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IDEAL STATISTICALLY LIMIT POINTS AND IDEAL STATISTICALLY CLUSTER POINTS OF TRIPLE SEQUENCES OF FUZZY NUMBERS

ÖMER KIŞI, VERDA GÜRDAL* AND MUALLA BIRGÜL HUBAN

Abstract. In this paper we extend the notions of ideal statistical limit points and ideal statistical cluster points for a sequence of fuzzy numbers. We introduce the notions ideal statistical limit points and ideal statistical cluster points of a triple sequence of fuzzy numbers, and give some correlation theorem.

1. Introduction

While preparing this article, we inspired from Tripathy et al. [37], where the concepts of ideal statistically limit points and ideal statistically cluster points of sequences of fuzzy numbers were introduced. We will often quote some results from [30] and [37] that can be easily transferred to the concepts of $\mathscr{I}$-statistical limit points and $\mathscr{I}$-statistical cluster points of triple sequences.

The idea of statistical convergence is introduced in [8] and developed in [28]. Some applications of statistical convergence in mathematical analysis can be found in [3, 5, 7, 9, 10, 13, 25, 36]. Kostyrko et al. [19] introduced the concept of $\mathscr{I}$-convergence of sequences in a metric space and studied some properties of this convergence. Note that the concept of $\mathscr{I}$-convergence is a generalization of statistical convergence and it is based on the notion of the ideal $\mathscr{I}$ of subsets of the set $\mathbb{N}$ of positive integers. Das et al. [4] introduced the concept of $\mathscr{I}$-convergence of double sequences in a metric space. In [29], Savaş and Das extended the notion of ideal convergence as introduced by Kostyrko et al. [19] to $\mathscr{I}$-statistical convergence and examined its some basic properties. For an extensive view of this article we refer to [11, 12, 14, 15, 18].

Theory of fuzzy sets was firstly given by Zadeh [38]. Matloka [22] defined the notion of fuzzy sequence and introduced bounded and convergent sequences of fuzzy real numbers and studied their some properties. After then, Nuray and Savaş [26] generalized the concept of ordinary convergence and investigated statistical convergence and statistically Cauchy sequence of fuzzy numbers. Since then, there has been increasing interest in the study of convergence of fuzzy sequences (see [6, 17, 20, 21, 35, 37]).

This paper consists of three sections with the new results in Section 3. In Section 3 the concepts of $\mathscr{I}_3$-statistically convergence, $\mathscr{I}_3$-statistical limit point and $\mathscr{I}_3$-statistical cluster point for fuzzy valued sequences are introduced and studied its fundamental properties.

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* Corresponding author.

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2. Definitions and notations

In this section, we give some basic notions which will be used throughout the paper

We denote by $\mathscr{D}$ the set of all closed and bounded intervals on real line $\mathbb{R}$, i.e., $\mathscr{D} = \left\{A \subset \mathbb{R} : A = \left[\underline{A}, \overline{A}\right]\right\}$. For $A, B \in \mathscr{D}$ we define $A \leqslant B$ iff $\underline{A} \leqslant \underline{B}$ and $\overline{A} \leqslant \overline{B}$ and $\overline{d} = \max\left\{|\underline{A} - \underline{B}|, \left|\overline{A} - \overline{B}\right|\right\}$. $\left(\mathscr{D}, \overline{d}\right)$ forms a complete metric space.

DEFINITION 1. A fuzzy number is a function X from $\mathbb{R}$ to $[0,1]$, which satisfies the following conditions:

(a) X is normal.

(b) X is fuzzy convex.

(c) X is upper semi-continuous.

(d) The closure of the set $\{x \in \mathbb{R} : X(x) > 0\}$ is compact.

The properties (a)–(d) imply that for each $\alpha \in [0,1]$, the α-level set,

$$X^{\alpha} = \{x \in \mathbb{R} : X(x) \geqslant \alpha\} = \left[\underline{X}^{\alpha}, \overline{X}^{\alpha}\right]$$

is a non-empty compact convex subset of $\mathbb{R}$. The 0-level set is the class of the strong 0-cut, i.e., $cl(\{x \in \mathbb{R} : X(x) \geqslant 0\})$. Let $L(\mathbb{R})$ denotes the set of all fuzzy numbers. Define a map $\overline{d}(X,Y) = \sup_{\alpha \in [0,1]} d(X^{\alpha}, Y^{\alpha})$. $\left(L(\mathbb{R}), \overline{d}\right)$ also forms a complete metric space [27].

