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ON THE GROWTH OF MEROMORPHIC SOLUTIONS OF HOMOGENEOUS AND NON–HOMOGENEOUS LINEAR DIFFERENCE EQUATIONS IN TERMS OF (p,q)–ORDER

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Abstract. In this paper we have studied the growth of meromorphic solutions of higher order homogeneous and non-homogeneous linear difference equations with entire and meromorphic coefficients. We have extended and improved some results of Zhou and Zheng (2017), Belaidi and Benkarouba (2019) by using (p,q)-order and (p,q)-type.

1. Introduction and definitions

Recently the properties of meromorphic solutions of complex difference equations

$$A_k(z)f(z+c_k)+A_{k-1}(z)f(z+c_{k-1})+\cdots+A_1(z)f(z+c_1)+A_0(z)f(z)=0 \tag{1}$$

and

$$A_k(z)f(z+c_k)+A_{k-1}(z)f(z+c_{k-1})+\cdots+A_1(z)f(z+c_1)+A_0(z)f(z)=F(z) \tag{2}$$

have become a subject of great interest from the view point of Nevanlinna's theory and achieved many valuable results where the coefficients $A_0, A_1, \dots, A_k \not\equiv 0$ and $F \not\equiv 0$ $(k \geqslant 2)$ in (1) or (2) are entire or meromorphic functions and $c_k, c_{k-1}, \dots, c_1$ are distinct nonzero complex numbers.

In 1976 Juneja, Kapoor and Bajpai [10] introduced the idea of (p,q)-order of an entire function and in 2010 Liu, Tu and Shi [7] modified the definition of the (p,q)-order to make it more suitable. Laine and Yang, in 2007, considered the equation (1) when more than one dominant coefficients exist but exactly one has its type strictly greater than the others ([6], Theorem 5.2.). In 2008 Chiang and Feng [14] investigated meromorphic solutions of (1) and established a theorem ([14], Theorem 9.2) taking exactly one coefficient of (1) with maximal order. In 2013, Liu and Mao used hyper order to establish the case when one or more coefficients of (1) or (2) having infinite order ([5], Theorem 1.4, Theorem 1.6). Finally in 2017, Zhou and Zheng ([15], Theorem 1.5) and in 2019, Belaïdi and Benkarouba ([3], Theorem 1.1–Theorem 1.4) used

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iterated order and iterated type to investigate the solutions of (1) or (2) and obtained some results which improved and generalized those previous results.

In this article we use the concept of (p,q)-order to investigate meromorphic solutions of (1) and (2). We also extend and improve some results of Zhou and Zheng [15], Belaïdi and Benkarouba [3]. Here we consider both cases, when (1) and (2) have entire coefficients and meromorphic coefficients. We also cover the cases when, either one of the coefficients have maximal (p,q)-order or more than one coefficients having maximal (p,q)-order.

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of Nevanlinna's value distribution theory [13].

Next we give some basic definitions which are used to prove our main results.

For all $r \in \mathbb{R}$, set $\exp_1 r = e^r$ and $\exp_{p+1} r = \exp\left(\exp_p r\right)$, $p \in \mathbb{N}$. Also for all sufficiently large values of r, $\log_1 r = \log r$ and $\log_{p+1} r = \log\left(\log_p r\right)$, $p \in \mathbb{N}$. Further $\exp_0 r = \log_0 r = r$, $\exp_{-1} r = \log_1 r$, $\exp_1 r = \log_{-1} r$.

DEFINITION 1. [9] Let $p \geqslant q \geqslant 1$ or $2 \leqslant q = p+1$ be integers. The (p,q)-order of a transcendental meromorphic function f is defined by

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log_p T(r,f)}{\log_q r}.$$

And if f is a transcendental entire function, then

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log_{p+1} M(r,f)}{\log_q r}.$$

Note that $0 \leqslant \rho_f(p,q) \leqslant \infty$. Also for a rational function $\rho_f(p,q) = 0$.

DEFINITION 2. [9] A transcendental meromorphic function is said to have index pair $[p,q]$ if $0 \leqslant \rho_f(p,q) \leqslant \infty$ and $\rho_f(p-1,q-1)$ is not a non-zero finite number.

DEFINITION 3. [9] The (p,q)-type of a meromorphic function f having non-zero finite (p,q)-order $\rho_f(p,q)$ is defined by

$$\tau_f(p,q) = \limsup_{r\to\infty} \frac{\log_{p-1} T(r,f)}{\left(\log_{q-1} r\right)^{\rho_f(p,q)}}.$$

And if f is a transcendental entire function, then

$$\tau_f(p,q) = \limsup_{r\to\infty} \frac{\log_p M(r,f)}{\left(\log_{q-1} r\right)^{\rho_f(p,q)}}.$$

DEFINITION 4. [9] Let $p \geqslant q \geqslant 1$ or $2 \leqslant q = p+1$ be integers. The (p,q)-exponent of convergence of the sequence of poles of a meromorphic function f is defined by

$$\lambda_{\frac{1}{f}}(p,q) = \limsup_{r\to\infty} \frac{\log_p N(r,f)}{\log_q r}.$$

Now we recall that the linear measure of a set $S \subset (0,+\infty)$ is defined by

$$m(S) = \int_0^\infty \chi_S(t)\, dt$$

and the logarithmic measure of a set $S \subset (1,+\infty)$ is defined by

$$lm(S) = \int_1^\infty \frac{\chi_S(t)}{t} dt,$$

where $\chi_S(t)$ is the characteristic function of a set S.

The upper density of a set $S \subset (0,+\infty)$ is defined by

$$\overline{dens} S = \limsup_{r\to\infty} \frac{m(S \cap [0,r])}{r}$$

and the upper logarithmic density of a set $S \subset (1,+\infty)$ is defined by

$$\overline{\log dens} S = \limsup_{r\to\infty} \frac{lm(S \cap [1,r])}{\log r}.$$

2. Preliminary lemmas

In this section we recall and proof some lemmas on which our main results depend.

