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SOME RESULTS ON UNIQUENESS OF MEROMORPHIC FUNCTIONS FOR FINITE ORDER IN AN ANGULAR DOMAIN

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Abstract. In this paper, we discuss how meromorphic functions are determined by their multiple values and deficient values in an angular domain and also we prove an at most 3-valued theorem for meromorphic function in an angular domain.

1. Introduction

We assume that the readers are familiar with the fundamental results in Nevanlinna's value distribution theory of meromorphic functions of single complex variable in the open complex plane. We say that f and $\widehat{f}$ share the value a CM (counting multiplicities) if f and $\widehat{f}$ have the same a-points with the same multiplicity and if f and $\widehat{f}$ share the value a IM (ignoring multiplicities) if we do not consider the multiplicities.

The uniqueness theory of meromorphic functions is an interesting problem in the value distribution theory. In 1929, R. Nevanlinna proved that, if f and $\widehat{f}$ be two non-constant meromorphic functions in $\mathbb{C}$ and if they share five distinct values IM, then $f \equiv \widehat{f}$; if they share four distinct values CM, then f is a Mobius transformation of $\widehat{f}$. After this work, many authors proved several results on uniqueness of meromorphic functions concerning shared values in the complex plane. In 2004, J. H. Zheng (see [3]) extended the uniqueness of meromorphic functions dealing with five shared values in an angular domains of $\mathbb{C}$. Also in 2010, He Ping proved some important results on the uniqueness of meromorphic functions sharing values in an angular domain (see [7]) and others have done lots of work in this area (see [3]–[22]). It is interesting to prove some important uniqueness results in the whole of the complex plane to an angular domain.

2. Basic notations and definitions

Nevanlinna theory in an angular domain will play a key role in the proof of theorems. Let $f(z)$ be a meromorphic function on the angular domain $\overline{\Omega}(\alpha,\beta) = \{z : \alpha \leqslant \arg z \leqslant \beta\}$,

$$A_{\alpha,\beta}(r,f) = \frac{\omega}{\pi}\int_1^r \left(\frac{1}{t^{\omega}} - \frac{t^{\omega}}{r^{2\omega}}\right)\left\{\log^+|f(te^{i\alpha})| + \log^+|f(te^{i\beta})|\right\}\frac{dt}{t},$$

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$$B_{\alpha,\beta}(r,f)=\frac{2\omega}{\pi r^{\omega}}\int_{\alpha}^{\beta}\log^{+}|f(re^{i\theta})|\sin\omega(\theta-a)d\theta,$$

$$C_{\alpha,\beta}(r,f)=\sum_{1<|b_n|<r}\left(\frac{1}{|b_n|^{\omega}}-\frac{|b_n|^{\omega}}{r^{2\omega}}\right)\sin\omega(\theta_n-a)d\theta,$$

where $\omega=\pi/(\beta-\alpha)$ and $b_n=|b_n|e^{i\theta_n}$ are the poles of $f(z)$ on $\overline{\Omega}(\alpha,\beta)$ appearing according to the multiplicities. $C_{\alpha,\beta}$ is called angular counting function of the poles of $f(z)$ on $\overline{\Omega}(\alpha,\beta)$ and Nevanlinna's angular characteristic function is defined as follows

$$S_{\alpha,\beta}(r,f)=A_{\alpha,\beta}(r,f)+B_{\alpha,\beta}(r,f)+C_{\alpha,\beta}(r,f).$$

Throughout, we denote by $R_{\alpha,\beta}(r,*)$ a quantity satisfying satisfying

$$R_{\alpha,\beta}(r,*)=O\{\log(rS_{\alpha,\beta}(r,*))\},\ \ r\in E,$$

where E denotes a set of positive real numbers with finite linear measure.

DEFINITION 1. Let $f(z)$ be a meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta)$ $=\{z:\alpha\leqslant\arg z\leqslant\beta\}$. Then function

$$S_{\alpha,\beta}(r,f)=A_{\alpha,\beta}(r,f)+B_{\alpha,\beta}(r,f)+C_{\alpha,\beta}(r,f)$$

is called angular Nevanlinna characteristic of $f(z)$.

