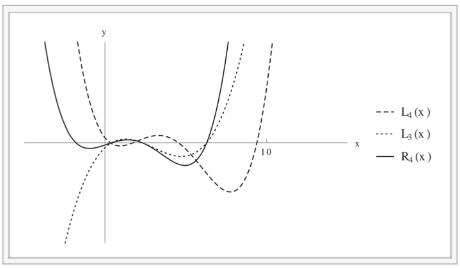

Figure 9: $a_4 = 2.2$, $b_4 = 0.75$.

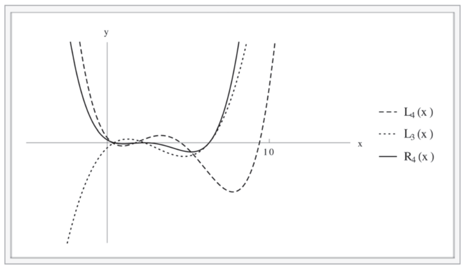

Figure 10: $a_4 = 1.176$ and $b_4 = 0.75$.

In Figures 11 and 12, $b_4 > \alpha_4 = 0.75$; observe that the zeros of $R_4(x)$ satisfy the interlacing property of Theorem 5. In Figure 11, where $a_4 = -1.7$, as $h_4(x_{1,4}) < 0$, by item 1 of Theorem 5 it follows that $R_4(x)$ has one real zero in the interval $(x_{1,3}, x_{1,4})$ and other in $(x_{1,4}, \infty)$. In Figure 12, $a_4 = -0.7983$, $h_4(x_{1,4}) = 0$ and $x_{1,4}$ is zero of $R_4(x)$.

EXAMPLE 3. The results on monotonicity of Theorem 6 we illustrate considering the following polynomials

$$R_7(x) = T_7(x) + a_7 T_6(x) + b_7 T_5(x) \text{ (Chebyshev case) and}$$
$$R_7(x) = -L_7(x) + a_7 L_6(x) - b_7 L_5(x) \text{ (Laguerre case)} .$$

To plot the graph in Figures 13 and 14 we consider, on the horizontal axis, $a_7 \in [0, 2]$ and, on the vertical axis, the values of the real zero $y_{1,7}$ of $R_7(x)$ (Chebyshev case, Figure 13) and, in the Laguerre case, the values of the zero $y_{6,7}$ of $R_7(x)$ (Figure 14). In both cases we consider four values of b_7 to show that the real zeros are decreasing

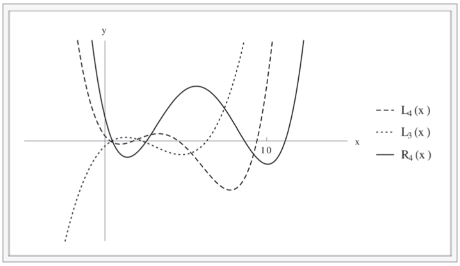

Figure 11: $a_4 = -1.7$ and $b_4 = 2$.

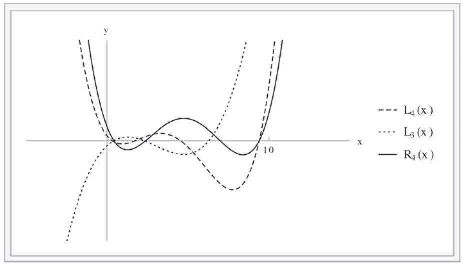

Figure 12: $a_4 = -0.7983$ and $b_4 = 1$.

functions of a_7: $b_7 = -0.25$ (line with points), $b_7 = 0$ (line with squares), $b_7 = 0.25$ (line with diamonds) and $b_7 = 0.5$ (line with triangles). Furthermore, in Tables 1 and 2 we present the values of $y_{1,7}$ and $y_{6,7}$, respectively, for certain values of a_7 and b_7.

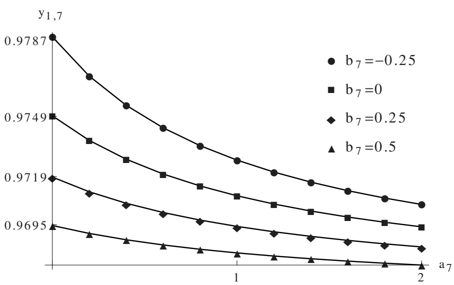

Figure 13: Representation of $y_{1,7}$ as a decreasing funtion of a_7, in Chebyshev case.

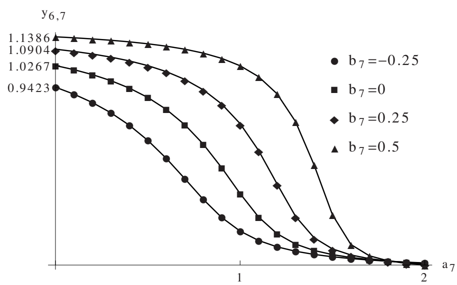

Figure 14: Representation of $y_{6,7}$ as a decreasing funtion of a_7, in Laguerre case.

Table 1: $y_{1,7}$ for certain values of a_7 and b_7, in Chebyshev case.

a_7 \ b_7	-0.25	0	0.25	0.5
0	0.9787	0.9749	0.9719	0.9695
0.4	0.9754	0.9728	0.9706	0.9687
0.8	0.9734	0.9714	0.9697	0.9683
1.2	0.9721	0.9705	0.9691	0.9679
1.6	0.9712	0.9699	0.9687	0.9677
2.0	0.9705	0.9694	0.9684	0.9675

Table 2: $y_{6,7}$ for certain values of a_7 and b_7, in Laguerre case.

a_7 \ b_7	-0.25	0	0.25	0.5
0	0.9423	1.0267	1.0904	1.1386
0.4	0.7941	0.9388	1.0448	1.1190
0.8	0.5056	0.7209	0.9323	1.0752
1.2	0.3202	0.3715	0.5655	0.9167
1.6	0.2745	0.2788	0.2885	0.3293
2.0	0.2575	0.2560	0.2537	0.2492

