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ON THE SPEED OF APPROXIMATION IN THE CLASSES OF $\overline{\psi}$–INTEGRALS

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Abstract. The concept of the $\overline{\psi}$-integrals introduced by A. I. Stepanets who has brought a new perspective in the theory of Fourier series, especially in approximation theory. The main objective of this study is to get the speed of approximation to the functions of the class $\overline{\psi}$-integrals by generalized Zygmund sums, Woronoi-Nörlund and Riesz means, responding to the solution of the Kolmogorov-Nikol'skii problem under the uniform norm.

1. Introduction

Assume that $L := L(0,2\pi)$ denotes the space of functions that are 2π-periodic and Lebesgue integrable on $[0,2\pi]$ and let

$$S[f] = \frac{a_o}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) \equiv \sum_{k=0}^{\infty} A_k(f;x)$$

be the Fourier series of a function $f \in L$ where

$$a_k = a_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt; \quad \text{for } k = 0,1,2,\cdots,$$

$$b_k = b_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,dt; \quad \text{for } k = 1,2,\cdots.$$

It is known that $C_{\infty}^{\overline{\psi}}$ is class of 2π-periodic continuous functions which is expressed by

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \frac{1}{\pi}\int_{-\pi}^{\pi}\theta(x-t)\Psi(t)dt \\ &= \frac{a_0}{2} + (f^{\overline{\psi}} * \Psi)(x), \end{aligned}$$

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where $\Psi(x)$ is a certain function that has the Fourier series

$$\sum_{k=1}^{\infty}(\psi_1(k)\cos kx+\psi_2(k)\sin kx),$$

$\overline{\psi}=(\psi_1,\psi_2)$ is a pair of arbitrary fixed systems of numbers $\psi_1(k)$ and $\psi_2(k)$, $(k=1,2,\cdots)$ [10]. Here, the function θ is called $\overline{\psi}$-derivative of function f, and is denoted by $f^{\overline{\psi}}(\cdot)$, $\operatorname{ess\,sup}_t|\theta(t)|\leqslant 1$, $\int_{-\pi}^{\pi}\theta(t)dt=0$.

Note that if $\psi_1(v)=\psi(v)\cos\frac{\beta\pi}{2}$ and $\psi_2(v)=\psi(v)\sin\frac{\beta\pi}{2}$, then the classes $C_{\infty}^{\overline{\psi}}$ coincide with the classes $C_{\beta,\infty}^{\psi}$. Moreover, if $\psi(v)=v^{-r}$, then the classes $C_{\infty}^{\overline{\psi}}$ coincide with the classes $W_{\beta,\infty}^{r}$-Weil-Nagy [10].

We are essentially interested in asymptotic equalities for the quantities

$$\mathscr{E}_n(\mathfrak{N},U_n(f))=\sup_{f\in\mathfrak{N}}\|f-U_n(f)\|_X \tag{1}$$

that realize solutions of the corresponding Kolmogorov-Nikol'skii problem. For a given method $U_n(f;\lambda)$ on the class $\mathfrak{N}$ in the space X, this problem is solved if the function $\xi(n)=\xi(n,\lambda;\mathfrak{N})$ is determined in obvious form such that

$$\mathscr{E}_n(\mathfrak{N},U_n(f;\lambda))=\sup_{f\in\mathfrak{N}}\|f(x)-U_n(f;x;\lambda)\|_X=\xi(n)+O(\xi(n))$$

as $n\to\infty$, where $\lambda=||\lambda_k^{(n)}||$ is a triangular matrix [10].

The value $\mathscr{E}_n(\mathfrak{N},U_n(f;\lambda))$ is examined for different $U_n(f;\lambda)$ methods in various spaces. Especially, some evaluations have been obtained in various subclasses of continuous functions space according to the $Z_n^s(f;x)$ method which has an important place in this study, where

$$Z_n^s(f;x)=\frac{a_0}{2}+\sum_{k=1}^{n-1}\left(1-\left(\frac{k}{n}\right)^s\right)A_k(f;x),\quad s>0$$

are Zygmund sums and $Z_n^1(f;x)=\sigma_n(f;x)$ are known as Fejér sums.

There are many studies focusing on the value $\mathscr{E}_n(\mathfrak{N},Z_n^s)_C$. Some of these studies have been conducted by A. Zygmund [13] in case of $\mathfrak{N}=W_{\infty}^r$, $r>0$; B. Nagy and S. A. Teljakovskií [8], [11] in case of $\mathfrak{N}=W_{\beta,\infty}^r$ under different conditions on β,s,r; A. I. Stepanets [10], D. N. Bushev [1], in case of $\mathfrak{N}=C_{\beta,\infty}^{\psi}$ under the conditions on function $\psi(\cdot)$; A. S. Fedorenko [4, 5] and U. Değer [2, 3] in case of $\mathfrak{N}=C_{\infty}^{\overline{\psi}}$ under different conditions on functions $\psi_1(\cdot)$ and $\psi_2(\cdot)$.

In this study, the results about the speed of approximation which gives the solution of (1) in case of $\mathfrak{N}=C_{\infty}^{\overline{\psi}}$ for Riesz and Woronoi-Nörlund methods will be given under some certain conditions. Let us remember these methods. The polynomials that have the form

$$R_n(f;x)=\frac{1}{P_n}\sum_{m=0}^{n}p_m s_m(f;x),$$

and

$$N_n(f;x) = \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m} s_m(f;x),$$

are called the Riesz means and Woronoi-Nörlund means [12], respectively, where $s_n(f;x)$ is the nth partial sum of Fourier series of a function $f \in L$ and

$$P_n = \sum_{k=0}^{n} p_k \neq 0 \ \ (n \geqslant 0)$$

by $p_{-1} = P_{-1} = 0$.

The polynomials that have the form

$$Z_n^{\varphi}(f;x) = \frac{a_0}{2} + \sum_{k=1}^{n-1} \left(1 - \frac{\varphi(k)}{\varphi(n)}\right)(a_k \cos kx + b_k \sin kx), \quad n \in \mathbb{N},$$

have been introduced in [6], [7] and called the generalized Zygmund sums, where $\varphi(k)$ are the values of a certain function $\varphi \in F$ at integer points, and F is the set of all continuous functions $\varphi(u)$ monotonically increasing to infinity on $[1,\infty)$. On the other hand, let F^+ show the class of functions that belong to F and satisfy the conditions $\varphi(u) \geqslant 0$, $u \geqslant 0$, such that $\varphi(0) = 0$ and $\varphi(u)$ is a function which is concave or convex on $[0,n]$ for any $n = 2,3,\ldots$. It is obvious that if $\varphi(t) = t^s$, $s > 0$, then $\varphi \in F^+$ and $Z_n^{\varphi}(f;x)$ coincide with the classical Zygmund sums.

According to [9], we know that the necessary and sufficient condition for the uniform convergence of the polynomials $Z_n^{\varphi}(f;x)$ to the function $f(x)$ in the complete space C is given in the following proposition, where C is the space of 2π-periodic continuous functions $f(t)$ with the norm $\|f\|_C = \max_t |f(t)|$.