DEFINITION 2. Let $f(z)$ be a meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta)$ $=\{z:\alpha\leqslant\arg z\leqslant\beta\}$. Then the order and lower order of $f(z)$ are defined by $\lambda(f)$ and $\mu(f)$, where

$$\lambda(f)=\limsup_{r\to\infty}\frac{S_{\alpha,\beta}(r,f)}{\log r}$$

and

$$\mu(f)=\liminf_{r\to\infty}\frac{S_{\alpha,\beta}(r,f)}{\log r}.$$

For positive integer k,m, we define that

$$\delta^{k}_{\alpha,\beta}(a,f)=1-\limsup_{r\to+\infty}\frac{C^{k}_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)}{S_{\alpha,\beta}(r,f)},$$

$$\Theta_{\alpha,\beta}(a,W)=1-\limsup_{r\to+\infty}\frac{\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)}{S_{\alpha,\beta}(r,f)},$$

where $C^{k}_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)$ is counting function of a-points of $f(z)$ in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leqslant\arg z\leqslant\beta\}$ where a-points of multiplicity m is counted m times if $m\leqslant k$ and $1+k$ times if $m>k$. In particular, if $k=\infty$, then

$$\delta_{\alpha,\beta}(a,f)=\liminf_{r\to+\infty}\frac{A_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)+B_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)}{S_{\alpha,\beta}(r,f)}=1-\limsup_{r\to+\infty}\frac{C_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)}{S_{\alpha,\beta}(r,f)}.$$

3. Some lemmas

LEMMA 1. [3] *Let* $f(z)$ *be a meromorphic function in an angular domain* $\overline{\Omega}(\alpha,\beta) = \{z : \alpha \leqslant \arg z \leqslant \beta\}$, $a \in \mathbb{C}$

$$S_{\alpha,\beta}\left(r, \frac{1}{f-a}\right) = S_{\alpha,\beta}(r,f) + O(1).$$

and for an integer $p \geqslant 0$,

$$S_{\alpha,\beta}(r, f^{(p)}) \leqslant 2pS_{\alpha,\beta}(r,f) + R_{\alpha,\beta}(r,f),$$

$$A_{\alpha,\beta}\left(r, \frac{f^{(p)}}{f}\right) + B_{\alpha,\beta}\left(r, \frac{f^{(p)}}{f}\right) = R_{\alpha,\beta}(r,f),$$

and $R_{\alpha,\beta}(r, f^{(p)}) = R_{\alpha,\beta}(r,f)$.

LEMMA 2. [3] *Let* $f(z)$ *be a meromorphic function in an angular domain* $\overline{\Omega}(\alpha,\beta) = \{z : \alpha \leqslant \arg z \leqslant \beta\}$. *Then for arbitrary* q *distinct* $a_j \in \overline{\mathbb{C}}$ $(1 \leqslant j \leqslant q)$, *we have*

$$(q-2)S_{\alpha,\beta}(r,f) \leqslant \sum_{j=1}^{q} \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-a_j}\right) + R_{\alpha,\beta}(r,f),$$

where the term $\overline{C}_{\alpha,\beta}(r, 1/f - a_j)$ *will be replaced by* $\overline{C}_{\alpha,\beta}(r,f)$ *when some* $a_j = \infty$.

We use $\overline{C}^{k)}_{\alpha,\beta}(r, 1/f - a_j)$ to denote the zeros of $f(z) - a$ in an angular domain $\overline{\Omega}(\alpha,\beta) = \{z : \alpha \leqslant \arg z \leqslant \beta\}$ whose multiplicities are no greater than k and are counted only once. Likewise, we use $\overline{C}^{(k+1}_{\alpha,\beta}(r, 1/f - a_j)$ to denote the zeros of $f(z) - a$ in $\overline{\Omega}(\alpha,\beta) = \{z : \alpha \leqslant \arg z \leqslant \beta\}$ whose multiplicities are greater than k and are counted only once.