DEFINITION 2. ([22]) A sequence (X_k) of fuzzy numbers is said to be convergent to a fuzzy number X_0 if for each $\varepsilon > 0$ there exists a positive number m such that $\overline{d}(X_k, X_0) < \varepsilon$ for every $k \geqslant m$. The fuzzy number X_0 is called the ordinary limit of the sequence (X_k) and we write $\lim_{k\to\infty} X_k = X_0$.

DEFINITION 3. ([22]) A fuzzy number X_0 is said to be a limit point of a sequence of fuzzy number (X_k) provided that there is a subsequence of (X_k) that converges to X_0. L_X denotes the set of all limit points of the sequence $X = (X_k)$.

DEFINITION 4. ([26]) A sequence (X_k) of fuzzy numbers is said to be statistical convergent to a fuzzy number X_0 if for each $\varepsilon > 0$ the set

$$A(\varepsilon) = \left\{k \in \mathbb{N} : \overline{d}(X_k, X_0) \geqslant \varepsilon\right\}$$

has natural density zero. The fuzzy number X_0 is called the statistical limit of the sequence (X_k) and we write $St - \lim_{k\to\infty} X_k = X_0$.

DEFINITION 5. ([19]) Take $S \neq \emptyset$. Then $\emptyset \neq \mathscr{I} \subseteq P(S)$ is called to be an *ideal* on S iff (i) $\emptyset \in \mathscr{I}$, (ii) for each $A, B \in \mathscr{I}$ one has $A \cup B \in \mathscr{I}$, (iii) for each $A \in \mathscr{I}$ and $B \subseteq A$ one has $B \in \mathscr{I}$. An ideal $\mathscr{I} \subseteq P(S)$ is called to be *non-trivial* if $\mathscr{I} \neq \emptyset$

and $S \notin \mathscr{I}$. A non-empty family of sets $\mathscr{F} \subseteq P(S)$ is named to be *filter* on S iff (i) $\emptyset \notin \mathscr{F}$, (ii) for each $A,B \in \mathscr{F}$ one has $A \cap B \in \mathscr{F}$, (iii) for each $A \in \mathscr{F}$ and each $B \supseteq A$ one has $B \in \mathscr{F}$. For every ideal $\mathscr{I}$, there is a *filter* $\mathscr{F}(\mathscr{I})$ corresponding to $\mathscr{I}$ i.e. $\mathscr{F}(\mathscr{I}) = \{K \subset S : K^c \in \mathscr{I}\}$, where $K^c = S - K$. We call that $\mathscr{I} \subseteq P(S)$ is (i) an *admissible ideal* on S iff it includes all singletons, i.e., if it includes $\{\{y\} : y \in S\}$.

DEFINITION 6. ([20]) Let $\mathscr{I} \subseteq P(\mathbb{N})$ be a non-trivial ideal on $\mathbb{N}$. A sequence (X_k) of fuzzy numbers is said to be $\mathscr{I}$-convergent to a fuzzy number X_0 if for each $\varepsilon > 0$ the set $A(\varepsilon) = \left\{k \in \mathbb{N} : \overline{d}(X_k, X_0) \geqslant \varepsilon\right\}$ belongs to $\mathscr{I}$. The fuzzy number X_0 is called the $\mathscr{I}$-limit of the sequence (X_k) and we write $\mathscr{I} - \lim_{k\to\infty} X_k = X_0$.

DEFINITION 7. ([21]) A fuzzy number X_0 is said to be $\mathscr{I}$-limit point of a sequence (X_k) of fuzzy numbers provided that there is a subset $K = \{k_1 < k_2 < \ldots\} \subset \mathbb{N}$ such that $K \notin \mathscr{I}$ and $\lim X_{k_n} = X_0$. Let $\mathscr{I}(\Lambda_X)$ denotes the set of all $\mathscr{I}$-limit points of the sequence X.

DEFINITION 8. ([21]) A fuzzy number X_0 is said to be $\mathscr{I}$-cluster point of a sequence (X_k) of fuzzy numbers provided that for each $\varepsilon > 0$ the set

$$\left\{k \in \mathbb{N} : \overline{d}(X_k, X_0) < \varepsilon\right\} \notin \mathscr{I}.$$

Let $\mathscr{I}(\Gamma_X)$ denotes the set of all $\mathscr{I}$-cluster points of the sequence X.

Recently, Mursaleen and Edely [24] presented the idea of statistical convergence for multiple sequences, and there are several papers dealing with the statistical and ideal convergence of double and triple sequences (see literature [1, 11, 15]). Also, the readers should refer to the monographs [2] and [23], and recent papers [32, 33, 34] and [16] for the background on the sequence spaces and on the classical sets of fuzzy valued sequences, and related topics.