LEMMA 1. [14] *Let f be a meromorphic function, ξ a nonzero complex number, and let $\nu > 1$, and $\varepsilon > 0$ be given real constants. Then there exists a subset $S \subset (1,+\infty)$ of finite logarithmic measure, and a constant K depending only on ν and ξ, such that for all z with $|z| = r \notin S \cup [0,1]$, we have*

$$\left|\log\left|\frac{f(z+\xi)}{f(z)}\right|\right| \leqslant K\left(\frac{T(\nu r, f)}{r} + \frac{n(\nu r)}{r}\log^\nu r \log^+ n(\nu r)\right),$$

where $n(t) = n(t,f) + n(t,\frac{1}{f})$.

LEMMA 2. [4] *Let f be a transcendental meromorphic function. Let j be a nonnegative integer and ξ be an extended complex number. Then for a real constant $\alpha > 1$, there exists a constant $R > 0$, such that for all $r > R$, we have*

$$n(r,\xi,f^{(j)}) \leqslant \frac{2j+6}{\log\alpha} T(\alpha r, f). \tag{3}$$

LEMMA 3. *Let f be a meromorphic function with finite (p,q)-order, $\rho_f(p,q) = \rho$. Let ξ be a nonzero complex number and $\varepsilon > 0$ be given real constant. Then there* *exists a subset* $S \subset (1,+\infty)$ *of finite logarithmic measure such that for all* z *with* $|z| = r \notin S \cup [0,1]$, *we have*

$$i)\exp\left\{-r^{\rho-1+\varepsilon}\right\} \leqslant \left|\frac{f(z+\xi)}{f(z)}\right| \leqslant \exp\left\{r^{\rho-1+\varepsilon}\right\}, \tag{4}$$

for $p = q = 1$, *and*

$$ii)\exp_p\left\{-\left(\log_{q-1} r\right)^{\rho+\varepsilon}\right\} \leqslant \left|\frac{f(z+\xi)}{f(z)}\right| \leqslant \exp_p\left\{\left(\log_{q-1} r\right)^{\rho+\varepsilon}\right\}, \tag{5}$$

for $p \geqslant q \geqslant 2$.

Proof. We prove only second part of the lemma. First part follows from [14].

By Lemma 1, there exists a subset $S \subset (1,+\infty)$ of finite logarithmic measure, and a constant K depending only on ν and ξ, such that for all z with $|z| = r \notin S \cup [0,1]$, we have

$$\left|\log\left|\frac{f(z+\xi)}{f(z)}\right|\right| \leqslant K\left(\frac{T(\nu r, f)}{r} + \frac{n(\nu r)}{r}\log^{\nu} r \log^{+} n(\nu r)\right), \tag{6}$$

where $n(t) = n(t,f) + n(t,\frac{1}{f})$. Now using (3) in (6), we obtain

$$\begin{aligned}
&\left|\log\left|\frac{f(z+\xi)}{f(z)}\right|\right| \\
&\leqslant K\left(\frac{T(\nu r, f)}{r} + \frac{12}{\log\alpha}\frac{T(\alpha\nu r, f)}{r}\log^{\nu} r \log^{+}\left(\frac{12}{\log\alpha}T(\alpha\nu r, f)\right)\right) \\
&\leqslant K_1\left(T(\eta r, f)\frac{\log^{\eta} r}{r}\log T(\eta r, f)\right),
\end{aligned} \tag{7}$$

where $K_1 > 0$ is some constant and we consider $\eta = \alpha\nu > 1$.

Now since f has finite (p,q)-order $\rho_f(p,q) = \rho$, so for given ε, $0 < \varepsilon < 2$, and for sufficiently large r we have

$$T(r,f) \leqslant \exp_{p-1}\left\{\left(\log_{q-1} r\right)^{\rho+\frac{\varepsilon}{2}}\right\} \tag{8}$$

Therefore using (8) in (7), we obtain

$$\begin{aligned}
&\left|\log\left|\frac{f(z+\xi)}{f(z)}\right|\right| \\
&\leqslant K_1\exp_{p-1}\left\{\log_{q-1}(\eta r)^{\rho+\frac{\varepsilon}{2}}\right\}\frac{\log^{\eta} r}{r}\exp_{p-2}\left\{\left(\log_{q-1}(\eta r)\right)^{\rho+\frac{\varepsilon}{2}}\right\} \\
&\leqslant \exp_{p-1}\left\{\left(\log_{q-1} r\right)^{\rho+\varepsilon}\right\}.
\end{aligned} \tag{9}$$

Hence from (9) we obtain the required result. $\square$

LEMMA 4. *Let* ξ_1 *and* ξ_2 *be two arbitrary distinct complex numbers and* f *be a meromorphic function with finite* (p,q)*-order,* $\rho_f(p,q)=\rho$*. Then for given* $\varepsilon>0$*, there exists a subset* $S\subset(1,+\infty)$ *of finite logarithmic measure such that for all* z *with* $|z|=r\notin S\cup[0,1]$*, we have*

$$i)\ \exp\left\{-r^{\rho-1+\varepsilon}\right\}\leqslant\left|\frac{f(z+\xi_1)}{f(z+\xi_2)}\right|\leqslant\exp\left\{r^{\rho-1+\varepsilon}\right\},$$

for $p=q=1$*, and*

$$ii)\ \exp_p\left\{-\left(\log_{q-1}r\right)^{\rho+\varepsilon}\right\}\leqslant\left|\frac{f(z+\xi_1)}{f(z+\xi_2)}\right|\leqslant\exp_p\left\{\left(\log_{q-1}r\right)^{\rho+\varepsilon}\right\},$$

for $p\geqslant q\geqslant 2$.

