Journal of Classical Analysis
Volume 20, Number 1 (2022), 49–57
doi:10.7153/jca-2022-20-05

PATH CONNECTEDNESS OF VOLTERRA TYPE INTEGRAL OPERATORS ON BERGMAN AND DIRICHLET TYPE SPACES

CUI WANG

Abstract. Let $V(A^p, A^p)$ be the class of all bounded Volterra type integral operators acting on Bergman spaces. The paper studies the topological structure of $V(A^p, A^p)$. We obtained that it has the same (path) connected components, while it has no isolated point and no essentially isolated Volterra type integral operator. The same is true for Dirichet type spaces.

1. Preliminary and introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ denote the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ denotes the set of all holomorphic functions on the open disk $\mathbb{D}$.

We say the analytic functions f on $\mathbb{D}$ is a Bloch function if

$$\|f\|_* = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

The Bloch space is the set of all Bloch functions. The norm of $f \in \mathscr{B}$ is

$$\|f\|_{\mathscr{B}} = |f(0)| + \|f\|_*.$$

$\|\cdot\|_*$ is a complete semi-norm on $\mathscr{B}$, and $\mathscr{B}$ is a Banach space with the above norm. The little Bloch space $\mathscr{B}_0$ is the closed subspace of $\mathscr{B}$, a function $f \in \mathscr{B}$ belongs to $\mathscr{B}_0$ if

$$\lim_{|z|\to 1^-}(1-|z|^2)f'(z) = 0.$$

See [2], [22] for more introductions of Bloch spaces.

Let $0 < p < +\infty$, a function $f \in H(\mathbb{D})$ belongs to the Bergman spaces A^p, if

$$\int_{\mathbb{D}} |f(z)|^p dA(z) < \infty.$$

For $1 \leqslant p < \infty$, it is known that A^p is a Banach space with the norm

$$\|f\|_{A^p} = \left(\int_{\mathbb{D}} |f(z)|^p dA(z)\right)^{1/p},$$

Mathematics subject classification (2020): 30B40, 30D10.
Keywords and phrases: Path connected component, Volterra type integral, isolated, topological structures.

and $dA(z)$ denotes the area measure on $\mathbb{D}$ [7], [13], [26]; while for $0<p<1$, A^p is a complete metric space with the distance $d(f,g)=\|f-g\|_{A^p}^p$.

For $0<p<\infty$, it follows from [3], [15], [19] that an analytic function f belongs to A^p if and only if

$$\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^p dA(z)<\infty.$$

The Hardy space $H^p(0<p<+\infty)$ contains functions $f\in H(\mathbb{D})$ with

$$\|f\|_{H^p}^p=\sup_{0\leqslant r<1}\int_{\partial\mathbb{D}}|f(r\xi)|^p dm(\xi)<\infty,$$

where dm denotes the normalized Lebesgue measure on $\partial\mathbb{D}$.

The Dirichlet type space $\mathscr{D}_{p-1}^p$ $(0<p<\infty)$ consists of these analytic functions f for which

$$\|f\|_{\mathscr{D}_{p-1}^p}^p=|f(0)|^p+\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^{p-1}dA(z)<\infty.$$

If $p=2$, $\mathscr{D}_{p-1}^p$ equals to the Hardy space H^2; for the case $p>2$, $H^p\subset\mathscr{D}_{p-1}^p$, which comes from a classical result by Littlewood and Paley [12], [16]; otherwise, $\mathscr{D}_{p-1}^p\subset H^p$ [8], [23], [24].

For $a\in\mathbb{D}$, $\varphi_a(z)=\frac{a-z}{1-\bar{a}z}$ $(z\in\mathbb{D})$ is defined as the Möbius function, in which $\varphi_a:\mathbb{D}\to\mathbb{D}$ and $\varphi_a\circ\varphi_a(z)=z$, $\varphi_a^{-1}(z)=\varphi_a(z)$.

The class $F(p,q,s)$ is consisted of $f:\mathbb{D}\to\mathbb{C}$ for which

$$\|f\|_{F(p,q,s)}^p=\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s dA(z)<\infty,$$

with the parameters $0<p<\infty$, $-2<q<\infty$, $0<s<\infty$.

We say f belongs to $F_0(p,q,s)$ if

$$\lim_{|a|\to1^-}\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s dA(z)=0.$$

You can see [25] for more related theory of the space $F(p,q,s)$ and $F_0(p,q,s)$.