PROPOSITION 1. *Let* $\varphi \in F^+$. *Then the condition*

$$\frac{1}{\varphi(n)} \sum_{k=1}^{n-1} \frac{\varphi(n) - \varphi(k)}{n-k} \leqslant K$$

is necessary and sufficient for the uniform convergence of the polynomials $Z_n^{\varphi}(f;x)$ *to the function* $f(x)$ *from* $C[0,\pi]$ *space.*

Also we note that the method Z_n^{φ} generated by a positive φ is saturated in the space C with saturation order $\frac{1}{\varphi(n)}$ [9].

The values

$$\mathscr{E}_n(C_\infty^{\overline{\psi}}, R_n)_C = \sup_{f \in C_\infty^{\overline{\psi}}} \|f(x) - R_n(f;x)\|_C, \tag{2}$$

$$\mathscr{E}_n(C_\infty^{\overline{\psi}}, N_n)_C = \sup_{f \in C_\infty^{\overline{\psi}}} \|f(x) - N_n(f;x)\|_C \tag{3}$$

and

$$\mathscr{E}_n(C_\infty^{\overline{\psi}}, Z_n^{\varphi})_C = \sup_{f \in C_\infty^{\overline{\psi}}} \|f(x) - Z_n^{\varphi}(f;x)\|_C \tag{4}$$

are the main subject of our study that aims at obtaining asymptotic equalities under different conditions on functions $\varphi(\cdot)$, $\psi_1(\cdot)$ and $\psi_2(\cdot)$, where $\|\rho\|_C = \max_x |\rho(x)|$.

The values of (2) and (3) depend on the functions $g_i(v) = v^s \psi_i(v)$, $i = 1,2$, (for the value (4): $g_i(v) = \varphi(v)\psi_i(v)$) which are convex or concave on $v \geqslant b \geqslant 1$. The functions $g_i(v)$, $i = 1,2$ have five probable cases:

a) $g_i(v)$ are convex functions with $\lim\limits_{v\to\infty} g_i(v) = \infty$,

b) $g_i(v)$ are convex functions with $\lim\limits_{v\to\infty} g_i(v) = C > 0$,

c) $g_i(v)$ are convex functions with $\lim\limits_{v\to\infty} g_i(v) = 0$,

d) $g_i(v)$ are concave functions with $\lim\limits_{v\to\infty} g_i(v) = c > 0$,

e) $g_i(v)$ are concave functions with $\lim\limits_{v\to\infty} g_i(v) = \infty$.

The problem will only be addressed for the states d) and e). Throughout this paper, $\mathfrak{M}$ denotes the set of continuous positive functions $\psi(t)$ which is convex downward for $t \geqslant 1$ and satisfying the conditions $\lim\limits_{t\to\infty} \psi(t) = 0$, i.e., for $\Delta(\psi, t_1, t_2) = \psi(t_1) - 2\psi(\frac{t_1+t_2}{2}) + \psi(t_2)$,

$$\mathfrak{M} = \left\{ \psi(t), t \geqslant 1 : \psi(t) > 0, \Delta(\psi, t_1, t_2) \geqslant 0, \forall t_1, t_2 \in [1,\infty), \lim_{t\to\infty} \psi(t) = 0 \right\}.$$

$\mathfrak{M}'$ shows the subset of functions $\psi(\cdot)$ from $\mathfrak{M}$ that satisfies the following condition:

$$\int_1^\infty \frac{\psi(t)}{t} dt < \infty.$$

On the other hand $\mathfrak{M}''$ denotes the subset of functions from $\mathfrak{M}'$ that $x \int_x^\infty \frac{\psi(v)}{v} dv$ is increasing for $x \geqslant 1$.

We also set

$$\mathfrak{M}_0 = \{\psi \in \mathfrak{M} : 0 < \kappa(\psi, t) \leqslant K < \infty, \forall t \geqslant 1\},$$

where

$$\kappa(\psi, t) = \frac{t}{\zeta(\psi, t) - t},$$

$$\zeta(\psi, t) = \psi^{-1}\left(\frac{\psi(t)}{2}\right),$$

$\psi^{-1}(\cdot)$ is the inverse of function $\psi(\cdot)$, and the constant K may depend on the function ψ.

2. Solution of the Kolmogorov-Nikols'kii problem by the generalized Zygmund sums

In this section we will give some main results with respect to the generalized Zygmund sums for the states d) and e). Throughout this paper, $O(1)$ indicates a properly bounded identity with respect to n and $\overline{\psi}(n) = (\psi_1^2(n) + \psi_2^2(n))^{1/2}$.

THEOREM 1. *Suppose that $\varphi \in F^+$ and is a concave function, $\psi_1 \in \mathfrak{M}_0$, $\psi_2 \in \mathfrak{M}'$ and $g_i(v) = \varphi(v)\psi_i(v)$, $i = 1,2$, are concave functions on $v \geqslant b \geqslant 1$ with $\lim\limits_{v\to\infty} g_i(v) = \infty$ or $\lim\limits_{v\to\infty} g_i(v) = c > 0$. Then as $n \to \infty$, we get*

$$\mathscr{E}_n(C_\infty^{\overline{\psi}}, Z_n^{\varphi})_C = \frac{2}{\pi\varphi(n)}\int_1^n \frac{\varphi(v)\psi_2(v)}{v}dv + \frac{2}{\pi}\int_n^\infty \frac{\psi_2(v)}{v}dv + O(1)\overline{\psi}(n). \tag{5}$$

Before giving the proof of Theorem 1, we need to prove the next propositions.

PROPOSITION 2. *Let $\varphi \in F^+$, $\psi_1 \in \mathfrak{M}_0$ and $g_1(v) = \varphi(v)\psi_1(v)$ be concave function on $v \geqslant b \geqslant 1$ with $\lim\limits_{v\to\infty} g_1(v) = \infty$ or $\lim\limits_{v\to\infty} g_1(v) = c > 0$. Then as $n \to \infty$, we have*

$$\int_{-\infty}^{\infty}\left|\frac{1}{\pi}\int_0^\infty \tau_1(v)\cos vt\,dv\right|dt = O(1)\psi_1(n), \tag{6}$$

where

$$\tau_1(v) = \begin{cases} \dfrac{\varphi(v)\psi_1(1)}{\varphi(n)}, & 0 \leqslant v \leqslant 1 \\ \dfrac{\varphi(v)\psi_1(v)}{\varphi(n)}, & 1 \leqslant v \leqslant n \\ \psi_1(v), & v \geqslant n \end{cases}.$$