4. Applications: some classes of real self–reciprocal polynomials

In this section we present some results related to the location and monotonicity of the zeros of the following real self–reciprocal polynomials

$$P_T(z) = z^{2n} + a_{n_T} z^{2n-1} + b_{n_T} z^{2n-2} + b_{n_T} z^2 + a_{n_T} z + 1 \tag{14}$$

$$\begin{aligned} P_U(z) = {} & 2[z^{2n} + a_{n_U}(z^{2n-1} + z^{2n-3} + \ldots + z) + (b_{n_U}+1) \\ & (z^{2n-2} + z^{2n-4} \ldots + z^2) + 1] \end{aligned} \tag{15}$$

$$\begin{aligned} P_V(z) = {} & 2[z^{2n} + (a_{n_V}-1)z^{2n-1} + (-a_{n_V} + b_{n_V} + 1) \\ & (z^{2n-2} - z^{2n-3} + \ldots + (-1)^n z^n + \ldots + z^2) + (a_{n_V}-1)z + 1] \end{aligned} \tag{16}$$

$$\begin{aligned} P_W(z) = {} & 2[z^{2n} + (a_{n_W}+1)z^{2n-1} + (a_{n_W} + b_{n_W} + 1)(z^{2n-2} + \ldots + z^2) + \\ & (a_{n_W}+1)z + 1]. \end{aligned} \tag{17}$$

PROPOSITION 1. *The polynomials $P_\delta(z)$, $\delta = T,U,V,W$, represented by (14), (15), (16) and (17), respectively, are generated by the Chebyshev polynomials of the first, second, third and fourth kinds, T_n, U_n, V_n and W_n, respectively.*

Proof. Let $C_n^{(\delta)}(x) = Q_n^{(\delta)}(x) + a_{n_\delta} Q_{n-1}^{(\delta)}(x) + b_{n_\delta} Q_{n-2}^{(\delta)}(x)$ be a polynomial of degree n, where $\delta = T,U,V,W$ represents each family of Chebyshev polynomials, i.e.,

$$C_n^{(T)}(x) = T_n(x) + a_{n_T} T_{n-1}(x) + b_{n_T} T_{n-2}(x), \tag{18}$$

$$C_n^{(U)}(x) = U_n(x) + a_{n_U} U_{n-1}(x) + b_{n_U} U_{n-2}(x), \tag{19}$$

$$C_n^{(V)}(x) = V_n(x) + a_{n_V} V_{n-1}(x) + b_{n_V} V_{n-2}(x), \tag{20}$$

$$C_n^{(W)}(x) = W_n(x) + a_{n_W} W_{n-1}(x) + b_{n_W} W_{n-2}(x). \tag{21}$$

From the known relations

$$\begin{aligned} U_n(x) - U_{n-2}(x) &= 2T_n(x) \\ V_n(x) + V_{n-1}(x) &= 2T_n(x) \\ W_n(x) - W_{n-1}(x) &= 2T_n(x), n = 2,3,\ldots, \end{aligned}$$

and basic manipulations we can rewrite the equations (19), (20) and (21) in terms of Chebyshev polynomials of the first kind as

$$C_n^{(U)}(x) = \begin{cases} 2[T_n(x) + a_{n_U}(T_{n-1}(x) + T_{n-3}(x) + \ldots + T_1(x)) + \\ (b_{n_U}+1)(T_{n-2}(x) + T_{n-4}(x) + \ldots + (T_0(x))/2], \text{ for } n \text{ even}, \\ 2[T_n(x) + a_{n_U}(T_{n-1}(x) + T_{n-3}(x) + \ldots + (T_0(x))/2) + \\ (b_{n_U}+1)(T_{n-2}(x) + T_{n-4}(x) + \ldots + T_1(x)], \text{ for } n \text{ odd}, \end{cases} \tag{22}$$

$$\begin{aligned} C_n^{(V)}(x) &= 2[T_n(x) + (a_{n_V}-1)T_{n-1}(x) + (b_{n_V} - a_{n_V}+1)T_{n-2}(x) \\ &\quad -(b_{n_V} - a_{n_V}+1)T_{n-3}(x) + \ldots + (-1)^{n-2}(b_{n_V} - a_{n_V}+1)T_1(x) \\ &\quad +(-1)^{n-1}(b_{n_V} - a_{n_V}+1)(T_0(x))/2], \end{aligned} \tag{23}$$

$$\begin{aligned} C_n^{(W)}(x) &= 2[T_n(x) + (a_{n_W}+1)T_{n-1}(x) + (a_{n_W} + b_{n_W}+1)(T_{n-2}(x) \\ &\quad +T_{n-3}(x) + \ldots + T_1(x) + (T_0(x))/2]. \end{aligned} \tag{24}$$

Comparing the coefficients of (6) with (18), (22), (23) and (24), respectively, follows the result. □

The location of the zeros of polynomials $C_n^{(T)}(x)$ and $C_n^{(U)}(x)$ is determined by Corollaries 1, 2 and 3, considering $c = 1$, $\alpha_n = 1$ for $n = 2, \ldots$, and $\gamma_n = 2$ for $n = 1, \ldots$. Because of the non-symmetry of Chebyshev polynomials of the third and fourth kind, respectively, Theorems 3, 4 and 5 will be used to show the behaviour of the zeros of polynomials $C_n^{(V)}(x)$ and $C_n^{(W)}(x)$. Furthermore, Table 3 presents the values of $C_n^{(\delta)}(-1)$ and $C_n^{(\delta)}(1)$ for each family of Chebyshev polynomials.

Table 3: $C_n^{(\delta)}(-1)$ and $C_n^{(\delta)}(1)$ for the family of Chebyshev polynomials.

Chebyshev polynomial	$C_n^{(\delta)}(-1)$	$C_n^{(\delta)}(1)$
$T_n(x)$	$(-1)^{n+1}(a_{n_T} - b_{n_T} - 1)$	$a_{n_T} + b_{n_T} + 1$
$U_n(x)$	$(-1)^{n+1}[n(a_{n_U} - 2) - (n-1)(b_{n_U} - 1)]$	$n(a_{n_U} + 2) + (n-1)(b_{n_U} - 1)$
$V_n(x)$	$(-1)^{n+1}[(2n-1)(a_{n_V} - 2) - (2n-3)(b_{n_V} - 1)]$	$a_{n_V} + b_{n_V} + 1$
$W_n(x)$	$(-1)^{n+1}(a_{n_W} - b_{n_W} - 1)$	$(2n-1)(a_{n_W} + 2) + (2n-3)(b_{n_W} - 1)$

As a consequence of Theorems 3, 4 and 5, we present the following lemmas. The notation $x_{k,n}^{(\delta)}$, $k = 1, \ldots, n$, $\delta = T, U, V, W$, denotes the zeros of each family of Chebyshev polynomials of degree n, where $x_{1,n}^{(\delta)} > 0$, and $y_{k,n}^{(\delta)}$ are the zeros of $C_n^{(\delta)}(x)$.