Suppose that $A\subseteq X$. A point $x_0\in X$ is called an accumulation point of A, if for every $\varepsilon>0$, there exists an $x\in A$ such that $0<d(x_0,x)<\varepsilon$; in other words, every punctured open ball $B(x_0,\varepsilon)\setminus x_0$ contains a point of A. A point $c\in A$ is called an isolated point of A if it is not an accumulation point of A [14].

Path connectedness is a topological property which is stronger than connectivity. Suppose X is a topological space. If there is a continuous road connecting them between any two points in X, then X is said to be path connected.

Firstly, the Volterra type integral operator V_g was studied by Pommerenke [20] in order to investigate the exponents of functions $f\in BMOA$, and at the same time, he demonstrated that V_g on H^2 is bounded if and only if $g\in BMOA$. Many researchers devoted great enthusiasm to the theory of Volterra type integral operators in the past decades. The operators have investigated quite widely on a variety of functional spaces over various domains. Aleman and Siskakis [3] have given a description of the boundedness and compactness of V_g on the Bergman spaces while Galanopoulos et al. [10], [11] determined to study the boundedness of V_g on the Dirichlet type spaces. Recently, Miihkinen, Pau, Perälä, and Wang [19] have completely demonstrated the boundedness of the Volterra type operators V_g acting from the weighted Bergman spaces A^p_α to the Hardy spaces H^q of the unit ball of $\mathbb{C}^n$ for $0<p,q<\infty$. Moreover, Qian and Hu [21] studied the Volterra integral operators and characterize the boundedness and compactness of embedding from Dirichlet-Morrey spaces into tent spaces. However, few researches do the topological structure of the space of the operators. Recent years, the compact difference structure of these operators has been studied in [18]. In this paper, we devote to give a description of the (path) connected components of the space of the operators acting on Bergman spaces and Dirichlet type spaces.

If $g\in H(\mathbb{D})$, Volterra-type integral operator V_g is defined by

$$V_g f(z)=\int_0^z f(w)g'(w)dw.$$

For $0<s<\infty$, the positive measure μ on $\mathbb{D}$ is called a bounded s-Carleson measure, if there exists a positive number C such that

$$\mu(S(I))\leqslant C|I|^s, \tag{1.1}$$

where $|I|$ denotes the arc length of a subarc I of $\partial\mathbb{D}$.

Define

$$S(I)=\{z\in\mathbb{D}:\frac{z}{|z|}\in I, 1-|z|\leqslant |I|/2\pi\}, \tag{1.2}$$

and it is called the Carleson box whose supremum takes over all subarcs I of $\partial\mathbb{D}$ with $|I|\leqslant 1$. Moreover, μ is a compact (or vanishing) s-Carleson measure, if

$$\frac{\mu(S(I))}{|I|^s}\to 0 \tag{1.3}$$

as $|I|\to 0$.

In addition, it is known that a positive measure μ on $\mathbb{D}$ is a bounded s-Carleson measure if and only if

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|\varphi_a'(z)|^s d\mu(z)<\infty, \tag{1.4}$$

and (1.1) and (1.4) are comparable. See [1], [11], [12] for the above results.

Carleson [4], [5] has stated that the injection map from the Hardy space H^p into the measure space $L^p(d\mu)$ is bounded if and only if the positive measure μ on $\mathbb{D}$ is a bounded Carleson measure. Later, the characterization was generalized by Duren [6].

Throughout this article, $C_1A\leqslant B\leqslant C_2A$ and $A\lesssim B$ (or $B\gtrsim A$) means that there is a constant C such that $A\leqslant CB$.

2. The main results

Hereafter, for convenience of notation, $V(A^p, A^p)$ denotes the space of all bounded Volterra type integral operators $V_g : A^p \to A^p$ endowed with the operator norm topology unless otherwise specified.

Inspired by Mengestie [17], we investigate the path connected components of the space $V(A^p, A^p)$ and $V(\mathscr{D}^p_{p-1}, \mathscr{D}^p_{p-1})$.

THEOREM 1. *Let $0 < p < \infty$ and $V_g : A^p \to A^p$ be a compact operator. Then V_g and $V_{g(0)}$ belong to the same path connected components of the space $V(A^p, A^p)$.*

The above theorem yields that when $0 < p < \infty$, the set of all compact Volterra type operators is path connected. Indeed, suppose V_{g_1} and V_{g_2} are two compact Volterra-type integral operators, we obtain that V_{g_1} and $V_{g_1(0)}$ belong to the same path connected components by employing Theorem 1, the same circumstance holds for V_{g_2} and $V_{g_2(0)}$, while $V_{g_1(0)}$ and $V_{g_2(0)}$ are zero operators. Hence the implication follows.