Proof. We prove only second part of the lemma. First part follows from [14]. For the second part we consider the following expression

$$\left|\frac{f(z+\xi_1)}{f(z+\xi_2)}\right|=\left|\frac{f(z+\xi_2+\xi_1-\xi_2)}{f(z+\xi_2)}\right|,\ (\xi_1\neq\xi_2).$$

Now by using Lemma 3, for any given $\varepsilon>0$ and for all z with $|z+\xi_2|=R\notin S\cup[0,1]$ such that $lm(S)<\infty$, we have

$$\begin{aligned}\exp_p\left\{-\left(\log_{q-1}(r)\right)^{\rho+\varepsilon}\right\}&\leqslant\exp_p\left\{-\left(\log_{q-1}(|z|+\xi_2)\right)^{\rho+\frac{\varepsilon}{2}}\right\}\\&\leqslant\exp_p\left\{-\left(\log_{q-1}R\right)^{\rho+\frac{\varepsilon}{2}}\right\}\\&\leqslant\left|\frac{f(z+\xi_1)}{f(z+\xi_2)}\right|=\left|\frac{f(z+\xi_2+\xi_1-\xi_2)}{f(z+\xi_2)}\right|\\&\leqslant\exp_p\left\{\left(\log_{q-1}R\right)^{\rho+\frac{\varepsilon}{2}}\right\}\\&\leqslant\exp_p\left\{\left(\log_{q-1}(|z|+|\xi_2|)\right)^{\rho+\frac{\varepsilon}{2}}\right\}\\&\leqslant\exp_p\left\{\left(\log_{q-1}(r)\right)^{\rho+\varepsilon}\right\},\end{aligned}$$

for z with $|z|=r\notin S\cup[0,1]$, where $S\subset(1,+\infty)$ is a set of finite logarithmic measure. Hence we obtain the required result. $\square$

LEMMA 5. [11] *Let* f *be a nonconstant meromorphic function. Suppose* $z_1\in\mathbb{C}$, $\delta<1$ *and* $\varepsilon>0$*. Then for all* r *outside of a possible exceptional set* S *with finite logarithmic measure* $\int_S\frac{dr}{r}<\infty$*, we have*

$$m\left(r,\frac{f(z+z_1)}{f(z)}\right)=o\left(\frac{\left(T(r+|z_1|,\ f)\right)^{1+\varepsilon}}{r^{\delta}}\right).$$

LEMMA 6. [1] *Let f be a nonconstant meromorphic function and z_1, z_2 be nonzero complex constants. Then for $r \to +\infty$ we have*

$$(1+o(1))T(r-|z_1|,\ f) \leqslant T(r,\ f(z+z_1)) \leqslant (1+o(1))T(r+|z_1|,\ f).$$

Consequently,

$$\rho_{f(z+z_2)}(p,q) = \rho_f(p,q),$$

for $p \geqslant q,\ p,q \in \mathbb{N}$.

LEMMA 7. [11] *Let f be a nonconstant meromorphic function. Suppose $z_1, z_2 \in \mathbb{C}$, such that $z_1 \neq z_2,\ \delta < 1,\ \varepsilon > 0$. Then*

$$m\left(r, \frac{f(z+z_1)}{f(z+z_2)}\right) = o\left(\frac{\{T(r+|z_1-z_2|+|z_2|,\ f)\}^{1+\varepsilon}}{r^{\delta}}\right)$$

for all r outside of a possible exceptional set S with finite logarithmic measure $\int\limits_S \frac{dr}{r} < \infty$.

LEMMA 8. [9] *Let f be a nonconstant meromorphic function with nonzero finite (p,q)-order $\rho_f(p,q)$ and nonzero finite (p,q)-type $\tau_f(p,q)$. Then for any given $b < \tau_f(p,q)$, there exists a subset $S \subset [1,+\infty)$ of infinite logarithmic measure such that for all $r \in S$, we have*

$$\log_{p-1} T(r,\ f) > b\left(\log_{q-1} r\right)^{\rho_f(p,q)}.$$

LEMMA 9. [14] *Let $\alpha,\ R,\ R'$ be real numbers such that $0 < \alpha < 1,\ R,\ R' > 0$ and let ξ be a nonzero complex number. Then there exists a positive constant K_α which depends only on α such that for a given meromorphic function f we have, when $|z| = r$, $\max\{1, r+|\xi|\} < R < R'$, the estimate*

$$\begin{aligned} & m\left(r, \frac{f(z+\xi)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\xi)}\right) \\ \leqslant\ & \frac{2|\xi|R}{(R-r-|\xi|)^2}\left(m(R,\ f) + m\left(R, \frac{1}{f}\right)\right) \\ & + \frac{2R'}{(R'-R)}\left(\frac{|\xi|}{R-r-|\xi|} + \frac{K_\alpha |\xi|^\alpha}{(1-\alpha)r^\alpha}\right)\left(N(R',\ f) + N\left(R', \frac{1}{f}\right)\right). \end{aligned}$$

LEMMA 10. *Let ξ_1, ξ_2 be two complex numbers such that $\xi_1 \neq \xi_2$ and suppose f be of finite (p,q)-order meromorphic function. Consider the (p,q)-order of f as $\rho_f(p,q) = \rho < +\infty$. Then for each $\varepsilon > 0$, we have*

$$i)\ m\left(r, \frac{f(z+\xi_1)}{f(z+\xi_2)}\right) = O\left(r^{\rho-1+\varepsilon}\right),$$

for $p=q=1$, *and*

$$ii)\; m\left(r,\frac{f(z+\xi_1)}{f(z+\xi_2)}\right)=O\left(\exp_{p-1}\left[\left\{\log_{q-1}(r)\right\}^{\rho+\varepsilon}\right]\right),$$

for $p\geqslant q\geqslant 2$.

