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(λ,μ)–UNIFORMLY DISTRIBUTED DOUBLE SEQUENCES

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Abstract. In this paper, we introduce and study (λ,μ)-uniformly distributed and (λ,μ)-well distributed double sequences. We also define the concept of A-uniformly distributed double sequence.

1. Introduction and background

The theory of uniform distribution mod 1 is concerned, at least in its classical setting, with the distribution of fractional parts of real numbers in the unit interval $(0,1)$. The development of this theory started with Hermann Weyl's celebrated paper of 1916 titled: "Uber die Gleichverteilung von Zahlen mod. Eins." Weyl's work was primarily intended as a refinement of Kronecker's approximation theorem, and, therefore, in its initial stage, the theory was deeply rooted in diophantine approximations. During the last decades the theory has unfolded beyond that framework (cf. [5]). The reader can refer to the recent monographs [2] and [9] on the spaces of single and double sequences and summability theory, and applications.

Let (x_{jk}) be a double sequence of real numbers. We may decompose x_{jk} as the sum of its integer part $[x_{jk}]$ (i.e. the largest integer which is less than or equal to x_{jk} and its fractional part $\overline{x}_{jk} = x_{jk} - [x_{jk}]$. Clearly, $0 \leqslant \overline{x}_{jk} < 1$. The study of (x_{jk}) mod 1 is the study of the sequence $(\overline{x}_{jk})$ in $[0,1)$.

We say that the double sequence (x_{jk}) is uniformly distributed mod 1 if for every $a,b \in \mathbb{R}$ with $0 \leqslant a < b < 1$, we have that

$$\lim_{m,n\to\infty} \frac{1}{mn} card\{(j,k) : 0 \leqslant j \leqslant m-1, 0 \leqslant k \leqslant n-1, \quad \overline{x}_{jk} \in [a,b]\} = b-a.$$

The condition is saying that the frequency with which the sequence $(\overline{x}_{jk})$ lies in $[a,b]$ converges to $b-a$, the length of the interval.

If $\chi_{[a,b]}$ is the characteristic function of the interval $[a,b]$, then we may rewrite the definition of uniform distribution as

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \chi_{[a,b]}(\overline{x}_{jk}) = \int_0^1 \chi_{[a,b]}(x)dx.$$

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From this we deduce that

$$\lim_{m,n\to\infty}\frac{1}{mn}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}g(\overline{x}_{jk})=\int_0^1 g(x)dx,$$

whenever g is a step function, i.e., a finite linear combination of characteristic functions of intervals.

The following theorems are known.

THEOREM 1. ([5]) *The necessary and sufficient condition that* (x_{jk}) *is uniformly distributed mod 1 is that for any Riemann integrable function* $f(x)$ *in* $[0,1]$

$$\lim_{m,n\to\infty}\frac{1}{mn}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\chi_{[a,b]}(\overline{x}_{jk})=\int_0^1 f(x)dx.$$

THEOREM 2. ([5]) *The necessary and sufficient condition that* (x_{jk}) *is uniformly distributed mod 1 is that, for all* $h\in\mathbb{Z}\setminus\{0\}$,

$$\lim_{m,n\to\infty}\frac{1}{mn}\sum_{j=1}^{m}\sum_{k=1}^{n}e^{2\pi h\overline{x}_j i}=0.$$

EXAMPLE 1. Let ζ be irrational and ξ an arbitrary real number. Then, the double sequence $(\zeta j+\xi k)$, $j=1,2,3,\ldots$, $k=1,2,3,\ldots$, is uniformly distributed mod 1.

A double sequence $x=(x_{jk})$ has a Pringsheim limit ℓ provided that, given an $\varepsilon>0$ there exists an $N\in\mathbb{N}$ such that $\left|x_{jk}-\ell\right|<\varepsilon$ whenever $j,k>N$. In this case we write $P-\lim x_{jk}=\ell$.