LEMMA 3. *Let* $f(z)$ *be a meromorphic function in an angular domain* $\overline{\Omega}(\alpha,\beta) = \{z : \alpha \leqslant \arg z \leqslant \beta\}$ *with* $\lambda(f)$ *as the order of* $f(z)$ *and* $\mu(f)$ *as the lower order of* $g(z)$. *If* $\lambda(f) < \mu(f) < \infty$, *then*

$$S_{\alpha,\beta}(r,f) = o(S_{\alpha,\beta}(r,g)) \quad (r \to \infty).$$

Proof. Picking up a number $\varepsilon \in (0, \frac{1}{2}(\mu(f) - f(z)))$, then we have

$$\lambda(f) < \lambda(f) + \varepsilon < \mu(g) - \varepsilon < \mu(g).$$

By the definitions of order and lower order for meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta) = \{z : \alpha \leqslant \arg z \leqslant \beta\}$, there exists a positive number r such that $S_{\alpha,\beta}(r,f) < r^{\lambda(f)+\varepsilon}$ and $S_{\alpha,\beta}(r,g) > r^{\mu(f)-\varepsilon}$. Therefore, we obtain

$$\frac{S_{\alpha,\beta}(r,f)}{S_{\alpha,\beta}(r,g)} < r^{-(\mu(f)-\lambda(f)-2\varepsilon)}.$$

Hence, we have

$$S_{\alpha,\beta}(r,f) = o(S_{\alpha,\beta}(r,g)) \quad (r \to \infty). \quad \square$$

4. Main results

We say that $f = a \Leftrightarrow g = a$ when f and g share a IM, then it is said that $f - a$ and $g - a$ have the same zeros (ignoring multiplicities); We say that $f = a \leftrightharpoons g = a$ when $f - a$ and $g - a$ have the same zeros (counting multiplicities). Let k be a positive integer, then we denote by $\overline{C}^{k)}_{\alpha,\beta}\left(r, \frac{1}{f-a}\right)$ the counting function of a-points of f with multiplicity $\leqslant k$ (ignoring multiplicities); let $\overline{E}_{k)}(a,f)$ denote the set of zeros of $f - a$ with multiplicity $\leqslant k$ (ignoring multiplicities).

Now, the main theorems of this paper are listed as follows

THEOREM 1. *Let $f(z)$ and $g(z)$ be two transcedental meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta) = \{z : \alpha \leqslant \arg z \leqslant \beta\}$ such that $\mu(f)$ is finite, then $\frac{f}{g} = H$, and let $\lambda(f) \neq \mu(H)$. Assume that $f = 0 \Leftrightarrow g = 0$. If there exist 4 distinct finite non-zero $a_j \in \mathbb{C}$ $(j = 1,2,\dots,4)$ such that*

$$\overline{E}_{1)}(a_{j,f}) = \overline{E}_{1)}(a_j, g) \tag{1}$$

and

$$\sum_{j=1}^{4} max\{\Theta_{\alpha,\beta}(0,f), \delta_{\alpha,\beta}(0,f) > 0\} > 0, \tag{2}$$

then, $f(z) \equiv g(z)$.

THEOREM 2. *Let $f(z)$ and $g(z)$ be two transcedental meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta) = \{z : \alpha \leqslant \arg z \leqslant \beta\}$ such that $\lambda(f)$ is finite, then $\frac{f}{g} = H$, and $\mu(f) \neq \mu(H) = \lambda(H)$. Assume that $f = 0 \Leftrightarrow g = 0$. If there exist 4 distinct finite non-zero $a_j \in \mathbb{C}$ $(j = 1,2,\dots,4)$ such that (1) and 2 hold, then $f(z) \equiv g(z)$.*

THEOREM 3. *Let $f(z)$ and $g(z)$ be two transcedental meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta) = \{z : \alpha \leqslant \arg z \leqslant \beta\}$ such that $\lambda(f)$ is finite, then $\frac{f}{g} = H$, and $\lambda(f) \neq \lambda(H)$. If there exist q distinct finite $a_j \in \mathbb{C}$ and integers $k_j(\geqslant j)$ such that*

$$\overline{E}_{k_j}(a_j, f) = \overline{E}_{k_j}(a_j, g), \ \ (j = 1,2,\dots,q), \tag{3}$$

where

$$q > 2 + \sum_{j=1}^{q} \frac{1}{j+1}, \tag{4}$$

then $f(z) \equiv g(z)$.