LEMMA 1. *The zeros of $C_n^{(\delta)}(x)$ are located in $[-1, 1]$ if*

1. $b_{n_\delta} < 1$, $a_{n_\delta} < -2b_{n_\delta} x_{n,n}^{(\delta)}$ *and* $C_n^{(\delta)}(1) \geqslant 0$;

2. $0 \leqslant b_{n_\delta} < 1$, $a_{n_\delta} > -2b_{n_\delta}x_{n,n}^{(\delta)}$, $C_n^{(\delta)}(-1) \leqslant 0$ *(n odd) and* $C_n^{(\delta)}(-1) \geqslant 0$ *(n even);*

3. $b_{n_\delta} < 0$, $a_{n_\delta} > -2b_{n_\delta}x_{1,n}^{(\delta)}$, $C_n^{(\delta)}(-1) \leqslant 0$ *(n odd) and* $C_n^{(\delta)}(-1) \geqslant 0$ *(n even);*

4. $b_{n_\delta} < 0$, $-2b_{n_\delta}x_{n,n}^{(\delta)} < a_{n_\delta} < -2b_{n_\delta}x_{1,n}^{(\delta)}$, $C_n^{(\delta)}(1) \geqslant 0$, $C_n^{(\delta)}(-1) \leqslant 0$ *(n odd) and* $C_n^{(\delta)}(-1) \geqslant 0$ *(n even);*

5. $0 < b_{n_\delta} < 1$ *and* $-2b_{n_\delta}x_{1,n}^{(\delta)} < a_{n_\delta} < -2b_{n_\delta}x_{n,n}^{(\delta)}$*;*

6. $b_{n_\delta} \geqslant 1$, $a_{n_\delta} < -2b_{n_\delta}x_{1,n}^{(\delta)}$ *and* $C_n^{(\delta)}(1) \geqslant 0$*;*

7. $b_{n_\delta} \geqslant 1$, $a_{n_\delta} > -2b_{n_\delta}x_{n,n}^{(\delta)}$, $C_n^{(\delta)}(-1) \leqslant 0$ *(n odd) and* $C_n^{(\delta)}(-1) \geqslant 0$ *(n even).*

LEMMA 2. *$C_n^{(\delta)}(x)$ has only one zero outside the interval $[-1,1]$ if*

1. $b_{n_\delta} < 0$, $a_{n_\delta} < -2b_{n_\delta}x_{n,n}^{(\delta)}$ *and* $C_n^{(\delta)}(1) < 0$*;*

2. $0 \leqslant b_{n_\delta} < 1$, $a_{n_\delta} < -2b_{n_\delta}x_{1,n}^{(\delta)}$ *and* $C_n^{(\delta)}(1) < 0$*;*

3. $b_{n_\delta} < 0$, $a_{n_\delta} > -2b_{n_\delta}x_{1,n}^{(\delta)}$, $C_n^{(\delta)}(-1) > 0$ *(n odd) and* $C_n^{(\delta)}(-1) < 0$ *(n even);*

4. $0 \leqslant b_{n_\delta} < 1$, $a_{n_\delta} > -2b_{n_\delta}x_{n,n}^{(\delta)}$, $C_n^{(\delta)}(-1) > 0$ *(n odd) and* $C_n^{(\delta)}(-1) < 0$ *(n even);*

5. $b_{n_\delta} < 0$, $-2b_{n_\delta}x_{n,n}^{(\delta)} < a_{n_\delta} < -2b_{n_\delta}x_{1,n}^{(\delta)}$, $C_n^{(\delta)}(1) < 0$, $C_n^{(\delta)}(-1) \leqslant 0$ *(n odd) and* $C_n^{(\delta)}(-1) \geqslant 0$ *(n even);*

6. $b_{n_\delta} < 0$, $-2b_{n_\delta}x_{n,n}^{(\delta)} < a_{n_\delta} < -2b_{n_\delta}x_{1,n}^{(\delta)}$, $C_n^{(\delta)}(1) \geqslant 0$, $C_n^{(\delta)}(-1) > 0$ *(n odd) and* $C_n^{(\delta)}(-1) < 0$ *(n even);*

7. $b_{n_\delta} \geqslant 1$, $a_{n_\delta} < -2b_{n_\delta}x_{1,n}^{(\delta)}$ *and* $C_n^{(\delta)}(1) < 0$*;*

8. $b_{n_\delta} \geqslant 1$, $a_{n_\delta} > -2b_{n_\delta}x_{n,n}^{(\delta)}$, $C_n^{(\delta)}(-1) > 0$ *(n odd) and* $C_n^{(\delta)}(-1) < 0$ *(n even);*

9. $b_{n_\delta} > 1$, $a_{n_\delta} > -2b_{n_\delta}x_{1,n}^{(\delta)}$ *and* $C_n^{(\delta)}(1) < 0$*;*

10. $b_{n_\delta} > 1$, $a_{n_\delta} < -2b_{n_\delta}x_{n,n}^{(\delta)}$, $C_n^{(\delta)}(-1) > 0$ *(n odd) and* $C_n^{(\delta)}(-1) < 0$ *(n even).*

LEMMA 3. *$C_n^{(\delta)}(x)$ has exactly two zeros outside the interval $[-1,1]$ if $b_n^{(\delta)} < 0$, $-2b_{n_\delta}x_{n,n}^{(\delta)} < a_{n_\delta} < -2b_{n_\delta}x_{1,n}^{(\delta)}$, $C_n^{(\delta)}(1) < 0$, $C_n^{(\delta)}(-1) > 0$ (n odd) and $C_n^{(\delta)}(-1) < 0$ (n even).*

As a consequence of Lemmas 1, 2, 3 and transformation (5), we have the following results.