Let $\lambda=(\lambda_m)$ and $\mu=(\mu_n)$ be two nondecreasing sequences of positive numbers tending to ∞, and $\lambda_{m+1}-\lambda_m\leqslant 1$, $\lambda_1=1$ and $\mu_{n+1}-\mu_n\leqslant 1$, $\mu_1=1$. The generalized double de la Vallée-Poussin mean of (x_{jk}) is defined by

$$\frac{1}{\lambda_m\mu_n}\sum_{j=m-\mu_m+1}^{m}\sum_{k=n-\mu_n+1}^{n}x_{jk}.$$

A double sequence $x=(x_{jk})$ is said to be (V,λ,μ)-summable to a number ℓ

$$\lim_{m,n\to\infty}\frac{1}{\mu_m\lambda_n}\sum_{j=m-\mu_m+1}^{m}\sum_{k=n-\lambda_n+1}^{n}x_{jk}=\ell.$$

If $\mu_m=m$ and $\lambda_n=n$, then (V,λ,μ)-summability of double sequences is reduced to (C,1,1)-summability of double sequences. Also, the reader can refer to papers [1]–[17] for more information on almost convergent double sequences, strong regularity of summability matrices, la Vallée-Poussin mean, uniformly distributed and well distributed single sequences.

2. (λ,μ)-uniformly distributed double sequences

In this section, we will generalize the notion of uniform distribution mod 1 to (λ,μ)-uniform distribution mod 1.

DEFINITION 1. Let (λ_n) and (μ_m) be two non-decreasing sequences of positive real numbers. Then, we say that the double sequence (x_{jk}) is (λ,μ)-uniformly distributed mod 1 if for every $a,b \in \mathbb{R}$ with $0 \leqslant a < b < 1$, we have that

$$\lim_{m,n\to\infty} \frac{1}{\mu_m\lambda_n} card\{(j,k) : n-\mu_m+1 \leqslant j \leqslant m, \quad n-\lambda_n+1 \leqslant j \leqslant n, \quad \overline{x}_j \in [a,b]\} = b-a.$$

We may rewrite the definition of (λ,μ)-uniform distribution as

$$\lim_{m,n\to\infty} \frac{1}{\mu_m\lambda_n} \sum_{j=m-\mu_m+1}^{m} \sum_{j=n-\lambda_n+1}^{n} \chi_{[a,b]}(\overline{x}_{jk}) = \int_0^1 \chi_{[a,b]}(x)dx.$$

From this we deduce that

$$\lim_{m,n\to\infty} \frac{1}{\mu_m\lambda_n} \sum_{j=m-\mu_m+1}^{m} \sum_{k=n-\lambda_n+1}^{n} g(\overline{x}_{jk}) = \int_0^1 g(x)dx,$$

whenever g is a step function. If $\mu_m = m$ and $\lambda_n = n$, then (λ,μ)-uniformly distribution mod 1 is reduced to the uniformly distribution mod 1.

THEOREM 3. *Let (λ_n) and (μ_m) be two non-decreasing sequences of positive real numbers. Then, the following statements are equivalent:*

(i) the double sequence (x_n) is (λ,μ)-uniformly distributed mod 1;

(ii) for any continuous function $f : [0,1] \to \mathbb{R}$ with $f(0) = f(1)$

$$\lim_{m,n\to\infty} \frac{1}{\mu_m\lambda_n} \sum_{j=m-\mu_m+1}^{m} \sum_{k=n-\lambda_n+1}^{n} f(\overline{x}_{jk}) = \int_0^1 f(x)dx;$$

(iii) for each $h \in \mathbb{Z}\backslash\{0\}$, we have

$$\lim_{m,n\to\infty} \frac{1}{\mu_m\lambda_n} \sum_{j=m-\mu_m+1}^{m} \sum_{k=n-\lambda_n+1}^{n} e^{2\pi h\overline{x}_{jk} i} = 0.$$