The notion of statistically convergent triple sequences was extended to $\mathscr{I}$-convergent triple sequences by Şahiner and Tripathy in [31]. We recall that a subset K of $\mathbb{N}^3$ is said to have natural density $d_3(K)$ if

$$d_3(K) = \lim_{m,n,o\to\infty} \frac{K(m,n,o)}{m.n.o},$$

where $K(m,n,o) = |\{(j,k,l) \in \mathbb{N}^3 : j \leqslant m, k \leqslant n, l \leqslant o\}|$.

Throughout the paper we take $\mathscr{I}_3$ as an admissible ideal in $\mathbb{N}^3$.

Let (X,ρ) be a metric space. A triple sequence $x = (x_{mno})$ in X is said to be $\mathscr{I}_3$-convergent to $L \in X$, if for any $\varepsilon > 0$ we have $A(\varepsilon) = \{(m,n,o) \in \mathbb{N}^3 : \rho(x_{mno}, L) \geqslant \varepsilon\} \in \mathscr{I}_3$. In this case, we say that x is $\mathscr{I}_3$-convergent and we write

$$\mathscr{I}_3 - \lim_{m,n,o\to\infty} x_{mno} = L.$$

3. Main results

Now we study some properties of the set of all $\mathscr{I}_3$-statistical cluster points and the set of all $\mathscr{I}_3$-statistical limit points of a triple sequence of fuzzy numbers including their interrelationship.

THEOREM 1. *If* (X_{jkl}) *be a triple sequence of fuzzy numbers such that* $\mathscr{I}_3 - st\lim X_{jkl} = X_0$, *then* X_0 *identified uniquely.*

Proof. Let a sequence (X_{jkl}) be $\mathscr{I}_3$-statistically convergent to two different fuzzy numbers X_0 and Y_0, i.e., for any $\varepsilon > 0$, $\delta > 0$, we get

$$K_1(m,n,o) = \left\{(m,n,o) \in \mathbb{N}^3 : \frac{1}{mno}\left|\left\{j \leqslant m, k \leqslant n, l \leqslant o : \overline{d}\left(X_{jkl}, X_0\right) \geqslant \varepsilon\right\}\right| < \delta\right\} \in \mathscr{F}(\mathscr{I}_3),$$
$$K_2(m,n,o) = \left\{(m,n,o) \in \mathbb{N}^3 : \frac{1}{mno}\left|\left\{j \leqslant m, k \leqslant n, l \leqslant o : \overline{d}\left(X_{jkl}, Y_0\right) \geqslant \varepsilon\right\}\right| < \delta\right\} \in \mathscr{F}(\mathscr{I}_3).$$

Therefore, $K_1 \cap K_2 \neq \emptyset$, since $K_1 \cap K_2 \in \mathscr{F}(\mathscr{I}_3)$. Let $(p,r,s) \in K_1 \cap K_2$ and take $\varepsilon := \frac{\overline{d}(X_0,Y_0)}{3} > 0$ such that we have

$$\frac{1}{prs}\left|\left\{j \leqslant p, k \leqslant r, l \leqslant s : \overline{d}\left(X_{jkl}, X_0\right) \geqslant \varepsilon\right\}\right| < \delta$$

and it follows that

$$\frac{1}{prs}\left|\left\{j \leqslant p, k \leqslant r, l \leqslant s : \overline{d}\left(X_{jkl}, Y_0\right) \geqslant \varepsilon\right\}\right| < \delta,$$

i.e., for maximum $j \leqslant p$, $k \leqslant r$, $l \leqslant s$ we have $\overline{d}\left(X_{jkl}, X_0\right) < \varepsilon$ and $\overline{d}\left(X_{jkl}, Y_0\right) < \varepsilon$ for a very small $\delta > 0$. Thus, we have to acquire

$$\left\{j \leqslant p, k \leqslant r, l \leqslant s : \overline{d}\left(X_{jkl}, X_0\right) < \varepsilon\right\} \cap \left\{j \leqslant p, k \leqslant r, l \leqslant s : \overline{d}\left(X_{jkl}, Y_0\right) < \varepsilon\right\} \neq \emptyset,$$

a contradiction, as the nbd of X_0 and Y_0 are disjoint. Hence, X_0 is uniquely identified. □

THEOREM 2. *Let* (X_{jkl}) *be a fuzzy valued sequence. Then,* $st_3\lim X_{jkl} = X_0$ *implies* $\mathscr{I}_3 - st\lim X_{jkl} = X_0$.