PROPOSITION 2. *The zeros of $P_\delta(z)$, $\delta = T,U,V,W$, represented by (14), (15), (16) and (17), respectively, are located on the unit circle when*

1. $b_{n_\delta} < 1$, $a_{n_\delta} < -2b_{n_\delta}x_{n,n}^{(\delta)}$ *and* $C_n^{(\delta)}(1) \geqslant 0$;
2. $0 \leqslant b_{n_\delta} < 1$, $a_{n_\delta} > -2b_{n_\delta}x_{n,n}^{(\delta)}$, $C_n^{(\delta)}(-1) \leqslant 0$ *(n odd) and* $C_n^{(\delta)}(-1) \geqslant 0$ *(n even);*
3. $b_{n_\delta} < 0$, $a_{n_\delta} > -2b_{n_\delta}x_{1,n}^{(\delta)}$, $C_n^{(\delta)}(-1) \leqslant 0$ *(n odd) and* $C_n^{(\delta)}(-1) \geqslant 0$ *(n even);*
4. $b_{n_\delta} < 0$, $-2b_{n_\delta}x_{n,n}^{(\delta)} < a_{n_\delta} < -2b_{n_\delta}x_{1,n}^{(\delta)}$, $C_n^{(\delta)}(1) \geqslant 0$, $C_n^{(\delta)}(-1) \leqslant 0$ *(n odd) and* $C_n^{(\delta)}(-1) \geqslant 0$ *(n even);*
5. $0 < b_{n_\delta} < 1$ *and* $-2b_{n_\delta}x_{1,n}^{(\delta)} < a_{n_\delta} < -2b_{n_\delta}x_{n,n}^{(\delta)}$;
6. $b_{n_\delta} \geqslant 1$, $a_{n_\delta} < -2b_{n_\delta}x_{1,n}^{(\delta)}$ *and* $C_n^{(\delta)}(1) \geqslant 0$;
7. $b_{n_\delta} \geqslant 1$, $a_{n_\delta} > -2b_{n_\delta}x_{n,n}^{(\delta)}$, $C_n^{(\delta)}(-1) \leqslant 0$ *(n odd) and* $C_n^{(\delta)}(-1) \geqslant 0$ *(n even).*

Moreover, their zeros are represented by $z_{j,2n}^{(\delta)} = e^{i\theta_{j,n}^{(\delta)}}$, *with* $\theta_{j,n}^{(\delta)} = \arccos(y_{j,n}^{(\delta)})$, $\theta_{j,n}^{(\delta)} \in [0,\pi]$, $j = 1,\ldots,n$, *where*

$$0 \leqslant \theta_{1,n}^{(\delta)} < \theta_{2,n}^{(\delta)} < \ldots < \theta_{n,n}^{(\delta)} \leqslant \pi. \tag{25}$$

The remaining zeros are the complex conjugate of $z_{j,2n}^{(\delta)}$, *i.e.,* $z_{2n+1-j,2n}^{(\delta)} = \overline{z_{j,2n}^{(\delta)}}$, $j = 1,\ldots,n$. *Furthermore, if* $b_{n_\delta} < 1$, $\theta_{j,n}^{(\delta)}$ *are increasing functions of* a_{n_δ}.

Proof. The first part of the proof follows directly from Lemma 1 and transformation (5).

Under the hypothesis of proposition, from Theorems 3, 4 and 5 it is easy to see that all the zeros of $C_n^{(\delta)}(x)$ are simple and we can represent them by

$$y_{n,n}^{(\delta)} < y_{n-1,n}^{(\delta)} < \ldots < y_{2,n}^{(\delta)} < y_{1,n}^{(\delta)}.$$

From (7) and using the mapping (5), half the zeros $z_{j,2n}^{(\delta)}$ of the polynomials $P_\delta(z)$, for $\delta = T,U,V,W$, respectively, are represented by $z_{j,2n}^{(\delta)} = e^{i\theta_{j,n}^{(\delta)}}$, with $\theta_{j,n}^{(\delta)} = \arccos(y_{j,n}^{(\delta)})$, $\theta_{j,n}^{(\delta)} \in [0,\pi]$, where $y_{j,n}^{(\delta)}$, $j = 1,\ldots,n$, are the zeros of $C_n^{(\delta)}(x)$. The remaining zeros are the complex conjugate of $z_{j,2n}^{(\delta)}$.

Since $y_{n,n}^{(\delta)} < y_{n-1,n}^{(\delta)} < \ldots < y_{2,n}^{(\delta)} < y_{1,n}^{(\delta)}$ and $\theta_{j,n}^{(\delta)} = \arccos(y_{j,n}^{(\delta)})$ is a decreasing function in $[-1,1]$, we have

$$0 \leqslant \theta_{1,n}^{(\delta)} < \theta_{2,n}^{(\delta)} < \ldots < \theta_{n,n}^{(\delta)} \leqslant \pi.$$

To prove that $\theta_{j,n}^{(\delta)}$ are increasing functions of a_{n_δ} if $b_{n_\delta} < 1$, we consider $\varepsilon \geqslant 0$, $C_{n,\varepsilon}^{(\delta)}(x) = Q_n^{(\delta)}(x) + (a_{n_\delta}+\varepsilon)Q_{n-1}^{(\delta)}(x) + b_{n_\delta}Q_{n-2}^{(\delta)}(x)$ with n zeros $y_{j,n,\varepsilon}^{(\delta)}$, and the polynomials $P_{\delta,\varepsilon}(z)$ obtained from $P_\delta(z)$ such that $a_{n_\delta,\varepsilon} = a_{n_\delta}+\varepsilon$, whose zeros are represented by $z_{j,2n,\varepsilon}^{(\delta)} = e^{i\theta_{j,n,\varepsilon}^{(\delta)}}$, with $\theta_{j,n,\varepsilon}^{(\delta)} = \arccos(y_{j,n,\varepsilon}^{(\delta)})$, $j = 1,\ldots,n$. It is clear that $C_{n,0}^{(\delta)}(x) = C_n^{(\delta)}(x)$, $y_{j,n,0}^{(\delta)} = y_{j,n}^{(\delta)}$, $P_{\delta,0}(z) = P_\delta(z)$, $a_{n_\delta,0} = a_{n_\delta}$ and $z_{j,2n,0}^{(\delta)} = z_{j,2n}^{(\delta)}$.