Proof. We prove (i) implies (ii). Let f be a continuous function on $[0,1]$. Then, given $\varepsilon > 0$, we can find a step function g with $\|f-g\| \leqslant \varepsilon$. We have

$$\begin{aligned}
&\left|\frac{1}{\mu_m\lambda_n}\sum_{j=m-\mu_m+1}^{m}\sum_{k=n-\lambda_n+1}^{n} f(\overline{x}_{jk}) - \int_0^1 f(x)dx\right| \\
&\leqslant \left|\frac{1}{\mu_m\lambda_n}\sum_{j=m-\mu_m+1}^{m}\sum_{k=n-\lambda_n+1}^{n} (f(\overline{x}_{jk}) - g(\overline{x}_{jk}))\right| \\
&\quad + \left|\frac{1}{\mu_m\lambda_n}\sum_{j=m-\mu_m+1}^{m}\sum_{k=n-\lambda_n+1}^{n} g(\overline{x}_{jk}) - \int_0^1 g(x)dx\right| \\
&\quad + \left|\int_0^1 g(x) - \int_0^1 f(x)dx\right| \\
&\leqslant 2\varepsilon + \left|\frac{1}{\mu_m\lambda_n}\sum_{j=m-\mu_m+1}^{m}\sum_{k=n-\lambda_n+1}^{n} g(\overline{x}_{jk}) - \int_0^1 g(x)dx\right|
\end{aligned}$$

Since the last term converges to zero as $n \to \infty$, we obtain

$$\limsup_{m,n\to\infty}\left|\frac{1}{\mu_m\lambda_n}\sum_{j=m-\mu_m+1}^{m}\sum_{k=n-\lambda_n+1}^{n} f(\overline{x}_{jk}) - \int_0^1 f(x)dx\right| \leqslant 2\varepsilon$$

and since $\varepsilon > 0$ is arbitrary, hence we have

$$\lim_{m,n\to\infty}\frac{1}{\mu_m\lambda_n}\sum_{j=m-\mu_m+1}^{m}\sum_{k=n-\lambda_n+1}^{n} f(\overline{x}_{jk}) = \int_0^1 f(x)dx.$$

Condition (ii) trivially implies (iii) by taking $f(x) = e^{2\pi hxi}$, for each $h \in \mathbb{Z}\backslash\{0\}$. We prove (iii) implies (i). Suppose that (iii) holds. Then

$$\lim_{m,n\to\infty}\frac{1}{\mu_m\lambda_n}\sum_{j=m-\mu_m+1}^{m}\sum_{k=n-\lambda_n+1}^{n} g(\overline{x}_{jk}) = \int_0^1 g(x)dx$$

whenever $g(x) = \sum_{k=1}^{\ell}\alpha_k e^{2\pi h_k xi}$ is a trigonometric polynomial, i.e. a finite linear combination of exponential functions. Now consider the interval $[a,b] \subset [0,1]$. Given $\varepsilon > 0$, we can find continuous functions f_1, f_2 with $f_1(0) = f_1(1), f_2(0) = f_2(1)$ such that $f_1 \leqslant \chi_{[a,b]} \leqslant f_2$ and

$$\int_0^1 (f_2(x) - f_1(x))dx \leqslant \varepsilon.$$

We have that

$$\begin{aligned}
\liminf_{m,n\to\infty} \quad & \frac{1}{\mu_m\lambda_n} \sum_{j=m-\mu_m+1}^{m} \sum_{k=n-\lambda_n+1}^{n} \chi_{[a,b]}(\overline{x}_{jk}) \\
& \geqslant \liminf_{n\to\infty} \frac{1}{\mu_m\lambda_n} \sum_{j=m-\mu_m+1}^{m} \sum_{k=n-\lambda_n+1}^{n} f_1(\overline{x}_{jk}) = \int_0^1 f_1(x) \\
& \geqslant \int_0^1 f_2(x) - \varepsilon \\
& \geqslant \int_0^1 \chi_{[a,b]}(x)dx - \varepsilon
\end{aligned}$$