Proof. Let $st_3\lim X_{jkl} = X_0$. Then, for each $\varepsilon > 0$ the set

$$K(\varepsilon) = \left\{j \leqslant m, k \leqslant n, l \leqslant o : \overline{d}\left(X_{jkl}, X_0\right) \geqslant \varepsilon\right\}$$

has natural density zero, i.e.,

$$\lim_{m,n,o\to\infty} \frac{1}{mno}\left|\left\{j \leqslant m, k \leqslant n, l \leqslant o : \overline{d}\left(X_{jkl}, X_0\right) \geqslant \varepsilon\right\}\right| = 0.$$

Therefore, for every $\varepsilon > 0$ and $\delta > 0$,

$$\left\{(m,n,o) \in \mathbb{N}^3 : \frac{1}{mno} \left|\left\{j \leqslant m, k \leqslant n, l \leqslant o : \overline{d}\left(X_{jkl}, X_0\right) \geqslant \varepsilon\right\}\right| \geqslant \delta\right\}$$

is a finite set and so belongs to $\mathscr{I}_3$, as $\mathscr{I}_3$ is an admissible ideal. Hence, we get $\mathscr{I}_3 - st \lim X_{jkl} = X_0$. □

THEOREM 3. *For any sequence of fuzzy numbers* $\left(X_{jkl}\right)$, $\mathscr{I}_3 - \lim X_{jkl} = X_0$ *implies* $\mathscr{I}_3 - st \lim X_{jkl} = X_0$.

Proof. The proof is obvious. □

But the converse is not true. For example, take $\mathscr{I}_3 = \mathscr{I}_3^f$, the fuzzy valued sequence $\left(X_{jkl}\right)$, where

$$X_{jkl}(p) := \begin{cases} \frac{n+m+o+p}{n+m+o}, & -n-m-o \leqslant p \leqslant 0, \\ \frac{n+m+o-p}{n+m+o} & 0 \leqslant p \leqslant n+m+o \end{cases}$$

for $j = n^2$, $k = m^2, l = o^2$, $n,m,o \in \mathbb{N}$ and

$$X_{jkl}(p) := \begin{cases} 1 + pnmo, & -\frac{1}{nmo} \leqslant p \leqslant 0 \\ 1 - pnmo & 0 \leqslant p \leqslant \frac{1}{nmo} \end{cases}$$

for $j \neq n^2, k \neq m^2, l \neq o^2$, $n,m,o \in \mathbb{N}$. Then, $\left(X_{jkl}\right)$ is $\mathscr{I}_3$-statistically convergent, but not $\mathscr{I}_3$-convergent.

THEOREM 4. *Let* $\left(X_{jkl}\right)$ *and* $\left(Y_{jkl}\right)$ *be two fuzzy valued sequences. Then,*

(i) $\mathscr{I}_3 - st \lim X_{jkl} = X_0$, $c \in \mathbb{R}$ *implies* $\mathscr{I}_3 - st \lim cX_{jkl} = cX_0$.

(ii) $\mathscr{I}_3 - st \lim X_{jkl} = X_0$, $\mathscr{I}_3 - st \lim Y_{jkl} = Y_0$ *implies* $\mathscr{I}_3 - st \lim \left(X_{jkl} + Y_{jkl}\right) = X_0 + Y_0$.

Proof. (i) If $c = 0$, we have nothing to prove. So, suppose that $c \neq 0$. Now

$$\begin{aligned} &\tfrac{1}{mno} \left|\left\{j \leqslant m, k \leqslant n, l \leqslant o : \overline{d}\left(cX_{jkl}, cX_0\right) \geqslant \varepsilon\right\}\right| \\ &\quad = \tfrac{1}{mno} \left|\left\{j \leqslant m, k \leqslant n, l \leqslant o : |c|\overline{d}\left(X_{jkl}, X_0\right) \geqslant \varepsilon\right\}\right| \\ &\quad \leqslant \tfrac{1}{mno} \left|\left\{j \leqslant m, k \leqslant n, l \leqslant o : \overline{d}\left(X_{jkl}, X_0\right) \geqslant \tfrac{\varepsilon}{|c|}\right\}\right| < \delta. \end{aligned}$$

Therefore, we obtain

$$\left\{(m,n,o) \in \mathbb{N}^3 : \frac{1}{mno} \left|\left\{j \leqslant m, k \leqslant n, l \leqslant o : \overline{d}\left(cX_{jkl}, cX_0\right) \geqslant \varepsilon\right\}\right| < \delta\right\} \in \mathscr{F}\left(\mathscr{I}_3\right),$$

i.e., $\mathscr{I}_2 - st \lim cX_{jkl} = cX_0$.