Proof. We prove only second part of the lemma. First part follows from [14]. In the second part, for $p\geqslant q\geqslant 2$ we first write the following expression as

$$\begin{aligned}m\left(r,\frac{f(z+\xi_1)}{f(z+\xi_2)}\right)&\leqslant m\left(r,\frac{f(z+\xi_1)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+\xi_2)}\right)\\&\leqslant m\left(r,\frac{f(z+\xi_1)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+\xi_1)}\right)\\&\quad+m\left(r,\frac{f(z)}{f(z+\xi_2)}\right)+m\left(r,\frac{f(z+\xi_2)}{f(z)}\right).\end{aligned}\tag{10}$$

By Lemma 9 and using the concept given in [[3], lemma 2.9], we obtain from above

$$m\left(r,\frac{f(z+\xi_1)}{f(z+\xi_2)}\right)\leqslant 4\left[\begin{array}{c}\frac{4|\xi_1|r}{\left(r-|\xi_1|^2\right)}+\frac{4|\xi_2|r}{\left(r-|\xi_2|^2\right)}\\+6\left(\frac{|\xi_1|}{(r-|\xi_1|)}+\frac{|\xi_2|}{(r-|\xi_2|)}\right)\\+\frac{2K_\alpha\left(|\xi_1|^{1-\frac{\varepsilon}{2}}+|\xi_2|^{1-\frac{\varepsilon}{2}}\right)}{\varepsilon r^{1-\frac{\varepsilon}{2}}}\end{array}\right]T(3r,f).\tag{11}$$

Now since the (p,q)-order of f is $\rho_f(p,q)=\rho<+\infty$, so given $0<\varepsilon<2$, by definition we have

$$T(r,f)\leqslant\exp_{p-1}\left\{\left\{\log_{q-1}(r)\right\}^{\rho+\frac{\varepsilon}{2}}\right\}.$$

Using the above in (11) we obtain

$$\begin{aligned}&m\left(r,\frac{f(z+\xi_1)}{f(z+\xi_2)}\right)\\&\leqslant 4\left[\begin{array}{c}\frac{4|\xi_1|r}{\left(r-|\xi_1|^2\right)}+\frac{4|\xi_2|r}{\left(r-|\xi_2|^2\right)}\\+6\left(\frac{|\xi_1|}{(r-|\xi_1|)}+\frac{|\xi_2|}{(r-|\xi_2|)}\right)\\+\frac{2K_\alpha\left(|\xi_1|^{1-\frac{\varepsilon}{2}}+|\xi_2|^{1-\frac{\varepsilon}{2}}\right)}{\varepsilon r^{1-\frac{\varepsilon}{2}}}\end{array}\right]\exp_{p-1}\left[\left\{\log_{q-1}(3r)\right\}^{\rho+\frac{\varepsilon}{2}}\right]\\&\leqslant K\exp_{p-1}\left\{\log_{q-1}r^{\rho+\varepsilon}\right\},\ \text{where } K>0 \text{ is a constant.}\end{aligned}$$

This completes the proof. $\square$

LEMMA 11. *Let D be a complex set satisfying $\overline{\log dens}\{r=|z|:z\in D\}>0$ and let $A_0(z),A_1(z),\ldots,A_k(z)$ be entire functions of (p,q)-order satisfying $\max\limits_{0\leqslant j\leqslant k}\{\rho_{A_j}(p,q)\}\leqslant\rho$. If there exists an integer l $(0\leqslant l\leqslant k)$ such that for some constants a,b $(0\leqslant b<a)$ and δ $(0<\delta<\rho)$ sufficiently small with*

$$|A_l(z)|\geqslant\exp_p\left[a\left\{\log_{q-1}(r)\right\}^{\rho-\delta}\right] \tag{12}$$

and

$$\left|A_j(z)\right|\leqslant\exp_p\left[b\left\{\log_{q-1}(r)\right\}^{\rho-\delta}\right],\ j=0,1,\ldots,k,\ j\neq l$$

as $z\to\infty$ for $z\in D$, then we have $\rho_{A_l}(p,q)=\rho$.

Proof. By the stated condition we have $\rho_{A_l}(p,q)\leqslant\rho$. Let $\rho_{A_l}(p,q)=\alpha<\rho$. Then for given ε and sufficiently large r, by definition we have

$$|A_l(z)|\leqslant\exp_p\left[\left\{\log_{q-1}(r)\right\}^{\alpha+\varepsilon}\right]$$

Again by (12),

$$|A_l(z)|\geqslant\exp_p\left[a\left\{\log_{q-1}(r)\right\}^{\rho-\delta}\right].$$

Combining the above two for $z\in D$, $|z|\to+\infty$, we obtain

$$\exp_p\left[a\left\{\log_{q-1}(r)\right\}^{\rho-\delta}\right]\leqslant|A_l(z)|\leqslant\exp_p\left[\left\{\log_{q-1}(r)\right\}^{\alpha+\varepsilon}\right],$$

where ε is arbitrary and $0<\varepsilon<\rho-\alpha-2\delta$, which is a contradiction as $r\to+\infty$. Hence $\rho_{A_l}(p,q)=\rho$. □

LEMMA 12. *Let D be a complex set satisfying $\overline{\log dens}\{r=|z|:z\in D\}>0$ and let $A_0(z),A_1(z),\ldots,A_k(z)$ be entire functions of (p,q)-order satisfying $\max\limits_{0\leqslant j\leqslant k}\{\rho_{A_j}(p,q)\}\leqslant\rho$. If there exists an integer l $(0\leqslant l\leqslant k)$ such that for some constants a,b $(0\leqslant b<a)$ and δ $(0<\delta<\rho)$ sufficiently small with*

$$T(r,A_l)\geqslant\exp_{p-1}\left[a\left\{\log_{q-1}(r)\right\}^{\rho-\delta}\right]$$

and

$$T(r,A_j)\leqslant\exp_{p-1}\left[b\left\{\log_{q-1}(r)\right\}^{\rho-\delta}\right],\ j=0,1,\ldots,k,\ j\neq l$$

as $z\to\infty$ for $z\in D$, then we have $\rho_{A_l}(p,q)=\rho$.

Proof. The proof follows from the previous lemma, hence we omit it. □

3. Main results

Here we state and proof our main results of this paper.