PROPOSITION 3. *Assume that $\varphi \in F^+$, $\psi_2 \in \mathfrak{M}'$ and $g_2(v) = \varphi(v)\psi_2(v)$ is concave function on $v \geqslant b \geqslant 1$ with $\lim\limits_{v\to\infty} g_2(v) = \infty$ or $\lim\limits_{v\to\infty} g_2(v) = c > 0$. Then as $n \to \infty$, we have*

$$\int_{-\infty}^{\infty}\left|\frac{1}{\pi}\int_0^\infty \tau_2(v)\sin vt\,dv\right|dt = \frac{2}{\pi\varphi(n)}\int_1^n \frac{\varphi(v)\psi_2(v)}{v}dv + \frac{2}{\pi}\int_n^\infty \frac{\psi_2(v)}{v}dv + O(1)\psi_2(n), \tag{7}$$

where

$$\tau_2(v) = \begin{cases} \dfrac{\varphi(v)\psi_2(1)}{\varphi(n)}, & 0 \leqslant v \leqslant 1 \\ \dfrac{\varphi(v)\psi_2(v)}{\varphi(n)}, & 1 \leqslant v \leqslant n \\ \psi_2(v), & v \geqslant n \end{cases}.$$

Proof of Proposition 2. By partial integration, we have

$$\int_0^\infty \tau_1(v)\cos vt\,dv = \frac{1}{t}\int_0^\infty (-\tau_1'(v))\sin vt\,dv.$$

Hence, we can write

$$\int_{-\infty}^{\infty}\left|\frac{1}{\pi}\int_0^\infty \tau_1(v)\cos vt\,dv\right|dt = 2\int_0^\infty\left|\frac{1}{\pi}\int_0^\infty \tau_1(v)\cos vt\,dv\right|dt$$

$$\leqslant 2\int_0^\infty\left|\frac{1}{\pi t}\int_0^n (-\tau_1'(v))\sin vt\,dv\right|dt + 2\int_0^\infty\left|\frac{1}{\pi t}\int_n^\infty (-\tau_1'(v))\sin vt\,dv\right|dt. \tag{8}$$

Now let us estimate first integral on the right side of inequality (8):

$$2\int_0^\infty\left|\frac{1}{\pi t}\int_0^n (-\tau_1'(v))\sin vt\,dv\right|dt$$

$$\leqslant 2\int_0^\infty\left|\frac{1}{\pi t}\int_0^1 \tau_1'(v)\sin vt\,dv\right|dt + 2\int_0^\infty\left|\frac{1}{\pi t}\int_1^n \tau_1'(v)\sin vt\,dv\right|dt. \tag{9}$$

Since the function $\tau_1'(v)$ is a continuous, nonnegative and nonincreasing function on interval $[0,1]$ for all $t \geqslant 0$, the following inequality is provided:

$$\frac{1}{t}\int_0^1 \tau_1'(v)\sin vt\,dv > 0. \tag{10}$$

For the first integral on the right side of inequality (9), if we consider the statement of (10) and change the order of integration, we obtain

$$\frac{2}{\pi}\int_0^\infty\left|\frac{1}{t}\int_0^1 \tau_1'(v)\sin vt\,dv\right|dt = \frac{2}{\pi}\int_0^1 \tau_1'(v)\int_0^\infty \frac{\sin vt}{t}\,dt\,dv = O(1)\psi_1(n). \tag{11}$$

Let us estimate the second integral on right side of (9):

$$\frac{2}{\pi}\int_0^\infty\left|\frac{1}{t}\int_1^n \tau_1'(v)\sin vt\,dv\right|dt$$

$$\leqslant \frac{2}{\pi}\int_0^\pi\left|\frac{1}{t}\int_1^n \tau_1'(v)\sin vt\,dv\right|dt + \frac{2}{\pi}\int_\pi^\infty\left|\frac{1}{t}\int_1^n \tau_1'(v)\sin vt\,dv\right|dt$$

$$= \frac{2}{\pi}\int_0^\pi |J_1|\,dt + \frac{2}{\pi}\int_\pi^\infty |J_1|\,dt.$$

Then we can write the following equality for J_1:

$$\begin{aligned} J_1 &= \frac{1}{t}\int_1^n \tau_1^{'}(v)\sin vt\,dv = \frac{1}{t}\int_1^{\pi/t} \tau_1^{'}(v)\sin vt\,dv + \frac{1}{t}\int_{\pi/t}^{n} \tau_1^{'}(v)\sin vt\,dv \\ &= J_{11}+J_{12}. \end{aligned}$$

Hence, for $0 \leqslant t \leqslant \pi$ and $1 \leqslant v \leqslant \frac{\pi}{t}$, $J_{11} \geqslant 0$, and for $0 \leqslant t \leqslant \pi$ and $\frac{\pi}{t} \leqslant v \leqslant n$, $J_{12} \leqslant 0$ because $\tau_1^{'}(v)$ is nonnegative and nonincreasing on $[1,n]$. If we consider $J_1 = J_{11}+J_{12}$, we can write

$$\frac{2}{\pi}\int_0^{\pi} |J_1|dt \leqslant \frac{2}{\pi}\int_0^{\pi} |J_{11}|dt + \frac{2}{\pi}\int_0^{\pi} |J_{12}|dt. \tag{12}$$

Firstly, we will estimate the first integral on the right side of (12):

$$\begin{aligned} \frac{2}{\pi}\int_0^{\pi} |J_{11}|dt &= \frac{2}{\pi}\int_0^{\pi}\frac{1}{t}\int_1^{\pi/t} \tau_1^{'}(v)\sin vt\,dv\,dt = \frac{2}{\pi}\int_1^{\infty}\tau_1^{'}(v)\int_0^{\pi/v}\frac{\sin vt}{t}dt\,dv \\ &= \frac{2}{\pi}\int_1^{\infty}\tau_1^{'}(v)\int_0^{\pi}\frac{\sin u}{u}du\,dv = O(1)\psi_1(n). \end{aligned}$$

Therefore, we get

$$\frac{2}{\pi}\int_0^{\pi} |J_{11}|dt = O(1)\psi_1(n). \tag{13}$$

Now let us estimate the second integral on the rigt side of (12):

$$\begin{aligned} \frac{2}{\pi}\int_0^{\pi} |J_{12}|dt &= -\frac{2}{\pi}\int_0^{\pi}\frac{1}{t}\int_{\pi/t}^{n} \tau_1^{'}(v)\sin vt\,dv\,dt = -\frac{2}{\pi}\int_1^{n}\tau_1^{'}(v)\int_{\pi/v}^{\pi}\frac{\sin vt}{t}dt\,dv \\ &= -\frac{2}{\pi}\int_1^{n}\tau_1^{'}(v)\int_{\pi}^{\pi v}\frac{\sin u}{u}du\,dv = O(1)\psi_1(n). \end{aligned}$$