THEOREM 2. *Let $1 \leqslant p \leqslant 2$ and $V_g : \mathscr{D}^p_{p-1} \to \mathscr{D}^p_{p-1}$ be a compact operator. Then V_g and $V_{g(0)}$ belong to the same path connected components of the space $V(\mathscr{D}^p_{p-1}, \mathscr{D}^p_{p-1})$.*

Mengestie [17] recently had several work on the topology properties on the space of Volterra type integral operators acting on Fock spaces. When V_g acts on Bergman spaces and Dirichlet type spaces, the corresponding results are totally different.

THEOREM 3. *Let $0 < p < \infty$. Then $V(A^p, A^p)$ has no isolated point.*

THEOREM 4. *Let $1 \leqslant p \leqslant 2$. Then $V(\mathscr{D}^p_{p-1}, \mathscr{D}^p_{p-1})$ has no isolated point.*

We all know that essential norm topology is weaker than operator norm topology. Theorems 3 induces a natural question, whether there exists essentially isolated Volterra type integral operator in $V(A^p, A^p)$. Now, we are going to figure that out.

For two Banach spaces X_1 and X_2, the essential norm $\|V\|_e$ of a bounded linear operator $V : X_1 \to X_2$ is defined by

$$\|V\|_e = \inf_T \{\|V - T\|; T : X_1 \to X_2 \text{ is a compact operator}\}.$$

It is easy to see that, the operator V is compact if and only if its essential norm $\|V\|_e = 0$.

Now, we give the result on essentially isolated Volterra type integral operators.

THEOREM 5. *Let $0 < p < \infty$. Then there exists no essentially isolated Volterra-type integral operator in the space $V(A^p, A^p)$.*

Proof. Since there is no isolated point by Theorem 3, clearly there is no essentially isolated points (essential norm topology is weaker than operator norm topology).

We proof the theorem. □

When V_g acts on Dirichlet type spaces $\mathscr{D}^p_{p-1}$ for $1 \leqslant p \leqslant 2$, the conclusion is exactly the same as those provided above, we omit it.

3. Proof of the main results

LEMMA 1. *(i) Suppose that μ is a positive measure on the unit open disk $\mathbb{D}$. Then μ is a bounded q/p-Carleson measure if and only if there is a positive constant C such that*

$$\int_{\mathbb{D}} |f(z)|^q d\mu(z) \leqslant C\|f\|_{H^p}^q \tag{1.5}$$

for all analytic functions f in $\mathbb{D}$, where C depends only on p and q for $0 < p \leqslant q < \infty$.

(ii) Let $0 < p \leqslant 2$, then $\|f\|_{H^p} \lesssim \|f\|_{\mathscr{D}^p_{p-1}} + |f(0)|$ for $f \in H(\mathbb{D})$.

(iii) Let $t \in (0,1)$ and $0 < s < \infty$. Then

$$\sup_{|I| \leqslant 1-t} \frac{\mu(S(I))}{|I|^s} \leqslant 10^s \sup_{|a| \geqslant t} \int_{\mathbb{D}\setminus\Delta(0,t)} |\varphi_a'(z)|^s d\mu(z). \tag{1.6}$$

where μ is a positive measure on $\mathbb{D}$.

The above lemma follows from [15], which will be used in the proof of the main results.

Aleman and Siskakis [3] have characterized several aspects of Volterra type integral operators on Bergman spaces, from their work, we have the following result.

LEMMA 2. *Let $0 < p < \infty$ and $V_g : A^p \to A^p$ be a bounded operator. Then*

$$\|V_g\| \simeq \sup_{z \in \mathbb{D}} |g'(z)|(1-|z|^2).$$

LEMMA 3. *Let $1 \leqslant p \leqslant 2$ and $V_g : \mathscr{D}^p_{p-1} \to \mathscr{D}^p_{p-1}$ be a bounded operator. Then*

$$\|V_g\|^p \simeq \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |g'(z)|^p (1-|z|^2)^{p-2}(1-|\varphi_a(z)|^2) dA(z).$$

Lemma 3 can be found in Proposition 11 of [15].

Proof of Theorem 1. We first consider sequences of scaling functions $g_t(z) = g(tz)$, $t \in [0,1]$.