Proof of Theorem 2. For $j = 1,2,\dots,4$, we have

$$\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right) \leqslant \frac{1}{2}\overline{C}^{1)}_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right) + \frac{1}{2}C_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right), \tag{5}$$

and hence

$$\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right) \leqslant \frac{1}{2}\overline{C}^{1)}_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right) + \frac{1}{2}S_{\alpha,\beta}(r,f),$$

By Lemma 2, we have

$$\begin{aligned}(2\nu+1)S_{\alpha,\beta}(r,f) &< \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right) + \sum_{j=1}^{4}\overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right) + R_{\alpha,\beta}(r,f)\\ &< \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right) + \frac{1}{2}\sum_{j=1}^{4}\overline{C}^{1}_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right) + 2S_{\alpha,\beta}(r,f) + R_{\alpha,\beta}(r,f).\end{aligned}$$

Therefore,

$$\begin{aligned}S_{\alpha,\beta}(r,f) &< \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f}\right) + \frac{1}{2}\sum_{j=1}^{4}\overline{C}^{1}_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right) + R_{\alpha,\beta}(r,f)\\ &= \overline{C}_{\alpha,\beta}\left(r,\frac{1}{g}\right) + \frac{1}{2}\sum_{j=1}^{4}\overline{C}^{1}_{\alpha,\beta}\left(r,\frac{1}{g-a_j}\right) + R_{\alpha,\beta}(r,f)\\ &< 3S_{\alpha,\beta}(r,g) + R_{\alpha,\beta}(r,f).\end{aligned} \tag{6}$$

Similarly, we can prove

$$S_{\alpha,\beta}(r,g) < 3S_{\alpha,\beta}(r,f) + R_{\alpha,\beta}(r,g). \tag{7}$$

Therefore, we have

$$\mu(f) = \mu(g).$$

By (7) and assumptions, we have

$$S_{\alpha,\beta}(r,H) \leqslant S_{\alpha,\beta}(r,f) + S_{\alpha,\beta}(r,g) + O(1) \leqslant 4S_{\alpha,\beta}(r,f) + R_{\alpha,\beta}(r,f).$$

Thus,

$$\mu(H) \leqslant \mu(f).$$

Note that $\lambda(H) = \mu(H) \neq \mu(f)$, hence we have

$$\lambda(H) < \mu(f). \tag{8}$$

Assume to the contrary that $f(z) \equiv g(z)$. Let $\{z_n\}$ be all simple zeros of $f - a_1$ in an angular domain. As $\overline{E}_{1)}(a_1, f) = \overline{E}_{1)}(a_1, g)$, then $\{z_n\}$ are also all simple zeros of

$M - a_1$ in an angular doamin. Note that $\frac{W}{M} = H$, thus, $H \not\equiv 1$, but $H(z_n) = 1$. Hence, we have

$$\overline{C}^{1)}_{\alpha,\beta}\left(r, \frac{1}{f - a_1}\right) \leqslant C_{\alpha,\beta}\left(r, \frac{1}{H - 1}\right) \leqslant S_{\alpha,\beta}(r, H) + O(1). \tag{9}$$

It follows from (8) and (9) that

$$\limsup_{r\to+\infty} \frac{\overline{C}^{1)}_{\alpha,\beta}\left(r, \frac{1}{f-a_1}\right)}{\log r} \leqslant \lambda(H) \leqslant \mu(f).$$

According to the method used in Lemma 3, we can prove

$$\lim_{r\to+\infty} \frac{\overline{C}^{1)}_{\alpha,\beta}\left(r, \frac{1}{f-a_1}\right)}{S_{\alpha,\beta}(r, f)} \leqslant 0.$$

Similarly, we can obtain

$$\lim_{r\to+\infty} \frac{\overline{C}^{1)}_{\alpha,\beta}\left(r, \frac{1}{f-a_j}\right)}{S_{\alpha,\beta}(r, W)} \leqslant 0, \quad j = 1, 2, \ldots, 4\nu. \tag{10}$$

By (5) and the Lemma 2, we have

$$\begin{aligned} 3S_{\alpha,\beta}(r, f) &< \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f}\right) + \sum_{j=1}^{4} \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f - a_j}\right) + R_{\alpha,\beta}(r, f) \\ &< \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f}\right) + \frac{1}{2}\sum_{j=1}^{4\nu} \overline{C}^{1)}_{\alpha,\beta}\left(r, \frac{1}{f - a_j}\right) \\ &\quad + \frac{1}{2}\sum_{j=1}^{4} C_{\alpha,\beta}\left(r, \frac{1}{f - a_j}\right) + R_{\alpha,\beta}(r, f). \end{aligned} \tag{11}$$