THEOREM 1. *Let D be a complex set satisfying $\overline{\log dens}\{r=|z|:z\in D\}>0$ and let $A_0(z)$, $A_1(z)$,..., $A_k(z)$ be entire functions of (p,q)-order satisfying $\max\limits_{0\leqslant j\leqslant k}\{\rho_{A_j}(p,q)\}\leqslant\rho$. If there exists an integer l $(0\leqslant l\leqslant k)$ such that for some constants a,b $(0\leqslant b<a)$ and δ $(0<\delta<\rho)$ sufficiently small with*

$$|A_l(z)|\geqslant\exp_p\left[a\left\{\log_{q-1}(r)\right\}^{\rho-\delta}\right] \tag{13}$$

$$\left|A_j(z)\right|\leqslant\exp_p\left[b\left\{\log_{q-1}(r)\right\}^{\rho-\delta}\right],\ j=0,1,\ldots,k,\ j\neq l \tag{14}$$

as $z\to\infty$ for $z\in D$, then every meromorphic solution $f(\not\equiv 0)$ of equation (1) satisfies
(i) $\rho_f\geqslant\rho_{A_l}+1$, for $p=1,\ q=1$.
(ii) $\rho_f(p,q)\geqslant\rho_{A_l}(p,q)$, for $p\geqslant q\geqslant 2$.

Proof. For $p=q=1$, see [3]. We consider the case when $p\geqslant q\geqslant 2$.
First let $f(\not\equiv 0)$ be a meromorphic solution of (1) and if possible let $\rho_f(p,q)<\rho$. Now divide (1) by $f(z+c_l)$ we get

$$\begin{aligned}-A_l(z)&=A_k(z)\frac{f(z+c_k)}{f(z+c_l)}+\ldots+A_{l-1}(z)\frac{f(z+c_{l-1})}{f(z+c_l)}+\ldots\\&\quad+A_1(z)\frac{f(z+c_1)}{f(z+c_l)}+A_0(z)\frac{f(z)}{f(z+c_l)}.\end{aligned} \tag{15}$$

The above expression can be written as

$$-1=\sum_{j=1,j\neq l}^{k}\frac{A_j(z)f(z+c_j)}{A_l(z)f(z+c_l)}+\frac{A_0(z)f(z)}{A_l(z)f(z+c_l)}.$$

The above implies

$$1\leqslant\sum_{j=1,j\neq l}^{k}\left|\frac{A_j(z)f(z+c_j)}{A_l(z)f(z+c_l)}\right|+\left|\frac{A_0(z)f(z)}{A_l(z)f(z+c_l)}\right|. \tag{16}$$

By Lemma 4 (ii), for any given $\varepsilon>0$ $\left(\varepsilon<\rho-\rho_f(p,q)-2\delta\right)$, there exists a subset $S\subset(1,+\infty)$ of finite logarithmic measure such that for all $|z|=r\notin S\cup[0,1]$, we have

$$\left|\frac{f(z+c_j)}{f(z+c_l)}\right|\leqslant\exp_p\left[\left\{\log_{q-1}(r)\right\}^{\rho_f(p,q)+\varepsilon}\right]<\exp_p\left[\left\{\log_{q-1}(r)\right\}^{\rho-2\delta}\right] \tag{17}$$
$$(j\neq l,\ j=1,2,\ldots,k)$$

and

$$\left|\frac{f(z)}{f(z+c_l)}\right| \leqslant \exp_p\left[\left\{\log_{q-1}(r)\right\}^{\rho_f(p,q)+\varepsilon}\right] < \exp_p\left[\left\{\log_{q-1}(r)\right\}^{\rho-2\delta}\right]. \tag{18}$$

Now D is a complex set satisfying $\overline{\log dens}\{r=|z|:z\in D\}>0$ and for $|z|\to+\infty$, we have (13) and (14). Therefore we set $D_1=\{r=|z|:z\in D\}$.

Since $\overline{\log dens}\{r=|z|:z\in D\}>0$, thus D_1 is a set of r with $\int\limits_{D_1}\frac{dr}{r}=\infty$.

Now for $z\in D_1\backslash S\cup[0,1]$, substituting (13),(14),(17) and (18) in (16), we obtain

$$1\leqslant k\frac{\exp_p\left[b\left\{\log_{q-1}(r)\right\}^{\rho-\delta}\right]}{\exp_p\left[a\left\{\log_{q-1}(r)\right\}^{\rho-\delta}\right]}.\exp_p\left[\left\{\log_{q-1}(r)\right\}^{\rho-2\delta}\right]\to 0 \text{ as } r\to\infty.$$

The above expression leads to a contradiction.

Hence we get $\rho_f(p,q)\geqslant\rho$.

Again by Lemma 11 we know $\rho_{A_l}(p,q)=\rho$, hence $\rho_f(p,q)\geqslant\rho_{A_l}(p,q)$. □

THEOREM 2. *Let D be a complex set satisfying $\overline{\log dens}\{r=|z|:z\in D\}>0$ and let $A_0(z)$, $A_1(z),\ldots,A_k(z)$ be entire functions satisfying $\max\limits_{0\leqslant j\leqslant k}\{\rho_{A_j}(p,q)\}\leqslant\rho$. If there exists an integer l $(0\leqslant l\leqslant k)$ such that for some constants a,b $(0\leqslant b<a)$ and δ $(0<\delta<\rho)$ sufficiently small with*

$$T(r,A_l)\geqslant\exp_{p-1}\left[\left\{\log_{q-1}(r)\right\}^{\rho-\delta}\right] \tag{19}$$

$$T(r,A_j)\leqslant\exp_{p-1}\left[\left\{\log_{q-1}(r)\right\}^{\rho-\delta}\right],\ j=0,1,\ldots,k,\ j\neq l \tag{20}$$

as $z\to\infty$ for $z\in D$, then every meromorphic solution $f\,(\not\equiv 0)$ of equation (1) satisfies

(i) $\rho_f\geqslant\rho_{A_l}+1$, for $p=1$, $q=1$ and $0\leqslant kb<a$.

(ii) $\rho_f(p,q)\geqslant\rho_{A_l}(p,q)$, for $p\geqslant q\geqslant 2$ and $0\leqslant b<a$.

Proof. For $p=q=1$, see [3]. We consider the case when $p\geqslant q\geqslant 2$.

First let $f(\not\equiv 0)$ be a meromorphic solution of (1) and if possible let $\rho_f(p,q)<\rho$.