For $\varepsilon_r < \varepsilon_s$, from Theorem 6 it follows that $y_{n,j,\varepsilon_r}^{(\delta)} > y_{n,j,\varepsilon_s}^{(\delta)}$. Hence, since $\theta_{j,n}^{(\delta)} = \arccos(y_{j,n}^{(\delta)})$ is a decreasing function in $[-1,1]$, we have $\theta_{j,n,\varepsilon_r}^{(\delta)} < \theta_{j,n,\varepsilon_s}^{(\delta)}$. Then, for $a_{n_\delta,\varepsilon_1} < a_{n_\delta,\varepsilon_2}$, $\theta_{j,n,\varepsilon_1}^{(\delta)} < \theta_{j,n,\varepsilon_2}^{(\delta)}$. $\square$

PROPOSITION 3. *The polynomials $P_\delta(z)$, $\delta = T,U,V,W$, represented by (14), (15), (16) and (17), respectively, have only two zeros outside the unit circle when*

1. $b_{n_\delta} < 0$, $a_{n_\delta} < -2b_{n_\delta}x_{n,n}^{(\delta)}$ *and* $C_n^{(\delta)}(1) < 0$;
2. $0 \leqslant b_{n_\delta} < 1$, $a_{n_\delta} < -2b_{n_\delta}x_{1,n}^{(\delta)}$ *and* $C_n^{(\delta)}(1) < 0$;
3. $b_{n_\delta} < 0$, $a_{n_\delta} > -2b_{n_\delta}x_{1,n}^{(\delta)}$, $C_n^{(\delta)}(-1) > 0$ *(n odd) and* $C_n^{(\delta)}(-1) < 0$ *(n even)*;
4. $0 \leqslant b_{n_\delta} < 1$, $a_{n_\delta} > -2b_{n_\delta}x_{n,n}^{(\delta)}$, $C_n^{(\delta)}(-1) > 0$ *(n odd) and* $C_n^{(\delta)}(-1) < 0$ *(n even)*;
5. $b_{n_\delta} < 0$, $-2b_{n_\delta}x_{n,n}^{(\delta)} < a_{n_\delta} < -2b_{n_\delta}x_{1,n}^{(\delta)}$, $C_n^{(\delta)}(1) < 0$, $C_n^{(\delta)}(-1) \leqslant 0$ *(n odd) and* $C_n^{(\delta)}(-1) \geqslant 0$ *(n even)*;
6. $b_{n_\delta} < 0$, $-2b_{n_\delta}x_{n,n}^{(\delta)} < a_{n_\delta} < -2b_{n_\delta}x_{1,n}^{(\delta)}$, $C_n^{(\delta)}(1) \geqslant 0$, $C_n^{(\delta)}(-1) > 0$ *(n odd) and* $C_n^{(\delta)}(-1) < 0$ *(n even)*;
7. $b_{n_\delta} \geqslant 1$, $a_{n_\delta} < -2b_{n_\delta}x_{1,n}^{(\delta)}$ *and* $C_n^{(\delta)}(1) < 0$;
8. $b_{n_\delta} \geqslant 1$, $a_{n_\delta} > -2b_{n_\delta}x_{n,n}^{(\delta)}$, $C_n^{(\delta)}(-1) > 0$ *(n odd) and* $C_n^{(\delta)}(-1) < 0$ *(n even)*;
9. $b_{n_\delta} > 1$, $a_{n_\delta} > -2b_{n_\delta}x_{1,n}^{(\delta)}$ *and* $C_n^{(\delta)}(1) < 0$;
10. $b_{n_\delta} > 1$, $a_{n_\delta} < -2b_{n_\delta}x_{n,n}^{(\delta)}$, $C_n^{(\delta)}(-1) > 0$ *(n odd) and* $C_n^{(\delta)}(-1) < 0$ *(n even)*.

In the cases 1, 2, 5, 7 and 9, $P_\delta(z)$ has two positive zeros $z_{k,2n}^{(\delta)} \in (1,\infty)$ and $\frac{1}{z_{k,2n}^{(\delta)}} \in (0,1)$. In the cases 3, 4, 6, 8 and 10, $P_\delta(z)$ has two negative zeros $z_{k,2n}^{(\delta)} \in (-\infty,-1)$ and $\frac{1}{z_{k,2n}^{(\delta)}} \in (-1,0)$.

Moreover, in the cases 1, 2, 5, 7 and 9, the positive zero $z_{k,2n}^{(\delta)}$ *is a decreasing function of* a_{n_δ} *and, consequently,* $\frac{1}{z_{k,2n}^{(\delta)}}$ *is an increasing function of* a_{n_δ}. *In the cases 3, 4, 6, 8 and 10, the negative zero* $z_{k,2n}^{(\delta)}$ *is an increasing function of* a_{n_δ} *and, consequently,* $\frac{1}{z_{k,2n}^{(\delta)}}$ *is a decreasing function of* a_{n_δ}.

Proof. The first part of the proof follows directly from Lemma 2 and transformation (5).

The positive (negative) zeros $z_{k,2n}^{(\delta)}$ and $\frac{1}{z_{k,2n}^{(\delta)}}$ we obtain from the fact that the correspondent zero $y_{i,n}^{(\delta)}$ of $C_n^{(\delta)}(x)$ is positive (negative), using the expression (8), i.e., for $y_{1,n}^{(\delta)} > 1$, we have $z_{k,2n}^{(\delta)} = y_{1,n}^{(\delta)} + \sqrt{(y_{1,n}^{(\delta)})^2 - 1} > 0$ and $z_{l,2n}^{(\delta)} = y_{1,n}^{(\delta)} - \sqrt{(y_{1,n}^{(\delta)})^2 - 1} = \frac{1}{z_{k,2n}^{(\delta)}} > 0$. For $y_{n,n}^{(\delta)} < -1$, we have $z_{k,2n}^{(\delta)} = y_{n,n}^{(\delta)} - \sqrt{(y_{n,n}^{(\delta)})^2 - 1} < 0$ and $z_{l,2n}^{(\delta)} = y_{n,n}^{(\delta)} + \sqrt{(y_{1,n}^{(\delta)})^2 - 1} = \frac{1}{z_{k,2n}^{(\delta)}} < 0$.