From (10) and (11), we get

$$3S_{\alpha,\beta}(r, f) < \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f}\right) + \frac{1}{2}\sum_{j=1}^{4} C_{\alpha,\beta}\left(r, \frac{1}{f - a_j}\right) + R_{\alpha,\beta}(r, f), \tag{12}$$

which gives

$$3 \leqslant (1 - \Theta_{\alpha,\beta}(0, f)) + \frac{1}{2}\left(4 - \sum_{j=1}^{4} \delta_{\alpha,\beta}(a_j, f)\right),$$

or

$$\Theta_{\alpha,\beta}(0, f) + \frac{1}{2}\sum_{j=1}^{4} \delta_{\alpha,\beta}(a_j, f) \leqslant 0.$$

This contradicts (2). Hence $f(z) \equiv g(z)$. This proves the Theorem 2. $\square$

Proof of Theorem 1. Assume that $f(z) \not\equiv g(z)$. According to the proof of Theorem 2, we also obtain

$$S_{\alpha,\beta}(r,H) \leqslant 4S_{\alpha,\beta}(r,f) + R_{\alpha,\beta}(r,f).$$

Note that $\lambda(H) \neq \lambda(f)$, hence, we have

$$\lambda(H) < \lambda(f). \tag{13}$$

Simiraly, (9) shows that

$$\overline{C}^{1)}_{\alpha,\beta}\left(r, \frac{1}{f-a_j}\right) \leqslant S_{\alpha,\beta}(r,H) + O(1), \quad (j = 1,2,\ldots,4). \tag{14}$$

From (11) and (14), we have

$$\begin{aligned} 3S_{\alpha,\beta}(r,f) < (1 - \Theta_{\alpha,\beta}(0,f) + o(1))S_{\alpha,\beta}(r,f) + 2S_{\alpha,\beta}(r,H) \\ + \frac{1}{2}\left(\sum_{j=1}^{4}(1 - \delta_{\alpha,\beta})(a_j,f) + o(1))\right) S_{\alpha,\beta}(r,f) + o(S_{\alpha,\beta}(r,f)), \end{aligned}$$

which implies that

$$\left(\Theta_{\alpha,\beta}(0,f) + \frac{1}{2}\sum_{j=1}^{4}\delta_{\alpha,\beta}(a_j,f) + o(1)\right) S_{\alpha,\beta}(r,f) < 2S_{\alpha,\beta}(r,H). \tag{15}$$

(2) and (15) yield $\lambda(f) \leqslant \lambda(H)$, which contradicts (13). Therefore, we have $f(z) \equiv g(z)$. □

Proof of Theorem 3. Without loss of generality, we may assume that $k_j = j$ $(j = 1,2,3,\ldots,q)$. Note that

$$\begin{aligned} \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-a_1}\right) &\leqslant \frac{1}{2}\overline{C}^{1)}_{\alpha,\beta}\left(r, \frac{1}{f-a_1}\right) + \frac{1}{2}C_{\alpha,\beta}\left(r, \frac{1}{f-a_j}\right), \\ &\leqslant \frac{1}{2}\overline{C}^{1)}_{\alpha,\beta}\left(r, \frac{1}{f-a_j}\right) + \frac{1}{2}S_{\alpha,\beta}(r,f) + O(1), \\ \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-a_2}\right) &\leqslant \frac{2}{3}\overline{C}^{2)}_{\alpha,\beta}\left(r, \frac{1}{f-a_2}\right) + \frac{1}{3}C_{\alpha,\beta}\left(r, \frac{1}{f-a_j}\right), \\ &\leqslant \frac{2}{3}\overline{C}^{2)}_{\alpha,\beta}\left(r, \frac{1}{f-a_j}\right) + \frac{1}{3}S_{\alpha,\beta}(r,f) + O(1), \\ &\vdots \\ \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-a_q}\right) &\leqslant \frac{q}{q+1}\overline{C}^{q)}_{\alpha,\beta}\left(r, \frac{1}{f-a_j}\right) + \frac{1}{q+1}C_{\alpha,\beta}\left(r, \frac{1}{f-a_q}\right), \\ &\leqslant \frac{q}{q+1}\overline{C}^{1)}_{\alpha,\beta}\left(r, \frac{1}{f-a_q}\right) + \frac{1}{q+1}S_{\alpha,\beta}(r,f) + O(1). \end{aligned} \tag{16}$$