Hence, we have

$$\frac{2}{\pi}\int_0^{\pi} |J_{12}|dt = O(1)\psi_1(n). \tag{14}$$

Owing to (13) and (14), we obtain

$$\frac{2}{\pi}\int_0^{\pi} |J_1|dt = O(1)\psi_1(n). \tag{15}$$

Now we will estimate that

$$\frac{2}{\pi}\int_{\pi}^{\infty}|J_1|dt = O(1)\psi_1(n).$$

For this purpose, we take into account the function

$$\phi_x(t) = \int_{x}^{n}\mu(v)\sin vt\,dv,\ \ x>0,\ \ t>0, \tag{16}$$

where $\mu(v)$ is nonnegative and nonincreasing function for all $v \geqslant 1$. The function $\phi_x(t)$ is a continuous function for every fixed t. Further, on each interval between the successive zeros v_k and v_{k+1} of the function $\sin vt$, the function $\phi_x(t)$ has one simple zero x_k [p. 227, [10]]. Therefore let's suppose that $x_k^{'}$ is zero the nearest from the right of the point 1. In view of this, if we set $\mu(v) = \tau_1^{'}(v)$ on interval $[1,n]$ in (16), we have

$$J_1 = \frac{1}{t}\int_{1}^{x_k^{'}}\tau_1^{'}(v)\sin vt\,dv.$$

Hence the following result is obtained:

$$\frac{2}{\pi}\int_{\pi}^{\infty}|J_1|dt \leqslant \frac{2}{\pi}\int_{\pi}^{\infty}\frac{1}{t}\int_{1}^{1+\frac{2\pi}{t}}|\tau_1^{'}(v)|dv dt \leqslant \frac{2}{\pi}\int_{\pi}^{\infty}\frac{2\pi}{t^2}|\tau_1^{'}(1)|dt = O(1)\psi_1(n)$$
$$\frac{2}{\pi}\int_{\pi}^{\infty}|J_1|dt = O(1)\psi_1(n).$$

Therefore from (15) and (2), we get

$$\frac{2}{\pi}\int_{0}^{\infty}\left|\frac{1}{t}\int_{1}^{n}\tau_1^{'}(v)\sin vt\,dv\right|dt = O(1)\psi_1(n).$$

Thus for the first integral on the right side of (8) we get that

$$2\int_{0}^{\infty}\left|\frac{1}{\pi t}\int_{0}^{n}\tau_1^{'}(v)\sin vt\,dv\right|dt = O(1)\psi_1(n). \tag{17}$$

Now we will show that

$$2\int_{0}^{\infty}\left|\frac{1}{\pi t}\int_{n}^{\infty}\tau_1^{'}(v)\sin vt\,dv\right|dt = O(1)\psi_1(n). \tag{18}$$

Firstly by partial integration we have

$$\frac{1}{t}\int_{n}^{\infty}\tau_1'(v)\sin vt\,dv=\frac{1}{t^2}\Big[-\tau_1'(n+0)\cos nt-\int_{n}^{\infty}(-\tau_1''(v))\cos vt\,dv\Big].$$

We know that $\tau_1''(v)>0$. Then we get

$$\begin{aligned}\Big|\frac{1}{t}\int_{n}^{\infty}\tau_1'(v)\sin vt\,dv\Big| &\leqslant \frac{1}{t^2}\Big[|\tau_1'(n+0)\cos nt|+\Big|\int_{n}^{\infty}\tau_1''(v)\cos vt\,dv\Big|\Big]\\ &\leqslant \frac{1}{t^2}[|\psi_1'(n)|+|\psi_1'(n)|]=\frac{2}{t^2}|\psi_1'(n)|.\end{aligned}$$

Hence since $\psi_1\in\mathfrak{M}_0$, we obtain

$$\frac{1}{\pi}\int_{t\geqslant\frac{1}{n}}\Big|\frac{1}{t}\int_{n}^{\infty}\tau_1'(v)\sin vt\,dv\Big|dt\leqslant\frac{1}{\pi}\int_{t\geqslant\frac{1}{n}}\frac{2}{t^2}|\psi_1'(n)|dt=O(1)\psi_1(n). \tag{19}$$

After this estimation we will show that

$$\frac{1}{\pi}\int_{t\leqslant\frac{1}{n}}\Big|\frac{1}{t}\int_{n}^{\infty}\tau_1'(v)\sin vt\,dv\Big|dt=O(1)\psi_1(n).$$

By partial integration, we obtain

$$\int_{n}^{\infty}\tau_1(v)\cos vt\,dv=-\psi_1(n)\frac{\sin nt}{t}-\frac{1}{t}\int_{n}^{\infty}\tau_1'(v)\sin vt\,dv.$$

$$\Big|\frac{1}{t}\int_{n}^{\infty}\tau_1'(v)\sin vt\,dv\Big|\leqslant\psi_1(n)\Big|\frac{\sin nt}{t}\Big|+\Big|\int_{n}^{\infty}\tau_1(v)\cos vt\,dv\Big|.$$

From here, we have

$$\frac{1}{\pi}\int_{t\leqslant\frac{1}{n}}\Big|\frac{1}{t}\int_{n}^{\infty}\tau_1'(v)\sin vt\,dv\Big|dt\leqslant\frac{2}{\pi}\psi_1(n)\int_{0}^{\frac{1}{n}}\Big|\frac{\sin nt}{t}\Big|dt+\frac{2}{\pi}\int_{0}^{\frac{1}{n}}\Big|\int_{n}^{\infty}\tau_1(v)\cos vt\,dv\Big|dt.$$

$\int_{0}^{\frac{1}{n}}\Big|\frac{\sin nt}{t}\Big|dt\leqslant K_1$ and owing to similar estimation of the integral in [3] we know that

$$\frac{2}{\pi}\int_{0}^{\frac{1}{n}}\Big|\int_{n}^{\infty}\tau_1(v)\cos vt\,dv\Big|dt=O(1)\psi_1(n).$$

Thus we find that

$$\int\limits_{t\leqslant\frac{1}{n}} \left|\frac{1}{\pi t}\int\limits_{n}^{\infty}\tau_1'(v)\sin vt\,dv\right|dt = O(1)\psi_1(n).$$

Therefore, the proof of proposition is completed. □

Proof of Proposition 3. $\tau_2(v)$ is nonnegative continuous function on interval $[0,\infty)$ and is increasing on intervals $[0,n]$ and

$$\lim_{v\to\infty}\tau_2(v) = \lim_{v\to\infty}\tau_2'(v) = 0.$$

By applying two times partial integration, we have

$$\frac{1}{\pi}\int\limits_0^\infty \tau_2(v)\sin vt\,dv$$

$$= \frac{1}{\pi t^2}\Big[(\tau_2'(1-0)-\tau_2'(1+0))\sin t + (\tau_2'(n-0)-\tau_2'(n+0))\sin nt$$
$$-\Big(\int\limits_0^1 \tau_2''(v)\sin vt\,dv + \int\limits_1^n \tau_2''(v)\sin vt\,dv + \int\limits_n^\infty \tau_2''(v)\sin vt\,dv\Big)\Big]. \tag{20}$$

From (20), since $g_2(v) = \varphi(v)\psi_2(v)$ is increasing, we get

$$\left|\frac{1}{\pi}\int\limits_0^\infty \tau_2(v)\sin vt\,dv\right| \leqslant \frac{-2\psi_2'(n)}{\pi t^2}. \tag{21}$$

Hence, accordingly (21) we obtain

$$\int\limits_{|t|\geqslant\pi/2}\left|\frac{1}{\pi}\int\limits_0^\infty \tau_2(v)\sin vt\,dv\right|dt = 2\int\limits_{\pi/2}^\infty\left|\frac{1}{\pi}\int\limits_0^\infty \tau_2(v)\sin vt\,dv\right|dt \leqslant \int\limits_{\pi/2}^\infty \frac{-2\psi_2'(n)}{\pi t^2}dt$$
$$= \frac{-8\psi_2'(n)}{\pi^2} = O(1)\psi_2(n). \tag{22}$$