and we have that

$$\begin{aligned}
\limsup_{m,n\to\infty} \quad & \frac{1}{\mu_m\lambda_n} \sum_{j=m-\mu_m+1}^{m} \sum_{k=n-\lambda_n+1}^{n} \chi_{[a,b]}(\overline{x}_{jk}) \\
& \leqslant \limsup_{m,n\to\infty} \frac{1}{\mu_m\lambda_n} \sum_{j=m-\mu_m+1}^{m} \sum_{k=n-\lambda_n+1}^{n} f_2(\overline{x}_{jk}) = \int_0^1 f_2(x) \\
& \leqslant \int_0^1 f_1(x) + \varepsilon \\
& \leqslant \int_0^1 \chi_{[a,b]}(x)dx + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\lim_{m,n\to\infty} \frac{1}{\mu_m\lambda_n} \sum_{j=m-\mu_m+1}^{m} \sum_{k=n-\lambda_n+1}^{n} \chi_{[a,b]}(\overline{x}_j) = \int_0^1 \chi_{[a,b]}(x)dx = b-a$$

that is, (x_{jk}) is (λ, μ)-uniformly distributed mod 1. □

Let UD and $UD_{(\lambda,\mu)}$ denote the set of all sequences of uniformly distributed and (λ, μ)-uniformly distributed mod 1.

THEOREM 4. *Let (λ_n) and (μ_m) be two non-decreasing sequences of positive real numbers. Then $UD = UD_{(\lambda,\mu)}$ if and only if*

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = 1 \ \ \textit{and} \ \ \lim_{m\to\infty} \frac{m}{\mu_m} = 1 \tag{1}$$

or equivalently

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = 1 \ \textit{and} \ \lim_{m\to\infty} \frac{\mu_m}{m} = 1.$$

Proof. Let assume that $\lim_{n\to\infty}\frac{n}{\lambda_n}=1$ and $\lim_{m\to\infty}\frac{m}{\mu_m}=1$. We can write the following equalities:

$$\begin{aligned}
&\frac{1}{\lambda_n\mu_m}\sum_{j=m-\mu_m+1}^{m}\sum_{k=n-\lambda_n+1}^{n}\chi_{[a,b]}(\overline{x}_{jk})\\
&=\frac{mn}{\mu_m\lambda_n}\frac{1}{mn}\sum_{j=1}^{m}\sum_{k=1}^{n}\chi_{[a,b]}(\overline{x}_{jk})\\
&\quad-\frac{m(n-\lambda_n)}{\mu_m\lambda_n}\frac{1}{m(n-\lambda_n)}\sum_{j=1}^{m}\sum_{k=1}^{n-\lambda_n}\chi_{[a,b]}(\overline{x}_{jk})\\
&\quad-\frac{n(m-\mu_m)}{\mu_m\lambda_n}\frac{1}{n(m-\mu_m)}\sum_{j=1}^{m-\mu_m}\sum_{k=1}^{n}\chi_{[a,b]}(\overline{x}_{jk})\\
&\quad+\frac{(m-\mu_m)(n-\lambda_n)}{\mu_m\lambda_n}\frac{1}{(m-\mu_m)(n-\lambda_n)}\sum_{j=1}^{m-\mu_m}\sum_{k=1}^{n-\lambda_n}\chi_{[a,b]}(\overline{x}_{jk})
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{mn}\sum_{j=1}^{m}\sum_{k=1}^{m}\chi_{[a,b]}(\overline{x}_{jk})\\
&=\frac{1}{mn}\sum_{j=1}^{m-\mu_m}\sum_{k=1}^{n}\chi_{[a,b]}(\overline{x}_{jk})+\frac{1}{mn}\sum_{j=m-\mu_m+1}^{m}\sum_{k=1}^{n}\chi_{[a,b]}(\overline{x}_{jk})\\
&=\frac{(m-\mu_m)(n-\lambda_n)}{mn}\frac{1}{(m-\mu_m)(n-\lambda_n)}\sum_{j=1}^{m-\mu_m}\sum_{k=1}^{n-\lambda_n}\chi_{[a,b]}(\overline{x}_{jk})\\
&\quad+\frac{(m-\mu_m)\lambda_n}{mn}\frac{1}{\lambda_n(m-\mu_m)}\sum_{j=1}^{m-\mu_m}\sum_{k=n-\lambda_n}^{n}\chi_{[a,b]}(\overline{x}_{jk})\\
&\quad+\frac{\mu_m(n-\lambda_n)}{mn}\frac{1}{\mu_m(n-\lambda_n)}\sum_{j=m-\mu_m}^{n-\lambda_n}\sum_{k=1}^{n-\lambda_n}\chi_{[a,b]}(\overline{x}_{jk})\\
&\quad+\frac{\mu_m\lambda_n}{mn}\frac{1}{\mu_m\lambda_n}\sum_{j=m-\mu_m}^{m}\sum_{k=n-\lambda_n}^{n}\chi_{[a,b]}(\overline{x}_{jk})
\end{aligned}$$