(ii) We have

$$K_1=\left\{(m,n,o)\in\mathbb{N}^3:\frac{1}{mno}\left|\left\{j\leqslant m,k\leqslant n,l\leqslant o:\overline{d}\left(X_{jkl},X_0\right)\geqslant\frac{\varepsilon}{2}\right\}\right|<\frac{\delta}{2}\right\}\in\mathscr{F}\left(\mathscr{I}_3\right),$$

and

$$K_2=\left\{(m,n,o)\in\mathbb{N}^3:\frac{1}{mno}\left|\left\{j\leqslant m,k\leqslant n,l\leqslant o:\overline{d}\left(Y_{jkl},Y_0\right)\geqslant\frac{\varepsilon}{2}\right\}\right|<\frac{\delta}{2}\right\}\in\mathscr{F}\left(\mathscr{I}_3\right).$$

Since, $K_1\cap K_2\neq\emptyset$, therefore for all $(m,n,o)\in K_1\cap K_2$, we get

$$\begin{aligned}&\tfrac{1}{mno}\left|\left\{j\leqslant m,k\leqslant n,l\leqslant o:\overline{d}\left(X_{jkl}+Y_{jkl},X_0+Y_0\right)\geqslant\varepsilon\right\}\right|\\&\quad\leqslant\tfrac{1}{mno}\left|\left\{j\leqslant m,k\leqslant n,l\leqslant o:\overline{d}\left(X_{jkl},X_0\right)\geqslant\tfrac{\varepsilon}{2}\right\}\right|\\&\quad+\tfrac{1}{mno}\left|\left\{j\leqslant m,k\leqslant n,l\leqslant o:\overline{d}\left(Y_{jkl},Y_0\right)\geqslant\tfrac{\varepsilon}{2}\right\}\right|<\delta,\end{aligned}$$

i.e.,

$$\left\{(m,n,o)\in\mathbb{N}^3:\frac{1}{mno}\left|\left\{j\leqslant m,k\leqslant n,l\leqslant o:\overline{d}\left(X_{jkl}+Y_{jkl},X_0+Y_0\right)\geqslant\varepsilon\right\}\right|<\delta\right\}\in\mathscr{F}\left(\mathscr{I}_3\right).$$

Hence, we have $\mathscr{I}_3-st\lim\left(X_{jkl}+Y_{jkl}\right)=X_0+Y_0$. □

DEFINITION 9. An element $X_0\in L(\mathbb{N})$ is said to be an $\mathscr{I}_3$-statistical limit point of a fuzzy valued sequence $X=\left(X_{jkl}\right)$ provided that for each $\varepsilon>0$ there is a set

$$M=\left\{(j_1,k_1,l_1)<(j_2,k_2,l_2)<\ldots<(j_p,k_r,l_s)<\ldots\right\}\subset\mathbb{N}^3$$

such that $M\notin\mathscr{I}_3$ and $st_3-\lim X_{j_p,k_r,l_s}=X_0$.

$\mathscr{I}_3-S\left(\Lambda_X\right)$ indicates the set of all $\mathscr{I}_3$-statistical limit point of a fuzzy valued sequence $\left(X_{jkl}\right)$.

THEOREM 5. *If $\left(X_{jkl}\right)$ is a sequence of fuzzy numbers such that $\mathscr{I}_3-st\lim X_{jkl}=X_0$, then $\mathscr{I}_3-S\left(\Lambda_X\right)=\left\{X_0\right\}$.*

Proof. Since $\left(X_{jkl}\right)$ is $\mathscr{I}_3$-statistically convergent to a fuzzy number X_0, for each $\varepsilon>0$ and $\delta>0$, the set

$$K=\left\{(m,n,o)\in\mathbb{N}^3:\frac{1}{mno}\left|\left\{j\leqslant m,k\leqslant n,l\leqslant o:\overline{d}\left(X_{jkl},X_0\right)\geqslant\varepsilon\right\}\right|\geqslant\delta\right\}\in\mathscr{I}_3,$$

where $\mathscr{I}_3$ is an admissible ideal.

Assume that $\mathscr{I}_3-S\left(\Lambda_X\right)$ includes Y_0 different from X_0, i.e., $Y_0\in\mathscr{I}_3-S\left(\Lambda_X\right)$. So, there is a $M\subset\mathbb{N}^3$ such that $M\notin\mathscr{I}_3$ and $st_3-\lim X_{j_p,k_r,l_s}=Y_0$.