To prove the monotonicity of the positive zeros of $P_\delta(z)$ such that the conditions 1, 2, 5, 7 and 9 are satisfied, we consider $a_{n_\delta,\varepsilon} = a_{n_\delta} + \varepsilon$, $\varepsilon \geqslant 0$, such that $a_{n_\delta,\varepsilon}$ satisfy the conditions 1, 2, 5, 7 and 9 (to guarantee the existence of two positive zeros) and

$$
\begin{aligned}
P_{T,\varepsilon}(z) &= P_T(z) + \varepsilon(z^{2n-1} + z)\\
P_{U,\varepsilon}(z) &= P_U(z) + 2\varepsilon(z^{2n-1} + z^{2n-3} + \ldots + z)\\
P_{V,\varepsilon}(z) &= P_V(z) + 2\varepsilon(z^{2n-1} - z^{2n-2} + \ldots - (-1)^n z^n + \ldots - z^2 + z)\\
P_{W,\varepsilon}(z) &= P_W(z) + 2\varepsilon(z^{2n-1} + z^{2n-2} + \ldots + z^2 + z)
\end{aligned}
$$

with its real zeros represented by $z_{k,2n}^{(\delta)}(\varepsilon)$ and $\frac{1}{z_{k,2n}^{(\delta)}(\varepsilon)}$.

Observe that $z_{k,2n}^{(\delta)} = z_{k,2n}^{(\delta)}(0)$ and

$$
\begin{aligned}
P_{T,\varepsilon}(z_{k,2n}^{(\delta)}) &= \varepsilon((z_{k,2n}^{(\delta)})^{2n-1} + z_{k,2n}^{(\delta)})\\
P_{U,\varepsilon}(z_{k,2n}^{(\delta)}) &= 2\varepsilon((z_{k,2n}^{(\delta)})^{2n-1} + (z_{k,2n}^{(\delta)})^{2n-3} + \ldots + z_{k,2n}^{(\delta)})\\
P_{V,\varepsilon}(z_{k,2n}^{(\delta)}) &= 2\varepsilon((z_{k,2n}^{(\delta)})^{2n-1} - (z_{k,2n}^{(\delta)})^{2n-2} + \ldots - (-1)^n(z_{k,2n}^{(\delta)})^n + \ldots - (z_{k,2n}^{(\delta)})^2 + z_{k,2n}^{(\delta)})\\
P_{W,\varepsilon}(z_{k,2n}^{(\delta)}) &= 2\varepsilon((z_{k,2n}^{(\delta)})^{2n-1} + (z_{k,2n}^{(\delta)})^{2n-2} + \ldots + (z_{k,2n}^{(\delta)})^2 + z_{k,2n}^{(\delta)}).
\end{aligned}
$$

Then, for $\varepsilon > 0$, $sign(P_{\delta,\varepsilon}(z_{k,2n}^{(\delta)})) = 1$.

Hence, $z_{k,2n}^{(\delta)}(0) > z_{k,2n}^{(\delta)}(\varepsilon)$, showing that $z_{k,2n}^{(\delta)}$ is a decreasing function of a_{n_δ}. Consequently, $\frac{1}{z_{k,2n}^{(\delta)}(\varepsilon)}$ is an increasing function of a_{n_δ}.

Using the same arguments, it is easy to prove the monotonicity of the negative zeros of $P_\delta(z)$ such that the conditions 3, 4, 6, 8 and 10 are satisfied. In these cases, for $\varepsilon > 0$, $sign(P_{\delta,\varepsilon}(z^{(\delta)}_{k,2n})) = -1$. □

PROPOSITION 4. *If $b_{n_\delta} < 0$, $-2b_{n_\delta}x^{(\delta)}_{n,n} < a_{n_\delta} < -2b_{n_\delta}x^{(\delta)}_{1,n}$, $C^{(\delta)}_n(1) < 0$, $C^{(\delta)}_n(-1) > 0$ (n odd) and $C^{(\delta)}_n(-1) < 0$ (n even), $P_\delta(z)$, $\delta = T,U,V,W$, represented by (14), (15), (16) and (17), respectively, have exactly four zeros outside the unit circle: two positive zeros $z^{(\delta)}_{k,2n} \in (1,\infty)$ and $\frac{1}{z^{(\delta)}_{k,2n}} \in (0,1)$ and two negative zeros $z^{(\delta)}_{k,2n} \in (-\infty,-1)$ and $\frac{1}{z^{(\delta)}_{k,2n}} \in (-1,0)$. Moreover, the positive zero $z^{(\delta)}_{k,2n}$ is a decreasing function of a_{n_δ} and, consequently, $\frac{1}{z^{(\delta)}_{k,2n}}$ is an increasing function of a_{n_δ}. The negative zero $z^{(\delta)}_{k,2n}$ is an increasing function of a_{n_δ} and, consequently, $\frac{1}{z^{(\delta)}_{k,2n}}$ is a decreasing function of a_{n_δ}.*

Proof. The proof follows directly from Lemma 3, transformation (5) and (8), using the same arguments of the proof of Proposition 3. □

PROPOSITION 5. *Related to the zeros of $P_\delta(z)$, $\delta = T,U,V,W$, with multiplicity, we have the following situations:*

1. *If $b_{n_\delta} = 1$ and $a_{n_\delta} = -2x^{(\delta)}_{j,n-1}$ for a fixed j, $j = 1,\ldots,n$, $z^{(\delta)}_{j,2n} = x^{(\delta)}_{j,n-1} + \sqrt{1-x^{(\delta)}_{j,n-1}}\,i$ is a double zero of $P_\delta(z)$ ($z^{(\delta)}_{2n+1-j,2n} = \overline{z^{(\delta)}_{j,2n}} = x^{(\delta)}_{j,n-1} - \sqrt{1-x^{(\delta)}_{j,n-1}}\,i$ is a double zero too).*
2. *If $C^{(\delta)}_n(\pm 1) = 0$, $z^{(\delta)}_{k,2n} = \pm 1$ is a double zero of $P_\delta(z)$.*

Proof. The proof of item 1 follows directly from Remark 1. The proof of item 3 follows directly from (8). □

4.1. Numerical examples

To clarify the results of Section 4, in this subsection we present some examples showing the behaviour of the zeros of polynomials $P_\delta(z)$, $\delta = T,U,V,W$, represented by (14), (15), (16) and (17), respectively. We consider $n = 5$ (consequently, the polynomials $P_\delta(z)$ have degree ten) and certain values of $a_{5\delta}$ and $b_{5\delta}$. The zeros are represented by black points.