Now since $A_0(z),A_1(z),\ldots,A_k(z)$ are entire, by (15) we have

$$\begin{aligned} m(r,A_l) &= T(r,A_l)\\ &\leqslant \sum_{j=0,j\neq l}^{k} m(r,A_j)+\sum_{j=1,j\neq l}^{k} m\left(r,\frac{f(z+c_j)}{f(z+c_l)}\right)\\ &\quad +m\left(r,\frac{f(z)}{f(z+c_l)}\right)+O(1)\\ &= \sum_{j=0,j\neq l}^{k} T(r,A_j)+\sum_{j=1,j\neq l}^{k} m\left(r,\frac{f(z+c_j)}{f(z+c_l)}\right)\\ &\quad +m\left(r,\frac{f(z)}{f(z+c_l)}\right)+O(1). \end{aligned} \tag{21}$$

For any given ε $\left(0<\varepsilon<\rho-\rho_f(p,q)-2\delta\right)$, from Lemma (10) the above implies

$$T(r,A_l)\leqslant \sum_{j=0,j\neq l}^{k} T(r,A_j)+\sum_{j=1,j\neq l}^{k}\exp_{p-1}\left\{\left\{\log_{q-1}(r)\right\}^{\rho_f(p,q)+\varepsilon}\right\} \\ +\exp_{p-1}\left\{\left\{\log_{q-1}(r)\right\}^{\rho_f(p,q)+\varepsilon}\right\}+O(1). \tag{22}$$

Substituting (19) and (20) in (22), we obtain

$$\begin{aligned}
&\exp_{p-1}\left[\left\{a\log_{q-1}(r)\right\}^{\rho-\delta}\right]\\
\leqslant &\sum_{j=0,j\neq l}^{k}\exp_{p-1}\left[b\left\{\log_{q-1}(r)\right\}^{\rho-\delta}\right]+\sum_{j=1,j\neq l}^{k}\exp_{p-1}\left\{\left\{\log_{q-1}(r)\right\}^{\rho_f(p,q)+\varepsilon}\right\}\\
&+\exp_{p-1}\left\{\left\{\log_{q-1}(r)\right\}^{\rho_f(p,q)+\varepsilon}\right\}+O(1) &(23)\\
\leqslant &\, k\exp_{p-1}\left[b\left\{\log_{q-1}(r)\right\}^{\rho-\delta}\right]+k\exp_{p-1}\left\{\left\{\log_{q-1}(r)\right\}^{\rho_f(p,q)+\varepsilon}\right\}+O(1). &(24)
\end{aligned}$$

By (24) it follows

$$(a-b)\left\{\log_{q-1}(r)\right\}^{\rho-\delta}\leqslant\left\{\log_{q-1}(r)\right\}^{\rho_f(p,q)+\varepsilon}+O(1).$$

Since $(a-b)>0$, the above implies

$$1\leqslant\frac{\left\{\log_{q-1}(r)\right\}^{\rho_f(p,q)+\varepsilon-\rho+\delta}}{(a-b)}+\frac{O(1)}{(a-b)\left\{\log_{q-1}(r)\right\}^{\rho-\delta}}\to 0 \text{ as } r\to+\infty,$$

which is a contradiction.

Again by Lemma 12 it follows that $\rho_{A_l}(p,q)=\rho$. Hence we have $\rho_f(p,q)\geqslant\rho_{A_l}(p,q)$ and the theorem is proved. $\square$

THEOREM 3. *Let $A_0(z)$, $A_1(z),\ldots,A_k(z)$ be entire functions and there exists an integer l $(0\leqslant l\leqslant k)$ such that*

$$\max\left\{\rho_{A_j}(p,q):j=0,1,\ldots,k,\ j\neq l\right\}\leqslant\rho_{A_l}(p,q),$$

$$\max\left\{\tau_{A_j}(p,q):\rho_{A_j}(p,q)=\rho_{A_l}(p,q)\right\}<\tau_{A_l}(p,q),$$

where $0<\rho_{A_l}(p,q)$, $\tau_{A_l}(p,q)<\infty$ and $p\geqslant q\geqslant 1$ are integers. Then every meromorphic solution $(f\not\equiv 0)$ of (1) satisfies $\rho_f(p,q)\geqslant\rho_{A_l}(p,q)$.

Proof. Suppose $f(\not\equiv 0)$ be a meromorphic solution of the equation (1).

Now from (21) by using Lemma 7, for all r outside of a possible exceptional set S_1 with finite logarithmic measure we obtain (see [3])

$$\begin{aligned}
&m(r,A_l)=T(r,A_l)\\
\leqslant &\sum_{j=0,j\neq l}^{k} m(r,A_j)+\sum_{j=1,j\neq l}^{k} m\left(r,\frac{f(z+c_j)}{f(z+c_l)}\right)+m\left(r,\frac{f(z)}{f(z+c_l)}\right)+O(1)\\
\leqslant &\sum_{j=0,j\neq l}^{k} T(r,A_j)+\sum_{j=1,j\neq l}^{k} o\left(\frac{\left(T\left(r+\left|c_j-c_l\right|+|c_l|,f\right)\right)^{1+\varepsilon}}{r^{\delta}}\right)\\
&+o\left(\frac{\left(T\left(r+2|c_l|,f\right)\right)^{1+\varepsilon}}{r^{\delta}}\right)+O(1)\\
\leqslant &\sum_{j=0,j\neq l}^{k} T(r,A_j)+o\left(\frac{\left(T\left(r+2|c_l|,f\right)\right)^{1+\varepsilon}}{r^{\delta}}\right). \qquad (25)
\end{aligned}$$

Consider two real numbers b_1,b_2 such that

$$\max\left\{\tau_{A_j}(p,q):\rho_{A_j}(p,q)=\rho_{A_l}(p,q)\right\}<b_1<b_2<\tau_{A_l}(p,q).$$

Now by Lemma 8 there exists a subset $S_2\subset[1,+\infty)$ of infinite logarithmic measure such that for all $r\in S_2,$ we have

$$\log_{p-1} T(r,A_l)>b_2\left(\log_{q-1} r\right)^{\rho_{A_l}(p,q)}.$$

Therefore for a sequence $\{r_n\}$ such that $r_n\in S_2, r_n\to\infty$ we have

$$\log_{p-1} T(r_n,A_l)>b_2\left(\log_{q-1} r_n\right)^{\rho_{A_l}(p,q)}. \qquad (26)$$