Let

$$P=\left\{(m,n,o)\in M:\frac{1}{mno}\left|\left\{j\leqslant m,k\leqslant n,l\leqslant o:\overline{d}\left(X_{jkl},Y_0\right)\geqslant\varepsilon\right\}\right|\geqslant\delta\right\}.$$

So P is a finite set and therefore $P\in\mathscr{I}_3$. So

$$P^c=\left\{(m,n,o)\in M:\frac{1}{mno}\left|\left\{j\leqslant m,k\leqslant n,l\leqslant o:\overline{d}\left(X_{jkl},Y_0\right)\geqslant\varepsilon\right\}\right|<\delta\right\}\in\mathscr{F}\left(\mathscr{I}_3\right).$$

Again let

$$K_1=\left\{(m,n,o)\in M:\frac{1}{mno}\left|\left\{j\leqslant m,k\leqslant n,l\leqslant o:\overline{d}\left(X_{jkl},X_0\right)\geqslant\varepsilon\right\}\right|\geqslant\delta\right\}.$$

So $K_1\subset K\in\mathscr{I}_3$, i.e., $K_1^c\in\mathscr{F}\left(\mathscr{I}_3\right)$. Therefore, $K_1^c\cap P^c\neq\emptyset$, since $K_1^c\cap P^c\in\mathscr{F}\left(\mathscr{I}_3\right)$.

Let $(u,v,y)\in K_1^c\cap P^c$ and take $\varepsilon:=\frac{\overline{d}(X_0,Y_0)}{3}>0$, so

$$\frac{1}{uvy}\left|\left\{j\leqslant u,k\leqslant v,l\leqslant y:\overline{d}\left(X_{jkl},X_0\right)\geqslant\varepsilon\right\}\right|<\delta \text{ and}$$
$$\frac{1}{uvy}\left|\left\{j\leqslant u,k\leqslant v,l\leqslant y:\overline{d}\left(X_{jkl},Y_0\right)\geqslant\varepsilon\right\}\right|<\delta,$$

i.e., for maximum $j\leqslant u,k\leqslant v,l\leqslant y$ will satisfy $\overline{d}\left(X_{jkl},X_0\right)<\varepsilon$ and $\overline{d}\left(X_{jkl},Y_0\right)<\varepsilon$ for a very small $\delta>0$. Thus, we have to get

$$\left\{j\leqslant u,k\leqslant v,l\leqslant y:\overline{d}\left(X_{jkl},X_0\right)<\varepsilon\right\}\cap\left\{j\leqslant u,k\leqslant v,l\leqslant y:\overline{d}\left(X_{jkl},Y_0\right)<\varepsilon\right\}\neq\emptyset,$$

a contradiction, as the nbd of X_0 and Y_0 are disjoint. Hence $\mathscr{I}_3-S\left(\Lambda_X\right)=\{X_0\}$. □

DEFINITION 10. An element $X_0\in L(\mathbb{R})$ is said to be an $\mathscr{I}_3$-statistical cluster point of a fuzzy valued sequence $X=\left(X_{jkl}\right)$ if for each $\varepsilon>0$ and $\delta>0$, the set

$$\left\{(m,n,o)\in\mathbb{N}^3:\frac{1}{mno}\left|\left\{j\leqslant m,k\leqslant n,l\leqslant o:\overline{d}\left(X_{jkl},X_0\right)\geqslant\varepsilon\right\}\right|<\delta\right\}\notin\mathscr{I}_3.$$

$\mathscr{I}_3-S\left(\Gamma_X\right)$ denotes the set of all $\mathscr{I}_3$-statistical cluster point of a fuzzy valued sequence $\left(X_{jkl}\right)$.

THEOREM 6. *For any sequence $\left(X_{jkl}\right)$ of fuzzy numbers $\mathscr{I}_3-S\left(\Gamma_X\right)$ is closed.*

Proof. Let the fuzzy number Y_0 be a limit point of the set $\mathscr{I}_3-S\left(\Gamma_X\right)$. Then, for any $\varepsilon>0$,

$$\mathscr{I}_3-S\left(\Gamma_X\right)\cap B\left(Y_0,\varepsilon\right)\neq\emptyset,$$

where

$$B\left(Y_0,\varepsilon\right)=\left\{W\in L(\mathbb{R}):\overline{d}\left(W,Y_0\right)<\varepsilon\right\}.$$