By partial integration, we have

$$\frac{1}{\pi}\int\limits_0^\infty \tau_2(v)\sin vt\,dv = \frac{1}{\pi}\int\limits_0^n \tau_2(v)\sin vt\,dv + \frac{1}{\pi}\int\limits_n^\infty \tau_2(v)\sin vt\,dv$$

$$= \frac{1}{\pi t}\Big(-\tau_2(v)\cos vt|_0^n + \int\limits_0^n \tau_2'(v)\cos vt\,dv\Big) + \Big(-\tau_2(v)\cos vt|_n^\infty + \int\limits_n^\infty \tau_2'(v)\cos vt\,dv\Big)$$

$$= \frac{1}{\pi t}\Big(\int\limits_0^n \tau_2'(v)\cos vt\,dv + \int\limits_n^\infty \tau_2'(v)\cos vt\,dv\Big). \tag{23}$$

To estimate the integral on right hand of (23), first of all we will consider integrals on $[0,n]$ and show that

$$\int_{\pi/2n}^{\pi/2}\left|\frac{1}{t}\int_0^n \tau_2'(v)\cos vt\,dv\right|dt=\frac{1}{\varphi(n)}\int_1^n\frac{\varphi(v)\psi_2(v)}{v}dv+O(1)\psi_2(n). \tag{24}$$

In this case, let us represent the function under the integral sign in left part of (24) in such form:

$$\begin{aligned}\frac{1}{t}\int_0^n \tau_2'(v)\cos vt\,dv &= \frac{1}{t}\int_0^{\pi/2t}\tau_2'(v)\cos vt\,dv+\frac{1}{t}\int_{\pi/2t}^{n}\tau_2'(v)\cos vt\,dv\\ &:= I_1(t)+I_2(t).\end{aligned} \tag{25}$$

In order to prove (24) it will suffice to prove following equalities

$$\int_{\pi/2n}^{\pi/2}|I_1(t)|dt=\frac{1}{\varphi(n)}\int_1^n\frac{\varphi(v)\psi_2(v)}{v}dv+O(1)\psi_2(n) \tag{26}$$

and

$$\int_{\pi/2n}^{\pi/2}|I_2(t)|dt\leqslant O(1)\psi_2(n). \tag{27}$$

As the function $\tau_2'(v)$ is nonnegative and nonincreasing on $[0,n]$, then $I_1(t)\geqslant 0$, $t\in[\pi/2n,\pi/2]$, and $I_2(t)\leqslant 0$, $t\in[\pi/2n,\pi/2]$. Therefore changing the order of integration, we obtain

$$\begin{aligned}\int_{\pi/2n}^{\pi/2}|I_1(t)|dt &= \int_{\pi/2n}^{\pi/2}\frac{1}{t}\int_0^{\pi/2t}\tau_2'(v)\cos vt\,dv\,dt\\ &=\int_0^1\int_{\pi/2n}^{\pi/2}\frac{\cos vt}{t}dt\,d\tau_2(v)+\int_1^n\int_{\pi/2n}^{\pi/2v}\frac{\cos vt}{t}dt\,d\tau_2(v)\\ &=\int_0^1\int_{\pi v/2n}^{\pi v/2}\frac{\cos z}{z}dz\,d\tau_2(v)+\int_1^n\int_{\pi v/2n}^{\pi/2}\frac{\cos z}{z}dz\,d\tau_2(v)\end{aligned}$$

$$= \left(\tau_2(v)\int\limits_{\pi v/2n}^{\pi v/2}\frac{\cos z}{z}dz\right)\Big|_0^1 - \int\limits_0^1 \frac{\tau_2(v)}{v}\left(\cos\frac{\pi v}{2} - \cos\frac{\pi v}{2n}\right)dv$$

$$+\left(\tau_2(v)\int\limits_{\pi v/2n}^{\pi/2}\frac{\cos z}{z}dz\right)\Big|_1^n + \int\limits_1^n \frac{\tau_2(v)}{v}\cos\frac{\pi v}{2n}dv$$

$$= \frac{1}{\varphi(n)}\int\limits_1^n \frac{\varphi(v)\psi_2(v)}{v}\cos\frac{\pi v}{2n}dv + O(1)\psi_2(n). \tag{28}$$

Now let's show that

$$\frac{1}{\varphi(n)}\int\limits_1^n \frac{\varphi(v)\psi_2(v)}{v}\cos\frac{\pi v}{2n}dv = \frac{1}{\varphi(n)}\int\limits_1^n \frac{\varphi(v)\psi_2(v)}{v}dv + O(1)\psi_2(n). \tag{29}$$

For proof of (29) we will obtain necessary estimation of following difference

$$\frac{1}{\varphi(n)}\int\limits_1^n \frac{\varphi(v)\psi_2(v)}{v}\left(1-\cos\frac{\pi v}{2n}\right)dv$$

$$= \frac{2}{\varphi(n)}\int\limits_1^n \frac{\varphi(v)\psi_2(v)}{v}v\frac{\sin \pi v/4n}{\pi v/4n}\frac{\pi}{4n}\sin\frac{\pi v}{4n}dv$$

$$\leqslant \frac{2\varphi(n)\psi_2(n)}{\varphi(n)}\frac{\pi}{4n}\int\limits_1^n \sin\frac{\pi v}{4n}dv \leqslant O(1)\psi_2(n).$$

Hence by combining (28) and (29), we get (26). Now we will obtain (27). Since $I_2(t)\leqslant 0, t\in[\pi/2n,\pi/2]$, then we have

$$\int\limits_{\pi/2n}^{\pi/2}|I_2(t)|dt = -\int\limits_{\pi/2n}^{\pi/2}\frac{1}{t}\int\limits_{\pi/2t}^{n}\tau_2'(v)\cos vt\,dv$$

$$= -\int\limits_1^n \tau_2'(v)\int\limits_{\pi/2v}^{\pi/2}\frac{\cos vt}{t}dtdv = -\int\limits_1^n \tau_2'(v)\int\limits_{\pi/2}^{\pi v/2}\frac{\cos z}{z}dzdv$$

$$\leqslant 2ci\left(\frac{\pi}{2}\right)\int\limits_1^n \tau_2'(v)dv = 2ci\left(\frac{\pi}{2}\right)\left(\psi_2(n) - \frac{\varphi(1)\psi_2(1)}{\varphi(n)}\right) \leqslant O(1)\psi_2(n). \tag{30}$$