In Figure 15, the small points represent the zeros of polynomial $P_U(z)$, where $a_{5U} = -1.55$ and $b_{5U} = 0.5$. The big points represent the zeros of polynomial $P_U(z)$ for $a_{5U} = 0.8$ and $b_{5U} = 0.5$. In both cases, $P_U(z)$ has all zeros on the unit circle, where the first condition of Proposition 2 is satisfied. Observe that relations (25) of Proposition 2 are satisfied. Furthermore, as $b_{5U} < 1$, $\theta_{j,5}^{(U)}$ are increasing functions of a_{5U}.

In Figure 16 we consider $a_{5T} = -3.1$ and $b_{5T} = 2$. The polynomial $P_T(z)$ has two positive zeros outside the unit circle ($z_{1,10}^{(T)} = 2.1786$ and $1/z_{1,10}^{(T)} = 0.459$). In this case, the condition 9 of Proposition 3 is satisfied. The fact that $z_{1,10}^{(T)}$ is a decreasing function of a_{5T} (consequently, $1/z_{1,10}^{(T)}$ is an increasing function of a_{5T}) can be observed in Table 4, where we give the values of $z_{1,10}^{(T)}$ and $1/z_{1,10}^{(T)}$ for $b_{5T} = 2$ and certain values of a_{5T}.

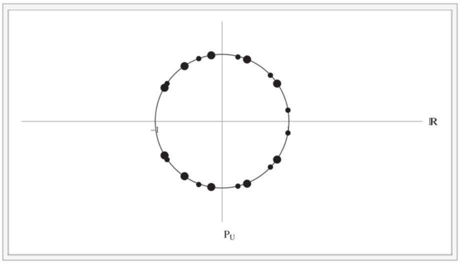

Figure 15: Zeros of $P_U(z)$ for $b_{5U} = 0.5$ and $a_{5U} = -1.55$ (black points) and $a_{5U} = 0.8$ (red points).

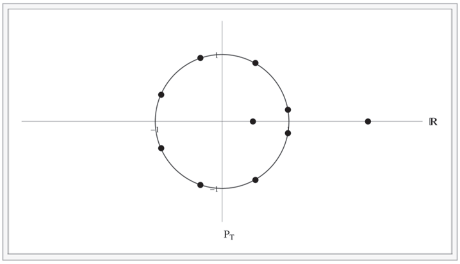

Figure 16: Zeros of $P_T(z)$ for $a_{5T} = -3.1$ and $b_{5T} = 2$.

Table 4: $z_{1,10}^{(T)}$ and $1/z_{1,10}^{(T)}$ for $b_{5T} = 2$ and certain values of a_{5T}.

a_{5T}	$z_{1,10}^{(T)}$	$1/z_{1,10}^{(T)}$
-3.1	2.1786	0.459
-3.3	2.498	0.4003
-3.5	2.7799	0.3597
-3.7	3.0422	0.3287

In Figure 17 we have $a_{5W} = 2$ and $b_{5W} = 0$. Observe that $P_W(z)$ has two negative zeros outside the unit circle: $z_{6,10}^{(W)} = -2.0029$ and $1/z_{6,10}^{(W)} = -0.4993$. In this case, the item 4 of Proposition 3 is satisfied. In Table 5 we can see the values of $z_{6,10}^{(W)}$ and $1/z_{6,10}^{(W)}$ for $b_{5W} = 0$ and certain values of a_{5W}, showing that $z_{6,10}^{(W)}$ is an increasing function of a_{5W} and $1/z_{6,10}^{(W)}$ is a decreasing function of a_{5W}.

In Figure 18, we represent the zeros of polynomial $P_U(z)$, where $a_{5U} = 0$ and $b_{5U} = -3$. The polynomial P_U has exactly four zeros outside the unit circle, according to Proposition 4 (two positive zeros $z_{1,10}^{(U)} = 1.7221$ and $1/z_{1,10}^{(U)} = 0.5807$ and two negative zeros $z_{7,10}^{(U)} = -1.7221$ and $1/z_{7,10}^{(U)} = -0.5807$). In Table 6 we can see the values of the zeros $z_{1,10}^{(U)}$, $1/z_{1,10}^{(U)}$, $z_{7,10}^{(U)}$ and $1/z_{7,10}^{(U)}$ for $b_{5U}=-3$ and certain values of a_{5U}.

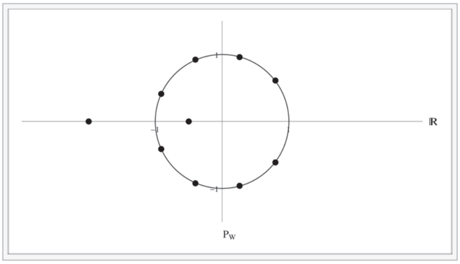

Figure 17: Zeros of $P_W(z)$ for $a_{5W}=2$ and $b_{5W}=0$.

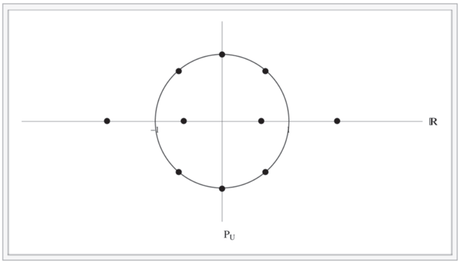

Figure 18: Zeros of $P_U(z)$ for $a_{5U}=0$ and $b_{5U}=-3$.