Let $Z_0 \in \mathscr{I}_3 - S(\Gamma_X) \cap B(Y_0, \varepsilon)$ and select $\varepsilon_1 > 0$ such that $B(Z_0, \varepsilon_1) \subseteq B(Y_0, \varepsilon)$. Then, we get

$$\left\{ j \leqslant m, k \leqslant n, l \leqslant o : \overline{d}\left(X_{jkl}, Z_0\right) \geqslant \varepsilon_1 \right\} \supseteq \left\{ j \leqslant m, k \leqslant n, l \leqslant o : \overline{d}\left(X_{jkl}, Y_0\right) \geqslant \varepsilon \right\},$$

which gives that

$$\begin{aligned} &\tfrac{1}{mno} \left| \left\{ j \leqslant m, k \leqslant n, l \leqslant o : \overline{d}\left(X_{jkl}, Z_0\right) \geqslant \varepsilon_1 \right\} \right| \\ &\quad \geqslant \tfrac{1}{mno} \left| \left\{ j \leqslant m, k \leqslant n, l \leqslant o : \overline{d}\left(X_{jkl}, Y_0\right) \geqslant \varepsilon \right\} \right|. \end{aligned}$$

Now, for any $\delta > 0$, we get

$$\begin{aligned} &\left\{ (m,n,o) \in \mathbb{N}^3 : \tfrac{1}{mno} \left| \left\{ j \leqslant m, k \leqslant n, l \leqslant o : \overline{d}\left(X_{jkl}, Z_0\right) \geqslant \varepsilon_1 \right\} \right| < \delta \right\} \\ &\quad \subseteq \left\{ (m,n,o) \in \mathbb{N}^3 : \tfrac{1}{mno} \left| \left\{ j \leqslant m, k \leqslant n, l \leqslant o : \overline{d}\left(X_{jkl}, Y_0\right) \geqslant \varepsilon \right\} \right| < \delta \right\}. \end{aligned}$$

Since $Z_0 \in \mathscr{I}_3 - S(\Gamma_X)$, we have

$$\left\{ (m,n,o) \in \mathbb{N}^3 : \frac{1}{mno} \left| \left\{ j \leqslant m, k \leqslant n, l \leqslant o : \overline{d}\left(X_{jkl}, Y_0\right) \geqslant \varepsilon \right\} \right| < \delta \right\} \notin \mathscr{I}_3,$$

i.e., $Y_0 \in \mathscr{I}_3 - S(\Gamma_X)$. This completes the proof. $\square$

THEOREM 7. *For any fuzzy valued sequence* $\left(X_{jkl}\right)$,

$$\mathscr{I}_3 - S(\Lambda_X) \subseteq \mathscr{I}_3 - S(\Gamma_X).$$

Proof. Let $X_0 \in \mathscr{I}_3 - S(\Lambda_X)$. Then, there is a set

$$M = \left\{ (j_1, k_1, l_1) < (j_2, k_2, l_2) < \ldots < (j_p, k_r, l_s) < \ldots \right\} \notin \mathscr{I}_3$$

such that $st_3 - \lim X_{j_p, k_r, l_s} = X_0$. So, we have

$$\lim_{j,k,l \to \infty} \frac{1}{jkl} \left| \left\{ j_p \leqslant j, k_r \leqslant k, l_s \leqslant l : \overline{d}\left(X_{j_p, k_r l_s}, X_0\right) \geqslant \varepsilon \right\} \right| = 0.$$

Take $\delta > 0$, so there exists $n_0 \in \mathbb{N}$ such that for $m, n, o > n_0$ we have

$$\frac{1}{mno} \left| \left\{ j_p \leqslant m, k_r \leqslant n, l_s \leqslant o : \overline{d}\left(X_{j_p, k_r l_s}, X_0\right) \geqslant \varepsilon \right\} \right| < \delta.$$

Let

$$K = \left\{ (m,n,o) \in \mathbb{N}^3 : \frac{1}{mno} \left| \left\{ j_p \leqslant m, k_r \leqslant n, l_s \leqslant o : \overline{d}\left(X_{j_p, k_r l_s}, X_0\right) \geqslant \varepsilon \right\} \right| < \delta \right\}.$$

Also we have

$$K \supset M \setminus \left\{ (j_1, k_1, l_1), \left((j_2, k_2, l_2), \ldots, \left(j_{n_0}, k_{n_0}, l_{n_0}\right)\right) \right\}.$$

Since $\mathscr{I}_3$ is an admissible ideal and $M \notin \mathscr{I}_3$, therefore $K \notin \mathscr{I}_3$. So, by the definition of $\mathscr{I}_3$-statistical cluster point $X_0 \in \mathscr{I}_3 - S(\Gamma_X)$. This completes the proof. $\square$