Taking the limit as $m,n\to\infty$ and using (1), we have $UD=UD_{(\lambda,\mu)}$.

Now suppose that $UD=UD_{(\lambda,\mu)}$ and $\lim_{n\to\infty}\frac{n}{\lambda_n}=\alpha\neq1$ and $\lim_{m\to\infty}\frac{m}{\mu_m}=\beta\neq 1$. Then

$$\frac{1}{\lambda_n\mu_m}\sum_{j=m-\mu_m+1}^{m}\sum_{k=n-\lambda_n+1}^{n}\chi_{[a,b]}(\overline{x}_{jk})=\alpha(b-a)-\beta(b-a)-(b-a).$$

This contradicts the fact that $UD=UD_{(\lambda,\mu)}$. Hence (1) is necessary. $\square$

3. (λ,μ)-well distributed double sequences

The notion of almost convergence for double sequences has been introduced in [8] as follows:

A double sequence (x_{kj}) of real numbers is called almost convergent to a limit ℓ if

$$\lim_{p,q\to\infty}\sup_{m,n\geqslant 0}\left|\frac{1}{pq}\sum_{j=m}^{m+p+1}\sum_{k=n}^{n+q-1}x_{jk}-\ell\right|=0.$$

DEFINITION 2. We say that a double sequence (x_{jk}) is well distributed mod 1 if for every $a,b\in\mathbb{R}$ with $0\leqslant a<b<1$, we have that

$$\lim_{m,n\to\infty}\frac{1}{mn}card\{(j,k):0\leqslant j\leqslant m-1,\ 0\leqslant k\leqslant n-1\ \overline{x}_{j+p,k+q}\in[a,b]\}=b-a$$

uniformly in $p=0,1,2,\ldots$ and $q=0,1,2,\ldots$.

We may rewrite the definition of well distribution as

$$\lim_{m,n\to\infty}\frac{1}{mn}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\chi_{[a,b]}(\overline{x}_{j+p,k+q})=\int_0^1\chi_{[a,b]}(x)dx$$

uniformly in $p=0,1,2,\ldots.$ and $q=0,1,2,\ldots.$

In other words, (x_{jk}) is well-distributed if $\left(\chi_{[a,b]}(\overline{x}_{jk})\right)$ is almost convergent to $b-a$ for every subinterval $[a,b]$ of $[0,1]$. Since any almost convergent double sequence is summable $(C,1,1)$, a well-distributed sequence is necessarily uniformly distributed. However, the converse is not true.

DEFINITION 3. Let (λ_n) and (μ_m) be two non-decreasing sequences of positive real numbers. Then, we say that a double sequence (x_{jk}) is (λ,μ)-well distributed mod 1 if for every $a,b\in\mathbb{R}$ with $0\leqslant a<b<1$, we have that

$$\lim_{m,n\to\infty}\frac{1}{\mu_m\lambda_n}card\{(j,k):m-\lambda_m+1\leqslant j\leqslant m,\ n-\lambda_n+1\leqslant k\leqslant n,\ \overline{x}_{j+p,k+q}\in[a,b]\}=b-a$$

uniformly in $p=0,1,2,\ldots$ and $q=0,1,2,\ldots$.