Therefore taking into account (26) and (30), we get (24). Now we will estimate integral on interval $[n,\infty)$ on right hand of (23). Taking into account that $\tau_2(v)$ is a convex function with $\lim_{v\to\infty}\tau_2(v)=0$ on interval $[n,\infty)$, by partial integration, we get

$$\frac{1}{\pi t}\Big|\int_n^\infty \tau_2'(v)\cos vt\,dv\Big| = \frac{1}{\pi t^2}\Big|\big(\tau_2'(v)\sin vt\big|_n^\infty\big)-\int_n^\infty \tau_2''(v)\sin vt\,dv\Big|$$
$$\leqslant \frac{2}{\pi n t^2}\Big(-\frac{n\psi_2'(n)}{\psi_2(n)}\Big)\psi_2(n) < \frac{2K\psi_2(n)}{\pi n t^2}. \quad (31)$$

According to (31), we obtain

$$\int_{\pi/2n}^{\pi/2}\Big|\frac{1}{\pi t}\int_n^\infty \tau_2'(v)\cos vt\,dv\Big|dt \leqslant \int_{\pi/2n}^{\pi/2}\frac{2K\psi_2(n)}{\pi n t^2}dt = O(1)\psi_2(n). \quad (32)$$

Taking into account to (24) and (32), we have

$$\int_{\pi/2n\leqslant|t|\leqslant\pi/2}\Big|\frac{1}{\pi}\int_0^\infty \tau_2(v)\sin vt\,dv\Big|dt = \frac{2}{\pi\varphi(n)}\int_1^n\frac{\varphi(v)\psi_2(v)}{v}dv+O(1)\psi_2(n). \quad (33)$$

Now let's investigate in neighborhood of origin: Since $\tau_2(v)=\psi_2(v)$ on $[n,\infty)$, in [page 226, [10]], there exist $a>0$ for all $n\geqslant 1$, such that we have

$$\int_{|t|\leqslant a/n}\Big|\frac{1}{\pi}\int_n^\infty \tau_2(v)\sin vt\,dv\Big|dt = \frac{2}{\pi}\int_n^\infty\frac{\psi_2(v)}{v}dv+O(1)\overline{\psi}(n). \quad (34)$$

After that we obtain

$$2\frac{1}{\pi}\Big|\int_{a/n}^{\pi/2n}\int_n^\infty \tau_2(v)\sin vt\,dv\Big|dt \leqslant \frac{2}{\pi}\int_{a/n}^{\pi/2n}\Big|\int_n^\infty \psi_2(v)\sin vt\,dv\Big|dt \quad (35)$$
$$\leqslant \frac{2}{\pi}\int_{a/n}^{\pi/2n}\int_n^{n+\frac{2\pi}{t}}\psi_2(v)\,dv\,dt \leqslant O(1)\psi_2(n).$$

Finally, we will estimate following integral:

$$\int_{-\pi/2n}^{\pi/2n}\Big|\frac{1}{\pi}\int_0^n \tau_2(v)\sin vt\,dv\Big|dt = 2\int_0^{\pi/2n}\Big|\frac{1}{\pi}\int_0^n \tau_2(v)\sin vt\,dv\Big|dt.$$

By considering that $\tau_2(v)$ is a continuous increasing function on interval $[0,n]$, we have $\tau_2(v)\leqslant\psi_2(n)$. Hence, we obtain

$$\int_{-\pi/2n}^{\pi/2n}\Big|\frac{1}{\pi}\int_0^n \tau_2(v)\sin vt\,dv\Big|dt \leqslant \frac{2\psi_2(n)}{\pi}\int_0^{\pi/2n} n\,dt = O(1)\psi_2(n). \quad (36)$$

Therefore, by using (24), (32) and (36), for $n \geqslant 1$, we get (7). □

Proof of Theorem 1. Similar to the statement in [2], it is known that

$$\mathscr{E}_n(C_\infty^{\overline{\Psi}}, Z_n^\varphi)_C = \int_{-\infty}^{\infty} |\hat{\tau}_n(t)| dt + \gamma(n), \tag{37}$$

where $\gamma(n) \leqslant 0$,

$$|\gamma(n)| = O\Big(\int_{|t| \geqslant \frac{\pi}{2}} |\hat{\tau}_n(t)| dt\Big)$$

and

$$\hat{\tau}_n(t) = \frac{1}{\pi} \int_0^\infty \tau_1(v) \cos vt dv + \frac{1}{\pi} \int_0^\infty \tau_2(v) \sin vt dv.$$

Now we are going to prove the Theorem 1 by using (37) and Propositions 2–3. Firstly, let us estimate $\gamma(n)$. Taking into account that

$$|\gamma(n)| \leqslant O(1) \int_{|t| \geqslant \frac{\pi}{2}} |\hat{\tau}_n(t)| dt \leqslant O(1) \int_{|t| \geqslant \frac{\pi}{2}} \left| \frac{1}{\pi} \int_0^\infty \tau_1(v) \cos vt dv \right| dt$$

$$+ O(1) \int_{|t| \geqslant \frac{\pi}{2}} \left| \frac{1}{\pi} \int_0^\infty \tau_2(v) \sin vt dv \right| dt := \gamma_1 + \gamma_2$$

and since we know that $\gamma_1 = O(1)\psi_1(n)$ and $\gamma_2 = O(1)\psi_2(n)$ from (8), (17), (18) and (22), it turns out that $|\gamma(n)| \leqslant O(1)\overline{\psi}(n)$. Finally, we have (5) by applying Propositions 2–3 to (37). □

3. Some results with respect to Riesz and Woronoi-Nörlund means

In this section, we are going to give some asymptotic results with respect to Woronoi-Nörlund and Riesz means taking into account of the results in Section 2. First of all, let us consider some notations. Assume that $\psi \in \mathfrak{M}$ and $\alpha(t) = \dfrac{\psi(t)}{t|\psi'(t)|}$ for $t \geqslant 1$. If $\lim\limits_{t\to\infty} \alpha(t)$ exists, let us show this limit by $\alpha_0(\psi) \overset{\text{df}}{=} \lim\limits_{t\to\infty} \alpha(t)$. Thus, we have seen that in case $\varphi(n) = n^s$, $s > 0$, Theorem 1 gives us the following results which coincide with the results in [2].

COROLLARY 1. *Let* $\psi_1 \in \mathfrak{M}$, $\psi_2 \in \mathfrak{M}'$ *and* $g_i(v) = v^s \psi_i(v)$, $s > 0$, $i = 1,2$, *be concave functions on* $v \geqslant b \geqslant 1$ *with* $\lim\limits_{v\to\infty} g_i(v) = \infty$. *If* $\alpha_0(\psi_2) = \infty$, *then as* $n \to \infty$, *we have*

$$\mathscr{E}_n(C_\infty^{\overline{\Psi}}, Z_n^s)_C = \frac{2}{\pi} \int_n^\infty \frac{\psi_2(v)}{v} dv + O(1)\psi_2(n).$$

COROLLARY 2. *Let* $\psi_1 \in \mathfrak{M}$, $\psi_2 \in \mathfrak{M}'$ *and* $g_i(v) = v^s\psi_i(v)$, $s > 0$, $i = 1,2$, *be concave functions on* $v \geqslant b \geqslant 1$ *with* $\lim_{v\to\infty} g_i(v) = \infty$ *or* $\lim_{v\to\infty} g_i(v) = c_i \geqslant 0$. *If* $\alpha_0(\psi_2) = 1/s$ *then as* $n \to \infty$, *we have*

$$\mathscr{E}_n(C_\infty^{\overline{\psi}}, Z_n^s)_C = \frac{2}{\pi n^s}\int_1^n v^{s-1}\psi_2(v)dv + O(1)\psi_2(n).$$