Table 5: $z_{6,10}^{(W)}$ and $1/z_{6,10}^{(W)}$ for $b_{5W}=0$ and certain values of a_{5W}.

a_{5W}	$z_{6,10}^{(W)}$	$1/z_{6,10}^{(W)}$
2	−2.0029	−0.4993
2.2	−2.2014	−0.4542
2.4	−2.4007	−0.4165
2.6	−2.6004	−0.3845

Table 6: $z_{1,10}^{(U)}$, $1/z_{1,10}^{(U)}$, $z_{7,10}^{(U)}$ and $1/z_{7,10}^{(U)}$ for $b_{5U}=-3$ and certain values of a_{5U}.

a_{5U}	$z_{1,10}^{(U)}$	$1/z_{1,10}^{(U)}$	$z_{7,10}^{(U)}$	$1/z_{7,10}^{(U)}$
0	1.7221	0.5807	−1.7221	−0.5807
0.2	1.6197	0.6174	−1.8284	−0.5469
0.4	1.5202	0.6578	−1.9393	−0.5156
0.6	1.422	0.7032	−2.0551	−0.4866

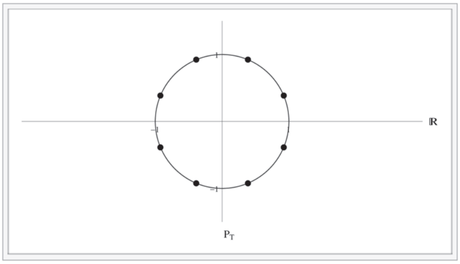

Figure 19: Zeros of $P_T(z)$ for $a_{5T}=-0.76$ and $b_{5T}=1$.

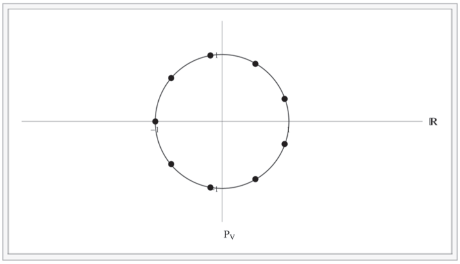

Figure 20: Zeros of $P_V(z)$ for $a_{5V}=2$ and $b_{5V}=1$.

In Figure 19, as $a_{5T}=-2x_{2,4}^{(T)}=-0.76$, $z_{2,10}^{(T)}=0.38+0.78i$ and $z_{8,10}^{(T)}=0.38-$

$0.78i$ are zeros of multiplicity two of $P_T(z)$. In Figure 20, $C_5(-1)^{(V)}(-1)=0$ and, consequently, -1 is zero of multiplicity two of $P_V(z)$.

5. Conclusions and open issues

In this work we presented new results on the location of zeros of polynomials $R_n(x)=Q_n(x)+a_nQ_{n-1}(x)+b_nQ_{n-2}(x)$, where a_n and b_n are real numbers and $\{Q_n(x)\}_{n\geqslant 0}$ is an orthogonal polynomial sequence on $[a,b]$ with respect to a given positive weight function $w(x)$.

For certain values of a_n and b_n (for example, $a_n=0$ or $b_n=0$), similar results of Theorems 3, 4 and 5 can be found in [5] and [9]. The case $a_n=0$ and $b_n=\lambda-1$, $\lambda\in\mathbb{R}$, for $Q_n(x)=U_n(x)$, was analised in [4].

A particular case when $\{Q_n(x)\}_{n\geqslant 0}$ is a symmetric orthogonal polynomial sequence on $[-c,c]$ with respect to a positive and even weight function $w(x)$, was also commented and from the family of Chebyshev polynomials we obtain some classes of real self-reciprocal polynomials. A complete study on the location of the zeros of these classes of real self-reciprocal polynomials is also presented, when the polynomials have even degree.

For real self-reciprocal polynomials of odd degree, as we mentioned before, we can represent by $S(z)=s_{2n+1}(z+1)P(z)$, where $P(z)$ is a real self–reciprocal polynomial of degree $2n$. So, from the polynomials $P_\delta(z)$, $\delta=T,U,V,W$, we have

$$\begin{aligned}
\tilde{P}_T(z) &= (z+1)P_T(z)=z^{2n+1}+(a_{n_T}+1)z^{2n}+(a_{n_T}+b_{n_T})z^{2n-1}+b_{n_T}z^{2n-2}+\\
&\quad b_{n_T}z^3+(a_{n_T}+b_{n_T})z^2+(a_{n_T}+1)z+1\\
\tilde{P}_U(z) &= (z+1)P_U(z)=2[z^{2n+1}+(a_{n_U}+1)z^{2n}+(a_{n_U}+b_{n_U}+1)(z^{2n-1}+\ldots+z^2)+\\
&\quad (a_{n_U}+1)z+1]\\
\tilde{P}_V(z) &= (z+1)P_V(z)=2[z^{2n+1}+a_{n_V}z^{2n}+b_{n_V}z^{2n-1}+b_{n_V}z^2+a_{n_V}z+1]\\
\tilde{P}_W(z) &= (z+1)P_W(z)=2[z^{2n+1}+(a_{n_W}+2)z^{2n}+(2a_{n_W}+b_{n_W}+2)z^{2n-1}+\\
&\quad 2(a_{n_W}+b_{n_W}+1)(z^{2n-2}+\ldots+z^3)+(2a_{n_W}+b_{n_W}+2)z^2+(a_{n_W}+2)z+1].
\end{aligned}$$

The results on the location and monotonicity of the zeros of $\tilde{P}_\delta(z)$, $\delta=T,U,V,W$, we obtain from the results applied to even polynomials.

If $b_n>\alpha_n$, from Theorem 5, $R_n(x)$ has $n-2$ distinct and real zeros which satisfy the interlacing property. Furthermore, if

1. $h_n(x_{1,n})>0$ and $R_n(b)>0$;
2. $h_n(x_{n,n})<0$ and $R_n(a)>0$ (for n even) and $R_n(a)<0$ (for n odd);
3. for a fixed i, $i=1,\ldots,n$, $h_n(x_{i,n})=0$ and $R_n'(x_{i,n})\neq 0$,

we need to investigate the conditions such that the two remaining zeros of $R_n(x)$ are both real or a pair of complex numbers.

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Vanessa Botta
São Paulo State University (UNESP)
School of Technology and Sciences
Department of Mathematics and Computer Science
Presidente Prudente, SP, Brazil
e-mail: vanessa.botta@unesp.br

Mijael Hancco Suni
São Paulo State University (Unesp)
Institute of Biosciences, Humanities and Exact Sciences
Department of Mathematics
São José do Rio Preto, SP, Brazil
e-mail: mijael.h.suni@unesp.br