Now if we take $b=\max\left\{\rho_{A_j}(p,q):j=0,1,\ldots,k,\ j\neq l\right\}<\rho_{A_l}(p,q)$, then for any given ε $\left(0<\varepsilon<\rho_{A_l}(p,q)-b\right)$ and sufficiently large r_n , we have

$$T(r_n,A_j)\leqslant \exp_{p-1}\left\{\left(\log_{q-1} r_n\right)^{b+\varepsilon}\right\}\leqslant \exp_{p-1}\left\{b_1\left(\log_{q-1} r_n\right)^{\rho_{A_l}(p,q)}\right\}. \qquad (27)$$

Again since

$$\max\left\{\tau_{A_j}(p,q):\rho_{A_j}(p,q)=\rho_{A_l}(p,q)\right\}<\tau_{A_l}(p,q),$$

Then for sufficiently large r_n , we have

$$T(r_n,A_j)\leqslant \exp_{p-1}\left\{b_1\left(\log_{q-1} r_n\right)^{\rho_{A_l}(p,q)}\right\}. \qquad (28)$$

Now for $r_n\in S_2\backslash S_1$, substituting (26) and (27) or (28) into (25) we obtain

$$\begin{aligned}
&\exp_{p-1}\left\{b_2\left(\log_{q-1} r_n\right)^{\rho_{A_l}(p,q)}\right\}<T(r_n,A_l)\\
\leqslant\ &k\exp_{p-1}\left\{b_1\left(\log_{q-1} r_n\right)^{\rho_{A_l}(p,q)}\right\}+o\left(\frac{\left(T\left(r_n+2|c_l|,f\right)\right)^{1+\varepsilon}}{r_n^{\delta}}\right).
\end{aligned}$$

The above implies

$$(1-o(1))\exp_{p-1}\left\{b_2\left(\log_{q-1} r_n\right)^{\rho_{A_l}(p,q)}\right\} < o\left(\frac{\left(T\left(r_n+2\left|c_l\right|,f\right)\right)^{1+\varepsilon}}{r_n^{\delta}}\right).$$

Hence the result follows. □

Next we consider the properties of meromorphic solutions of (2) where $A_0(z)$, $A_1(z),\ldots,A_k(z)$, F are entire functions.

THEOREM 4. *Let $A_0(z)$, $A_1(z),\ldots,A_k(z)$ be entire functions that satisfy the conditions stated in the Theorem 3 and let F be an entire function. Then the followings hold*

(i) If $\rho_F(p,q) < \rho_{A_l}(p,q)$ or $\rho_F(p,q) = \rho_{A_l}(p,q), \tau_F(p,q) \leqslant \tau_{A_l}(p,q)$, then every meromorphic solution $(f \not\equiv 0)$ of (2) satisfies $\rho_f(p,q) \geqslant \rho_{A_l}(p,q)$.

(ii) If $\rho_F(p,q) > \rho_{A_l}(p,q)$, then every meromorphic solution $(f \not\equiv 0)$ of (2) satisfies $\rho_f(p,q) \geqslant \rho_F(p,q)$.

Proof. We first consider Case (i), when $\rho_F(p,q) < \rho_{A_l}(p,q)$ or $\rho_F(p,q) = \rho_{A_l}(p,q)$, $\tau_F(p,q) \leqslant \tau_{A_l}(p,q)$.

First let $f\,(\not\equiv 0)$ be a meromorphic solution of (2) and divide (2) by $f(z+c_l)$ we get

$$\begin{aligned}-A_l(z) &= A_k(z)\frac{f(z+c_k)}{f(z+c_l)}+\ldots+A_{l-1}(z)\frac{f(z+c_{l-1})}{f(z+c_l)}+\ldots+A_1(z)\frac{f(z+c_1)}{f(z+c_l)}\\ &\quad +A_0(z)\frac{f(z)}{f(z+c_l)}-\frac{F(z)}{f(z+c_l)}.\end{aligned}$$

The above expression can be written as

$$-A_l(z) = \sum_{j=1,j\neq l}^{k} A_j(z)\frac{f(z+c_j)}{f(z+c_l)}+A_0(z)\frac{f(z)}{f(z+c_l)}-\frac{F(z)}{f(z+c_l)}. \tag{29}$$

Now for any given $\varepsilon > 0$ and sufficiently large r, using Lemma (6) and Lemma (7) in (29) we obtain (see [3])

$$\begin{aligned}T(r,A_l) = m(r,A_l(z)) &\leqslant m\left(r,\frac{F(z)}{f(z+c_l)}\right)+\sum_{j=0,j\neq l}^{k} m(r,A_j(z))\\ &\quad +\sum_{j=1,j\neq l}^{k} m\left(r,\frac{f(z+c_j)}{f(z+c_l)}\right)+m\left(r,\frac{f(z)}{f(z+c_l)}\right)+O(1)\\ &\leqslant T(r,F)+\sum_{j=0,j\neq l}^{k} T(r,A_j(z))+2T\left(r+\left|c_l\right|,f(z)\right)\\ &\quad +o\left(\frac{\left(T\left(r+2\left|c_l\right|,f\right)\right)^{1+\varepsilon}}{r^{\delta}}\right),\end{aligned} \tag{30}$$

for $r \to \infty,\ r \notin S_1,$ where S_1 is a set of finite logarithmic measure.