COROLLARY 3. *Let* $\psi_1 \in \mathfrak{M}$, $\psi_2 \in \mathfrak{M}'$ *and* $g_i(v) = v^s\psi_i(v)$, $s > 0$, $i = 1,2$, *be concave functions on* $v \geqslant b \geqslant 1$ *with* $\lim_{v\to\infty} g_i(v) = \infty$. *If* $\alpha_0(\psi_2) \in (1/s,\infty)$, *then as* $n \to \infty$, *we have*

$$\mathscr{E}_n(C_\infty^{\overline{\psi}}, Z_n^s)_C = O(1)\psi_2(n).$$

Under the perspective of these results, we will give some results related to (2) and (3) by taking into account of the results given in [2], [4].

THEOREM 2. *Assume that* $\psi_1 \in \mathfrak{M}$, $\psi_2 \in \mathfrak{M}''$ *and* $g_i(v) = v\psi_i(v)$, $i = 1,2$, *are concave functions on* $v \geqslant b \geqslant 1$ *with* $\lim_{v\to\infty} g_i(v) = \infty$ *and* $\alpha_0(\psi_2) = \infty$. *Let* $(n+1)p_n = O(P_n)$ *and* $\sum_{m=1}^{n-1} |\Delta p_m| = O(P_n n^{-1})$. *Then as* $n \to \infty$, *we have*

$$\mathscr{E}_n(C_\infty^{\overline{\psi}}, R_n)_C = O(1)\left\{\frac{2}{\pi}\int_n^\infty \frac{\psi_2(v)}{v}dv + \psi_2(n)\right\}, \tag{38}$$

where $\Delta p_m := p_m - p_{m+1}$.

Proof. We know that the following equality is satisfied:

$$f(x) - R_n(f;x) = \frac{1}{P_n}\sum_{m=0}^{n} p_m(f(x) - s_m(f;x)).$$

By Abel's transformation and taking norm, it turns out that

$$\| f(x) - R_n(f;x)\|_C$$
$$\leqslant \frac{1}{P_n}\Big(\sum_{m=0}^{n-1}(m+1)\,|\,\Delta p_m\,|\,\| f - \sigma_m(f)\|_C + (n+1)p_n\| f - \sigma_n(f)\|_C\Big).$$

According to Corollary 1, we know that

$$\| f - \sigma_m(f)\|_C = \frac{2}{\pi}\int_m^\infty \frac{\psi_2(v)}{v}dv + O(1)\psi_2(m).$$

By using this asymptotic equality, we have

$$\begin{aligned}
&\| f(x)-R_n(f;x)\|_C \\
&\leqslant \frac{1}{P_n}\Big(\sum_{m=0}^{n-1}(m+1)\mid \Delta p_m\mid \Big(\frac{2}{\pi}\int_m^\infty \frac{\psi_2(v)}{v}dv+O(1)\psi_2(m)\Big) \\
&\quad +(n+1)p_n\Big(\frac{2}{\pi}\int_n^\infty \frac{\psi_2(v)}{v}dv+O(1)\psi_2(n)\Big)\Big) \\
&= O(1)\frac{1}{P_n}\Big(\sum_{m=1}^{n-1} m\mid \Delta p_m\mid \Big(\frac{2}{\pi}\int_m^\infty \frac{\psi_2(v)}{v}dv+O(1)\psi_2(m)\Big) \\
&\quad +(n+1)p_n\Big(\frac{2}{\pi}\int_n^\infty \frac{\psi_2(v)}{v}dv+O(1)\psi_2(n)\Big)\Big) \\
&= O(1)\frac{1}{P_n}\sum_{m=1}^{n-1} m\mid \Delta p_m\mid \Big(\frac{2}{\pi}\int_m^\infty \frac{\psi_2(v)}{v}dv+O(1)\psi_2(m)\Big) \\
&\quad +O(1)\frac{1}{P_n}\sum_{m=1}^{n-1} m\mid \Delta p_m\mid O(1)\psi_2(m) \\
&:= J_1+J_2.
\end{aligned}$$

Since $g(m)=m\psi_2(m)$ is increasing with $\lim_{m\to\infty} g(m)=\infty$ and using the relation $\sum_{m=1}^{n-1}\mid \Delta p_m\mid = O(n^{-1}P_n)$, we get

$$J_2 := \frac{1}{P_n}\sum_{m=1}^{n-1} m\psi_2(m)\mid \Delta p_m\mid \leqslant \frac{1}{P_n}n\psi_2(n)\sum_{m=1}^{n-1}\mid \Delta p_m\mid = O(\psi_2(n)).$$

And also,

$$\begin{aligned}
J_1 &:= \frac{1}{P_n}\sum_{m=1}^{n-1} m\mid \Delta p_m\mid \Big(\frac{2}{\pi}\int_m^\infty \frac{\psi_2(v)}{v}dv+O(1)\psi_2(m)\Big) \\
&\leqslant \frac{1}{P_n}\sum_{m=1}^{n-1} m\mid \Delta p_m\mid \frac{2}{\pi}\int_m^\infty \frac{\psi_2(v)}{v}dv \\
&\leqslant \frac{2}{\pi}\frac{1}{P_n}\Big(n\int_n^\infty \frac{\psi_2(v)}{v}dv\Big)\sum_{m=1}^{n-1}\mid \Delta p_m\mid = O(1)\frac{2}{\pi}\int_n^\infty \frac{\psi_2(v)}{v}dv.
\end{aligned}$$

Taking into account J_1 and J_2 we obtain (38). □

THEOREM 3. *Assume that* $\psi_1 \in \mathfrak{M}$, $\psi_2 \in \mathfrak{M}'$ *and* $g_i(v)=v\psi_i(v)$, $i=1,2$, *are concave functions on* $v \geqslant b \geqslant 1$ *with* $\lim_{v\to\infty} g_i(v)=\infty$ *or* $\lim_{v\to\infty} g_i(v)=c_i \geqslant 0$ *and* $\alpha_0(\psi_2)=$

1. *Let* $(n+1)p_n = O(P_n)$ *and* $\sum\limits_{m=1}^{n-1} | \Delta p_m | = O(P_n n^{-1})$. *Then as* $n \to \infty$, *we get*

$$\mathscr{E}_n(C_\infty^{\overline{\Psi}}, R_n)_C = O(1)\left\{ \frac{2}{\pi n} \int\limits_1^n \psi_2(v)dv + \psi_2(n) \right\}. \tag{39}$$

Proof. According to Corollary 2, it is known that

$$\| f - \sigma_m(f)\|_C = \frac{2}{\pi m} \int\limits_1^m \psi_2(v)dv + O(1)\psi_2(m).$$