We may rewrite the definition of well distribution as

$$\lim_{mn\to\infty}\frac{1}{\mu_m\lambda_n}\sum_{j=m-\lambda_m+1}^{m}\sum_{k=n-\lambda_n+1}^{n}\chi_{[a,b]}(\overline{x}_{j+p,k+q})=\int_0^1\chi_{[a,b]}(x)dx$$

uniformly in $p=0,1,2,\ldots$ and $q=0,1,2,\ldots$.

In other words, (x_{jk}) is (μ,λ)-well distributed if $\left(\chi_{[a,b]}(\overline{x}_{jk})\right)$ is (μ,λ)-almost convergent to $b-a$ for every $[a,b]$. Since any (μ,λ)-almost convergent sequence is (V,μ,λ)-summable, a (λ,μ)-well distributed sequence is necessarily (λ,μ)-uniformly distributed. However, the converse is not true. If $\mu_m=m$ and $\lambda_n=n$ and then (λ,μ)-well distribution mod 1 is reduced to the well distribution mod 1.

THEOREM 5. *Let (λ_n) and (μ_m) be two non-decreasing sequences of positive real numbers. Then, the following are equivalent:*

(i) the double sequence (x_{jk}) is (λ,μ)-well distributed mod 1;

(ii) for any continuous function $f:[0,1]\to\mathbb{R}$ with $f(0)=f(1)$

$$\lim_{m,n\to\infty}\frac{1}{\mu_m\lambda_n}\sum_{j=m-\lambda_m+1}^{m}\sum_{k=n-\lambda_n+1}^{n} f(\overline{x}_{j+p,k+q})=\int_0^1 f(x)dx$$

uniformly in $p=0,1,2,\dots.$ and $q=0,1,2,\dots;$

(iii) for each $h\in\mathbb{Z}\backslash\{0\}$, we have

$$\lim_{m,n\to\infty}\frac{1}{\mu_m\lambda_n}\sum_{j=m-\lambda_m+1+p}^{m+p}\sum_{k=n-\lambda_n+1+q}^{n+q} e^{2\pi h\overline{x}_{jk}i}=0$$

uniformly in $p=0,1,2,\dots$ and $q=0,1,2,\dots.$

Proof. The proof is similar to the proof of Theorem 3, and we omit it. □

Let WD and $WD_{(\lambda,\mu)}$ denote the set of all sequences of well distributed and (λ,μ)-well distributed mod 1.

THEOREM 6. *Let (λ_n) and (μ_m) be two non-decreasing sequences of positive real numbers. $WD=WD_{(\lambda,\mu))}$ if and only if*

$$\lim_{n\to\infty}\frac{n}{\lambda_n}=1 \text{ and } \lim_{m\to\infty}\frac{\mu_m}{m}=1$$

or equivalently

$$\lim_{n\to\infty}\frac{\lambda_n}{n}=1 \text{ and } \lim_{m\to\infty}\frac{m}{\mu_m}=1.$$

Proof. The proof is similar to the proof of Theorem 4, and we omit it. □

4. A-uniformly distributed double sequences

Let $A=(a_{mnjk})_{j,k,m,n\in\mathbb{N}}$ be a four-dimensional summability matrix. For a double sequence (x_{jk}), the A-transform $((Ax)_{mn})$ of (x_{jk}) is given by

$$(Ax)_{mn}=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}a_{mnjk}x_{jk},$$

provided the double series convergence Pringsheim's sense for every $m,n\in\mathbb{N}$. In summability theory, a two-dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. The well-known characterization for two-dimensional matrix transformations is known as the Silverman-Toeplitz conditions. Robison [16] presented a four-dimensional analog of the regularity by considering an additional assumption of boundedness. This assumption was made because a double -convergent sequence is not necessarily bounded. The definition and the characterization of regularity for four-dimensional matrices is known as the Robison-Hamilton conditions, or for short, RH-regularity In [16] Robison presented the following notion of conservative four-dimensional matrix transformation and a Silverman-Toeplitz type characterization of such notion.

DEFINITION 4. A four-dimensional matrix $A=(a_{mnjk})_{j,k,m,n\in\mathbb{N}}$ is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

Robison and Hamilton presented a Silverman-Toeplitz type multidimensional characterization of regularity in [4] and [16].