THEOREM 8. *If* (X_{jkl}) *and* (Y_{jkl}) *are two sequences of fuzzy numbers such that*

$$\left\{(j,k,l)\in\mathbb{N}^3 : X_{jkl}\neq Y_{jkl}\right\}\in\mathscr{I}_3,$$

then

(*i*) $\mathscr{I}_3-S(\Lambda_X)=\mathscr{I}_3-S(\Lambda_Y)$.
(*ii*) $\mathscr{I}_3-S(\Gamma_X)=\mathscr{I}_3-S(\Gamma_Y)$.

Proof. (*i*) Let $X_0\in\mathscr{I}_3-S(\Lambda_X)$. So, by the definition there is a set

$$M=\left\{(j_1,k_1,l_1)<(j_2,k_2,l_2)<\ldots<(j_p,k_r,l_s)<\ldots\right\}\subset\mathbb{N}^3$$

such that $M\notin\mathscr{I}_3$ and $st_3-\lim X_{j_p,k_r,l_s}=X_0$. Since

$$\left\{(j,k,l)\in M : X_{jkl}\neq Y_{jkl}\right\}\subseteq\left\{(j,k,l)\in\mathbb{N}^3 : X_{jkl}\neq Y_{jkl}\right\}\in\mathscr{I}_3,$$

$$M'=\left\{(j,k,l)\in M : X_{jkl}=Y_{jkl}\right\}\notin\mathscr{I}_3 \text{ and } M'\subseteq M.$$

So, we have $st_3-\lim Y_{j'_p,k'_r,l'_s}=X_0$. This denotes that $X_0\in\mathscr{I}_3-S(\Lambda_Y)$ and therefore $\mathscr{I}_3-S(\Lambda_X)\subseteq\mathscr{I}_3-S(\Lambda_Y)$. By symmetry, $\mathscr{I}_3-S(\Lambda_Y)\subseteq\mathscr{I}_3-S(\Lambda_X)$. Hence, we obtain $\mathscr{I}_3-S(\Lambda_X)=\mathscr{I}_3-S(\Lambda_Y)$.

(*ii*) Let $X_0\in\mathscr{I}_3-S(\Gamma_X)$. So, by the definition for each $\varepsilon>0$ we have

$$K=\left\{(m,n,o)\in\mathbb{N}^3 : \frac{1}{mno}\left|\left\{j\leqslant m,k\leqslant n,l\leqslant o : \overline{d}\left(X_{jkl},X_0\right)\geqslant\varepsilon\right\}\right|<\delta\right\}\notin\mathscr{I}_3.$$

Let

$$L=\left\{(m,n,o)\in\mathbb{N}^3 : \frac{1}{mno}\left|\left\{j\leqslant m,k\leqslant n,l\leqslant o : \overline{d}\left(Y_{jkl},X_0\right)\geqslant\varepsilon\right\}\right|<\delta\right\}.$$

We have to prove that $L\notin\mathscr{I}_3$. Assume that $L\in\mathscr{I}_3$. So

$$L^c=\left\{(m,n,o)\in\mathbb{N}^3 : \frac{1}{mno}\left|\left\{j\leqslant m,k\leqslant n,l\leqslant o : \overline{d}\left(Y_{jkl},X_0\right)\geqslant\varepsilon\right\}\right|\geqslant\delta\right\}\in\mathscr{F}(\mathscr{I}_3).$$

By hypothesis,

$$P=\left\{(j,k,l)\in\mathbb{N}^3 : X_{jkl}=Y_{jkl}\right\}\in\mathscr{F}(\mathscr{I}_3).$$

Therefore $L^c\cap P\in\mathscr{F}(\mathscr{I}_3)$. Also, it is clear that $L^c\cap P\subseteq K^c\in\mathscr{F}(\mathscr{I}_3)$, i.e., $K\in\mathscr{I}_3$, which is a contradiction. Hence, $L\notin\mathscr{I}_3$ and thus the result proved. □

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Ömer Kişi
Department of Mathematics
Faculty of Science, Bartın University
Bartın, Turkey
e-mail: okisi@bartin.edu.tr

Verda Gürdal
Department of Mathematics
Suleyman Demirel University
32260, Isparta, Turkey
e-mail: verdagurdal@icloud.com

Mualla Birgül Huban
Isparta University of Applied Sciences
Isparta, Turkey
e-mail: muallahuban@isparta.edu.tr