By using it, we write

$$\begin{aligned} &\| f(x) - R_n(f;x)\|_C \\ &\leqslant \frac{1}{P_n}\Big(\sum_{m=0}^{n-1} (m+1) \mid \Delta p_m \mid \Big(\frac{2}{\pi m} \int\limits_1^m \psi_2(v)dv + O(1)\psi_2(m) \Big) \\ &\quad + (n+1)p_n \Big(\frac{2}{\pi n} \int\limits_1^n \psi_2(v)dv + O(1)\psi_2(n) \Big) \Big) \\ &\leqslant \frac{1}{P_n}\Big(\sum_{m=1}^{n-1} m \mid \Delta p_m \mid \Big(\frac{2}{\pi m} \int\limits_1^m \psi_2(v)dv + O(1)\psi_2(m) \Big) \\ &\quad + (n+1)p_n \Big(\frac{2}{\pi n} \int\limits_1^n \psi_2(v)dv + O(1)\psi_2(n) \Big) \Big) \\ &= \frac{1}{P_n} \sum_{m=1}^{n-1} m \mid \Delta p_m \mid \frac{2}{\pi m} \int\limits_1^m \psi_2(v)dv + \frac{1}{P_n} \sum_{m=1}^{n-1} m \mid \Delta p_m \mid O(1)\psi_2(m) := I_2 + I_1. \end{aligned}$$

Now let us evaluate these last two statements. Since $g_i(v) = v\psi_i(v)$, $i = 1,2$, are concave functions on $v \geqslant b \geqslant 1$ under the conditions of theorem, we obtain that

$$I_1 := \frac{1}{P_n} \sum_{m=1}^{n-1} m\psi_2(m) \mid \Delta p_m \mid \leqslant \frac{1}{P_n} n\psi_2(n) \sum_{m=1}^{n-1} \mid \Delta p_m \mid = O(\psi_2(n))$$

and

$$\begin{aligned} I_2 &:= \frac{1}{P_n} \sum_{m=1}^{n-1} m \mid \Delta p_m \mid \frac{2}{\pi m} \int\limits_1^m \psi_2(v)dv \leqslant \frac{2}{\pi P_n} \int\limits_1^n \psi_2(v)dv \sum_{m=1}^{n-1} \mid \Delta p_m \mid \\ &\leqslant O(1) \frac{2}{\pi n} \int\limits_1^n \psi_2(v)dv. \end{aligned}$$

Therefore, we get (39) from I_2 and I_1. □

THEOREM 4. *Assume that* $\psi_1 \in \mathfrak{M}$, $\psi_2 \in \mathfrak{M}'$ *and* $g_i(v) = v\psi_i(v)$, $i = 1,2$, *are concave functions on* $v \geqslant b \geqslant 1$ *with* $\lim_{v\to\infty} g_i(v) = \infty$ *and* $\alpha_0(\psi_2) \in (1,\infty)$. *Let* $(n+1)p_n = O(P_n)$ *and* $\sum_{m=1}^{n-1} |\Delta p_m| = O(P_n n^{-1})$. *Then as* $n \to \infty$, *we have*

$$\mathscr{E}_n(C_\infty^{\overline{\psi}}, R_n)_C = O(1)\psi_2(n). \tag{40}$$

Proof. We know that

$$\| f - \sigma_m(f)\|_C = O(1)\psi_2(m)$$

from Corollary 3. By considering this and hypothesis of Theorem 4, we get the desired result given below:

$$\begin{aligned}\| f(x) - R_n(f;x)\|_C &\leqslant \frac{1}{P_n}\Big(\sum_{m=0}^{n-1}(m+1)\,|\,\Delta p_m\,|\,(O(1)\psi_2(m)) + (n+1)p_n(O(1)\psi_2(n))\Big)\\ &\leqslant \frac{1}{P_n}\Big(\sum_{m=1}^{n-1} m\,|\,\Delta p_m\,|\,(O(1)\psi_2(m)) + (n+1)p_n(O(1)\psi_2(n))\Big)\\ &\leqslant \frac{1}{P_n}\sum_{m=1}^{n-1} m\,|\,\Delta p_m\,|\,\psi_2(m) + \frac{1}{P_n}(n+1)p_n O(1)\psi_2(n)\\ &\leqslant \frac{1}{P_n}O(1)n\psi_2(n)\sum_{m=1}^{n-1}|\,\Delta p_m\,| + O(1)\psi_2(n) \leqslant O(\psi_2(n)). \quad \square\end{aligned}$$

The subsequent results are related to the Woronoi-Nörlund means. Since the proofs of the next three results are similar to the proofs of the above results, we will omit them.

THEOREM 5. *Assume that* $\psi_1 \in \mathfrak{M}$, $\psi_2 \in \mathfrak{M}''$ *and* $g_i(v) = v\psi_i(v)$, $i = 1,2$, *are concave functions on* $v \geqslant b \geqslant 1$ *with* $\lim_{v\to\infty} g_i(v) = \infty$ *and* $\alpha_0(\psi_2) = \infty$. *Let* $(n+1)p_n = O(P_n)$ *and* $\sum_{m=1}^{n-1} |\Delta_m p_{n-m}| = O(n^{-1}P_n)$. *Then as* $n \to \infty$, *we have*

$$\mathscr{E}_n(C_\infty^{\overline{\psi}}, N_n)_C = \frac{2}{\pi}\int_n^\infty \frac{\psi_2(v)}{v}dv + O(1)\psi_2(n).$$

THEOREM 6. *Assume that* $\psi_1 \in \mathfrak{M}$, $\psi_2 \in \mathfrak{M}'$ *and* $g_i(v) = v\psi_i(v)$, $i = 1,2$, *are concave functions on* $v \geqslant b \geqslant 1$ *with* $\lim_{v\to\infty} g_i(v) = \infty$ *or* $\lim_{v\to\infty} g_i(v) = c_i \geqslant 0$ *and* $\alpha_0(\psi_2) = 1$. *Let* $(n+1)p_n = O(P_n)$ *and* $\sum_{m=1}^{n-1} |\Delta_m p_{n-m}| = O(n^{-1}P_n)$. *Then as* $n \to \infty$, *we have*

$$\mathscr{E}_n(C_\infty^{\overline{\psi}}, N_n)_C = \frac{2}{\pi n}\int_1^n \psi_2(v)dv + O(1)\psi_2(n).$$

THEOREM 7. *Assume that* $\psi_1 \in \mathfrak{M}$, $\psi_2 \in \mathfrak{M}'$ *and* $g_i(v) = v\psi_i(v)$, $i = 1,2$, *are concave functions on* $v \geqslant b \geqslant 1$ *with* $\lim_{v\to\infty} g_i(v) = \infty$ *and* $\alpha_0(\psi_2) \in (1,\infty)$. *Let* $(n+1)p_n = O(P_n)$ *and* $\sum_{m=1}^{n-1} |\Delta_m p_{n-m}| = O(n^{-1}P_n)$. *Then as* $n \to \infty$, *we have*

$$\mathscr{E}_n(C_\infty^{\overline{\psi}}, N_n)_C = O(1)\psi_2(n).$$

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