THEOREM 7. (Hamilton [4], Robison [16]) *The four-dimensional matrix A is RH-regular if and only if*

RH_1: $P\text{-}\lim_{m,n} a_{mnjk}=0$ *for each* j *and* k;
RH_2: $P\text{-}\lim_{m,n}\sum_{j,k=0,0}^{\infty,\infty} a_{mnjk}=1$;
RH_3: $P\text{-}\lim_{m,n}\sum_{j=0}^{\infty}\left|a_{mnjk}\right|=0$ *for each* k;
RH_4: $P\text{-}\lim_{m,n}\sum_{k=0}^{\infty}\left|a_{mnjk}\right|=0$ *for each* j;
RH_5: $\sum_{j,k=0,0}^{\infty,\infty}\left|a_{mnjk}\right|$ *is P-convergent;*
RH_6: *there exist finite positive integers* Δ *and* Γ *such that* $\sum_{j,k>\Gamma}\left|a_{mnjk}\right|<\Delta$.

The four-dimensional Cesàro matrix $(C,1,1)$ order one and one is defined by

$$(C,1,1)_{mnjk}:=\begin{cases}\frac{1}{mn}, & j\leqslant m \text{ and } k\leqslant n\\ 0, & \text{otherwise}\end{cases}$$

for all $m,n,j,k\in\mathbb{N}$.

DEFINITION 5. Let $A=(a_{mnjk})_{j,k,m,n\in\mathbb{N}}$ be a positive four-dimensional RH- regular matrix and let (x_{jk}) be a double sequence of real numbers. For $0\leqslant x\leqslant 1$, let χ_x be the characteristic function of the interval $[0,x)$. The function $g(x)$, $0\leqslant x\leqslant 1$, is the A- asymptotic distribution function mod 1 of (x_{jk}) if the double sequence $\chi_x(x_{jk})$ is summable by A to the value $g(x)$ for $0\leqslant x\leqslant 1$; that is, if

$$\lim_{mn\to\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{mnjk}\chi_x(x_{jk})=g(x)$$

for $0\leqslant x\leqslant 1$.

The function $g(x)$ is nondecreasing on $[0,1]$ with $g(0)=0$ and $g(1)=1$. In the case $g(x)=x$ for $0 \leqslant x \leqslant 1$, the sequence (x_{jk}) is called A-uniformly distributed mod 1. If we choose A as the matrix $(C,1,1)$, then we see that Definition 5 contains definition of uniformly distributed sequence mod 1 as a particular case and Theorem 1 and Theorem 2 can be generalized as follows.

THEOREM 8. *Let $A=(a_{mnjk})_{j,k,m,n\in\mathbb{N}}$ be a positive four-dimensional RH- regular matrix and let (x_{jk}) be a double sequence of real numbers. The necessary and sufficient condition that (x_{jk}) is A-uniformly distributed mod 1 is that for any Riemann integrable function $f(x)$ in $[0,1]$*

$$\lim_{m,n\to\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{mnjk}\chi_{[a,b]}(\overline{x}_{jk}) = \int_0^1 f(x)dx.$$

THEOREM 9. *Let $A=(a_{mnjk})_{j,k,m,n\in\mathbb{N}}$ be a positive four-dimensional RH- regular matrix and let (x_{jk}) be double sequence of real numbers. The necessary and sufficient condition that (x_{jk}) is A-uniformly distributed mod 1 is that, for all $h\in\mathbb{Z}\setminus\{0\}$,*

$$\lim_{m,n\to\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{mnjk}e^{2\pi h\overline{x}_j i} = 0.$$

If we choose the four-dimensional matrix $A=(a_{mnjk})$ as

$$a_{mnjk} := \begin{cases} \frac{1}{\mu_m\lambda_n}, & m-\mu_m+1 \leqslant j \leqslant m \text{ and } n-\lambda_n+1 \leqslant k \leqslant n, \\ 0, & \text{otherwise,} \end{cases}$$

then we obtain the definition of (λ,μ)-uniformly distributed double sequence and the given theorems about these definitions in the Section 2.

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