From [15, Corollary 7(i)], we know that $V_g : A^p \to A^p$ is compact if and only if $g \in \mathscr{B}_0$. Then for any $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that

$$(1-|z|^2)|g'(z)| < \varepsilon, \quad \delta^2 < |z| < 1.$$

If $\delta < t < 1$, $\delta < |z| < 1$, then $\delta^2 < t|z| < 1$ and hence

$$(1-|z|^2)|tg'(tz)| \leqslant (1-|z|^2)|g'(tz)| < \varepsilon, \quad \delta^2 < |tz| < 1.$$

Thus $g_t(z) \in \mathscr{B}_0$, and the assertion $V_{g_t} : A^p \to A^p$ is compact for all t follows with $V_g = V_{g_1}$ and $V_{g(0)} = V_{g_0}$.

We define an operator $T:[0,1]\to V(A^p,A^p)$ by $T(x)=V_{g_x}$. Then, to prove our results it suffices to show that for every x in $[0,1]$

$$\lim_{t\to x}\|V_{g_t}-V_{g_x}\|=0$$

for each fixed $t\in[0,1]$.

Let $\mathbb{D}_\delta=\{z\in\mathbb{D}:0\leqslant|z|\leqslant\delta\}$, $\mathbb{D}^c_\delta=\mathbb{D}\setminus\mathbb{D}_\delta$. Since $\mathscr{B}_0$ is a linear space, then $(g_t-g_x)\in\mathscr{B}_0$, thus

$$(1-|z|^2)|tg'(tz)-xg'(xz)|<\varepsilon,\ \ \delta<t,\ \ x<1,\quad \delta<|z|<1.$$

Applying the linearity of the integral, we have

$$\begin{aligned}\|V_{g_t}f-V_{g_x}f\|_{A^p}&\simeq\Big(\int_{\mathbb{D}}|f(z)|^p|g_t'(z)-g_x'(z)|^p(1-|z|^2)^p dA(z)\Big)^{\frac{1}{p}}\\&\lesssim\sup_{z\in\mathbb{D}}|g_t'(z)-g_x'(z)|(1-|z|^2)\|f\|_{A^p}\\&=\sup_{z\in\mathbb{D}}|tg'(tz)-xg'(xz)|(1-|z|^2)\|f\|_{A^p}\\&=\sup_{z\in\mathbb{D}_\delta}|tg'(tz)-xg'(xz)|(1-|z|^2)\|f\|_{A^p}\\&\quad+\sup_{z\in\mathbb{D}\setminus\mathbb{D}_\delta}|tg'(tz)-xg'(xz)|(1-|z|^2)\|f\|_{A^p}\\&<\varepsilon.\end{aligned}$$

The first term above approaches zero as $t\to x$ since $tg'(tz)\to xg'(xz)$ uniformly for $|z|\leqslant\delta$.

Since ε is arbitrary, we have

$$\lim_{t\to x}\|V_{g_t}-V_{g_x}\|=0.$$

We completes the proof of the theorem. $\square$

Proof of Theorem 2. Firstly, we consider function sequences $g_t(z)=g(tz)$, $t\in[0,1]$.

From [15, Corollary 13], we know that $V_g:\mathscr{D}^p_{p-1}\to\mathscr{D}^p_{p-1}$ is compact if and only if $g\in F_0(p,p-2,1)$. Then for any $\varepsilon>0$, there exists $\delta\in(0,1)$ such that

$$\int_{\mathbb{D}}|g'(z)|^p(1-|z|^2)^{p-2}(1-|\varphi_a(z)|^2)dA(z)<\varepsilon,\quad \delta<|a|<1.$$

Then, for $\delta<|a|<1$,

$$\begin{aligned}&\int_{\mathbb{D}}|(g_t(z))'|^p(1-|z|^2)^{p-2}(1-|\varphi_a(z)|^2)dA(z)\\&=\int_{\mathbb{D}}|tg'(tz)|^p(1-|z|^2)^{p-2}(1-|\varphi_a(z)|^2)dA(z)\\&\leqslant\int_{\mathbb{D}}|g'(tz)|^p(1-|z|^2)^{p-2}(1-|\varphi_a(z)|^2)dA(z)\\&<\varepsilon.\end{aligned}$$

Thus $g_t \in F_0(p,p-2,1)$ and $V_{g_t} : \mathscr{D}^p_{p-1} \to \mathscr{D}^p_{p-1}$ is compact for all t follows with $V_g = V_{g_1}$ and $V_{g(0)} = V_{g_0}$.