From (16) and Lemma 2, we have

$$
\begin{aligned}
(q-2)S_{\alpha,\beta}(r,f) &\leqslant \sum_{j=1}^{q} \overline{C}_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right)+R_{\alpha,\beta}(r,f),\\
&\leqslant \sum_{j=1}^{q} \frac{j}{j+1}\overline{C}^{j)}_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right)+\sum_{j=1}^{q}\frac{1}{j+1}S_{\alpha,\beta}(r,f)+R_{\alpha,\beta}(r,f).
\end{aligned}
$$

Hence, we have

$$
\left(q-2-\sum_{j=1}^{q}\frac{1}{j+1}\right)S_{\alpha,\beta}(r,f)\leqslant \sum_{j=1}^{q}\frac{j}{j+1}\overline{C}^{j)}_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right)+R_{\alpha,\beta}(r,f). \tag{17}
$$

By (17) and the assumptions, we have

$$
\begin{aligned}
\left(q-2-\sum_{j=1}^{q}\frac{1}{j+1}\right)S_{\alpha,\beta}(r,f) &\leqslant \sum_{j=1}^{q}\frac{j}{j+1}\overline{C}^{j)}_{\alpha,\beta}\left(r,\frac{1}{g-a_j}\right)+R_{\alpha,\beta}(r,f),\\
&\leqslant \left(\sum_{j=1}^{q}\frac{j}{j+1}\right)S_{\alpha,\beta}(r,g)+R_{\alpha,\beta}(r,f).
\end{aligned}
$$

Similarly, we also obtain

$$
\left(q-2-\sum_{j=1}^{q}\frac{1}{j+1}\right)S_{\alpha,\beta}(r,g)\leqslant \left(\sum_{j=1}^{q}\frac{j}{j+1}\right)S_{\alpha,\beta}(r,f)+R_{\alpha,\beta}(r,g). \tag{18}
$$

Hence the order of $f(z)$ and $g(z)$ satisfy $\lambda(f)=\lambda(g)$.

From (18) and the proof of Theorem 3, we also obtain

$$
\lambda(H)<\lambda(f). \tag{19}
$$

Assume that $f(z)\not\equiv g(z)$. By similar proof of (9), we can obtain

$$
\overline{C}^{j)}_{\alpha,\beta}\left(r,\frac{1}{f-a_j}\right)\leqslant C_{\alpha,\beta}\left(r,\frac{1}{H-1}\right)\leqslant S_{\alpha,\beta}(r,H)+O(1),\ \ (j=1,2,\dots,q). \tag{20}
$$

From (18) and (19) it follows that

$$
\left(q-2-\sum_{j=1}^{q}\frac{1}{j+1}\right)S_{\alpha,\beta}(r,f)\leqslant \left(\sum_{j=1}^{q}\frac{j}{j+1}\right)S_{\alpha,\beta}(r,H)+R_{\alpha,\beta}(r,g),
$$

which shows that $\lambda(f)\leqslant\lambda(H)$. This contradicts to (19), and hence $f(z)\equiv g(z)$. □

From Theorem 3, we have the following corollaries

COROLLARY 1. *Let $f(z)$ and $g(z)$ be two transcedental meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leqslant \arg z\leqslant\beta\}$ such that $\lambda(f)$ is finite, then $\frac{f}{g}=H$, and $\lambda(f)\neq\lambda(H)$. If there exist q distinct finite $a_j\in\mathbb{C}$ and integers $k_j(\geqslant j)$ such that*

$$\overline{E}_{k_j}(a_j,f)=\overline{E}_{k_j}(a_j,g),\ \ (j=1,2,\ldots,q),$$

then $f(z)\equiv g(z)$, where $q=4$; $q=5$; $q=6$ and $q\geqslant 7$.

By similar proof of Theorems 2 and 3, we can easily obtain the following corollary.

COROLLARY 2. *Let $f(z)$ and $g(z)$ be two transcedental meromorphic function in an angular domain $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\leqslant \arg z\leqslant\beta\}$ such that $\lambda(f)$ is finite, then $\frac{f}{g}=H$, and $\mu(f)=\lambda(f)\neq\lambda(H)$. If there exist q distinct finite $a_j\in\mathbb{C}$ and integers $k_j(\geqslant j)$ such that*

$$\overline{E}_{k_j}(a_j,f)=\overline{E}_{k_j}(a_j,g),\ \ (j=1,2,\ldots,q),$$

where

$$q>2+\sum_{j=1}^{q}\frac{1}{j+1},$$

then $f(z)\equiv g(z)$.

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