We define an operator $T : [0,1] \to V(\mathscr{D}^p_{p-1}, \mathscr{D}^p_{p-1})$ by $T(x) = V_{g_x}$. Then, to prove our results it is necessary to prove that for every x in $[0,1]$

$$\lim_{t\to x} \|V_{g_t} - V_{g_x}\| = 0$$

for each fixed $t \in [0,1]$.

By Lemma 1, and the fact that (1.1) and (1.4) are comparable,

$$\begin{aligned}
\|V_{g_t}f - V_{g_x}f\|^p_{\mathscr{D}^p_{p-1}} &\simeq \int_{\mathbb{D}} |f(z)|^p |g_t'(z) - g_x'(z)|^p (1-|z|^2)^{p-1} dA(z) \\
&\lesssim (\sup_I \frac{1}{|I|} \int_{S(I)} |tg'(tz) - xg'(xz)|^p (1-|z|^2)^{p-1} dA(z)) \|f\|^p_{H^p} \\
&\lesssim (\sup_I \frac{1}{|I|} \int_{S(I)} |tg'(tz) - xg'(xz)|^p (1-|z|^2)^{p-1} dA(z)) \\
&\quad \times (\|f\|_{\mathscr{D}^p_{p-1}} + |f(0)|)^p \\
&\lesssim (\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |tg'(tz) - xg'(xz)|^p (1-|z|^2)^{p-2} (1-|\varphi_a(z)|^2) dA(z)) \|f\|^p_{\mathscr{D}^p_{p-1}}.
\end{aligned}$$

Let $\mathbb{D}_r = \{z \in \mathbb{D} : 0 \leqslant |z| < r\}$, then $\mathbb{D}_r^c = \mathbb{D} \setminus \mathbb{D}_r$.

$$\begin{aligned}
&\int_{\mathbb{D}} |tg'(tz) - xg'(xz)|^p (1-|z|^2)^{p-2} (1-|\varphi_a(z)|^2) dA(z)) \\
&= \int_{\mathbb{D}_r} |tg'(tz) - xg'(xz)|^p (1-|z|^2)^{p-2} (1-|\varphi_a(z)|^2) dA(z) \\
&\quad + \int_{\mathbb{D}_r^c} |tg'(tz) - xg'(xz)|^p (1-|z|^2)^{p-2} (1-|\varphi_a(z)|^2) dA(z).
\end{aligned}$$

The first term above approaches zero as $t \to x$ since $tg'(tz) \to xg'(xz)$ uniformly for $|z| < r$. Since $g_t(z) \in F_0(p,p-2,1)$, we have

$$\int_{\mathbb{D}_r^c} |tg'(tz)|^p (1-|z|^2)^{p-2} (1-|\varphi_a(z)|^2) dA(z) < \varepsilon, \quad \delta < |a| < 1,$$

then

$$\int_{\mathbb{D}} |tg'(tz) - xg'(xz)|^p (1-|z|^2)^{p-2} (1-|\varphi_a(z)|^2) dA(z) < 2\varepsilon, \quad \delta < |a| < 1$$

with the fact that $F_0(p,p-2,1)$ is a linear space.

Since ε is arbitrary, then

$$\lim_{t\to x} \|V_{g_t} - V_{g_x}\| = 0$$

for each fixed $t \in [0,1]$. $\square$

Proof of Theorem 3. By the definition of accumulation points and Lemma 2, for any $\varepsilon > 0$, let $g_1(z) = g(z) + \varepsilon z$, then

$$\|V_g - V_{g_1}\| \simeq \sup_{z\in\mathbb{D}}(1-|z|^2)|\,(g(z)-g_1(z))'\,| < \varepsilon,$$

we complete it. □

Proof of Theorem 4. Similar to the proof of Theorem 3 and by Lemma 3, let $g_1(z) = g(z) + \varepsilon z$.

We get

$$\|V_g - V_{g_1}\|^p \simeq \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|\,(g(z)-g_1(z))'\,|^p(1-|z|^2)^{p-2}(1-|\varphi_a(z)|^2)dA(z) < \varepsilon$$

where $\varepsilon > 0$ is arbitrary, we prove the theorem. □

REFERENCES

[1] R. AULASKARI, D. STEGENGA, AND J. XIAO, *Some subclasses of BMOA and their characterization in terms of Carleson measures*, Rocky Mountain J. Math. **26** (1996), 485–506.

[2] K. ATTELE, *Interpolating sequences for the derivatives of Bloch functions*, Glasgow Math. J. **34** (1992) 35–41.

[3] A. ALEMAN, AND A. G. SISKAKIS, *Integration operators on Bergman spaces*, Indiana Univ. Math. J. **46** (1997), no. 2, 337–356.

[4] L. CARLESON, *An interpolation problem for bounded analytic functions*, Amer. J. Math. **80** (1958), 921–930.

[5] L. CARLESON, *Interpolations by bounded analytic functions and the corona problem*, Ann. of Math. (2) **76** (1962), 547–559.

[6] P. L. DUREN, *Extension of a Theorem of Carleson*, Bull. Amer. Math. Soc. **75** (1969), 143–146.

[7] P. L. DUREN AND A. SCHUSTER, *Bergman Spaces*, Math. Surveys Monogr. vol. **100**, Amer. Math. Soc. Providence, RI (2014).

[8] T. M. FLETT, *The dual of an inequality of Hardy and Littlewood and some related inequalitites*, J. Math. Anal. Appl. **38** (1972), 746–765.

[9] J. GARNETT, *Bounded analytic functions*, Academic Press, New York, 1981.

[10] P. GALANOPOULOS, D. GIRELA, J. PELAEZ, *Multipliers and integration operators on Dirichlet spaces*, Trans. Amer. Math. Soc. **363** (2011), no. 4, 1855–1886.

[11] D. GIRELA, J. PELAEZ, *Carleson measures, multipliers and integration operators for spaces of Dirichlet type*, J. Funct. Anal. **241** (2006), no. 1, 334–358.

[12] G. H. HARDY AND J. E. LITTLEWOOD, *Some more theorems concerning Fourier series and Fourier power series*, Duke Math. J. **2** (1936), 354–382.

[13] H. HEDENMALM, B. KORENBLUM, AND K. ZHU, *Theory of Bergman Spaces*, Grad. Texts in Math. vol. **199**, Springer, New York (2000).

[14] Y. HUANG, *Functional Analysis: An Introduction*, Science Press (2009).

[15] R. JOUNI, *Integration operator acting on Hardy and Weighted Bergman spaces*, Bull. Austral. Math. Soc. vol. **75** (2007), 431–446.

[16] D. H. LUECKING, *A new proof of an inequality of Littlewood and Paley*, Proc Amer. Math. Soc. **103** (1988), 887–893.

[17] T. MENGESTIE, *Path connected components of the space of Volterra-type integral operators*, Arch. Math. **111** (2018), 389–398.

[18] T. MENGESTIE AND M. WORKU, *Topological structures of generalized Volterra-type integral operators*, Mediterr. J. Math. **15** (2018), no. 2, Paper No. 42, 16 pp.

[19] S. MIIHKINEN, J. PAU, A. PERÄLÄ, AND M. WANG, *Volterra type integration operators from Bergman spaces to Hardy spaces*, J. Funct. Anal. (2020), 108564, 32 pp.
[20] CH. POMMERENKE, *Schlichte Funktionen und analytische Funktionen von beschränkter mittlerer Oszillation*, (German) Comment. Math. Helv. **52** (1977), no. 4, 591-602.
[21] R. QIAN, N. HU, *Embedding Dirichlet-Morrey Spaces $\mathscr{D}_p^\lambda$ into Tent Spaces and Volterra Integral Operator*, Iran. J. Sci. Technol. Trans. Sci. (2021) **45**: 1735–1741.
[22] M. TJANI, *Distance of a Bloch function to the little Bloch space*, Bull. Austral. Math. Soc. **74** (2006) 101–119.
[23] S. A. VINOGRADOV, *Multiplication and division in the space of analytic functions with an area-integrable derivative, and in some related spaces*, Issled. po Linein. Oper. iTeor. Funktsii. **23** (1995), 45–77. Translation in J. Math. Sci. (New York) 87 (1997), 3806–3827.
[24] Z. WU, *Carleson measures and multipliers for Dirichlet spaces*, J. Punc. Anal. **169** (1999), 148–163.
[25] R. ZHAO, *On a general family of function spaces*, Ann. Acad. Sci. Fenn. Math. Diss. **105**, 56 (1996).
[26] K. ZHU, *Operator Theory in Function Spaces*, Second Edition, Mathematical Surveys and Monographs, **138**, Amer. Math. Soc, Providence (2007).

(Received March 24, 2022)

Cui Wang
School of Mathematics and Statistics
Guangdong University of Technology
Guangzhou, Guangdong, 510520, P. R. China
e-mail: 2112014011@mail2.gdut.edu.cn