Consider two real numbers b_1, b_2 such that

$$\max\left\{\tau_{A_j}(p,q) : \rho_{A_j}(p,q) = \rho_{A_l}(p,q)\right\} < b_1 < b_2 < \tau_{A_l}(p,q).$$

Now by Lemma (8) there exists a subset $S_2 \subset [1,+\infty)$ of infinite logarithmic measure such that for all $r \in S_2,$ we have

$$\log_{p-1} T(r,F) \leqslant b_1 \left(\log_{q-1} r\right)^{\rho_{A_l}(p,q)}.$$

Therefore for a sequence $\{r_n\}$ such that $r_n \in S_2, r_n \to \infty$ we have

$$\log_{p-1} T(r_n,F) \leqslant b_1 \left(\log_{q-1} r_n\right)^{\rho_{A_l}(p,q)}. \tag{31}$$

Now for $r_n \in S_2 \backslash S_1,$ substituting (26), (27) or (28) and (31) into (30) we obtain

$$\begin{aligned} &\exp_{p-1}\left\{b_2 \left(\log_{q-1} r_n\right)^{\rho_{A_l}(p,q)}\right\} \\ &< T(r_n,A_l) \\ &\leqslant (k+1)\exp_{p-1}\left\{b_1 \left(\log_{q-1} r_n\right)^{\rho_{A_l}(p,q)}\right\} + 3\left(T(2r_n,f)\right)^2. \end{aligned} \tag{32}$$

By (32) first part of the theorem is proved.

Case (ii): Consider $\rho_F(p,q) > \rho_{A_l}(p,q)$ and let let $f(\not\equiv 0)$ be a meromorphic solution of (2).

Now for any given $\varepsilon > 0$ and sufficiently large $r,$ using Lemma (6) and Lemma (7) in (29) we obtain (see [3])

$$\begin{aligned} T(r,F) &\leqslant \sum_{j=0}^{k} T(r,A_j(z)) + \sum_{j=1}^{k} T(r, f(z+c_j)) + T(r, f(z)) + O(1) \\ &\leqslant \sum_{j=0}^{k} T(r,A_j(z)) + (2k+1)T(2r, f(z)) + O(1). \end{aligned} \tag{33}$$

Now by definition of the (p,q)-order there exists a sequence $\{r_n\}$ such that $r_n \to \infty$ and for any given ε $\left(0 < 2\varepsilon < \rho_F(p,q) - \rho_{A_l}(p,q)\right),$ we have

$$T(r_n,F) \geqslant \exp_{p-1}\left\{\left(\log_{q-1} r_n\right)^{\rho_F(p,q)-\varepsilon}\right\} \tag{34}$$

and for $j = 0,1,\ldots,k$

$$T(r_n,A_j(z)) \leqslant \exp_{p-1}\left\{\left(\log_{q-1} r_n\right)^{b+\varepsilon}\right\} \leqslant \exp_{p-1}\left\{\left(\log_{q-1} r_n\right)^{\rho_{A_l}(p,q)+\varepsilon}\right\}, \tag{35}$$

where $b = \max\left\{\rho_{A_j}(p,q) : j = 0,1,\ldots,k,\ j \neq l\right\} < \rho_{A_l}(p,q).$

Substituting (34) and (35) into (33) we obtain

$$\begin{aligned} \exp_{p-1}\left\{\left(\log_{q-1} r_n\right)^{\rho_F(p,q)-\varepsilon}\right\} &\leqslant (k+1)\exp_{p-1}\left\{\left(\log_{q-1} r_n\right)^{\rho_{A_l}(p,q)+\varepsilon}\right\} \\ &\quad + (2k+1)T(2r, f(z)). \end{aligned} \tag{36}$$

By (36) second part of the theorem is proved. □

Next two theorems, i.e. Theorem 5 and Theorem 6 are based on linear difference equation with meromorphic coefficients. In Theorem 5 we take homogeneous linear difference equation with one coefficient having maximal (p,q)-order.

THEOREM 5. *Let $A_j(z)(j=0,1,\ldots,k)$ be meromorphic functions. If there exits an $A_m(z)$ $(0 \leqslant m \leqslant k)$ such that*

$$\lambda_{\frac{1}{A_m}}(p,q) < \rho_{A_m}(p,q) < \infty,$$
$$\text{and} \quad \max\{\rho_{A_j}(p,q) : j=0,1,\ldots,k,\ j \neq m\} < \rho_{A_m}(p,q)$$

then for every nonzero meromorphic solution $f(z)$ of (1) satisfies $\rho_f(p,q) \geqslant \rho_{A_m}(p,q)$.

Proof. Let f be a meromorphic solution of (1) and put $c_0=0$. We divide (1) by $f(z+c_m)$ and we get

$$-A_m(z) = \sum_{j=0, j\neq m}^{k} \frac{A_j(z)f(z+c_j)}{f(z+c_m)}. \tag{37}$$

Now from Lemma 7 for any $\varepsilon > 0$ we get,

$$m\left(r, \frac{f(z+c_j)}{f(z+c_m)}\right) \leqslant o\left(\frac{\{T(r+3C,f)\}^{1+\varepsilon}}{r^{\delta}}\right), \quad j=0,1,\ldots,k,\ j\neq m,$$

where $C=\max_{0\leqslant \leqslant k}\{|c_j| : j=0,1,\ldots,k\}$, $r \notin S_1$ where S_1 is chosen as Lemma 7.

Using the above result, from (37) we get,

$$\begin{aligned}
&T(r,A_m)\\
&= m(r,A_m)+N(r,A_m)\\
&\leqslant \sum_{j=0,j\neq m}^{k} m(r,A_j) + \sum_{j=0,j\neq m}^{k} m\left(r, \frac{f(z+c_j)}{f(z+c_m)}\right) + N(r,A_m) + O(1)\\
&\leqslant \sum_{j=0,j\neq m}^{k} T(r,A_j) + o\left(\frac{\{T(r+3C,f)\}^{1+\varepsilon}}{r^{\delta}}\right) + N(r,A_m) + O(1)\\
&\leqslant \sum_{j=0,j\neq m}^{k} T(r,A_j) + \{T(2r,f)\}^2 + N(r,A_m) + O(1)
\end{aligned} \tag{38}$$

for $r \notin S_1$.

Now we denote, $\rho = \rho_{A_m}(p,q)$, $\rho_1 = \max\{\rho_{A_j}(p,q) : j=0,1,\ldots,k;\ j\neq m\}$. Then clearly $\rho_1 < \rho$.

For that ε we have

$$T(r,A_m) > \exp_{p-1}\left\{(\log_{q-1} r)^{\rho_1-\varepsilon}\right\} \tag{39}$$

for sufficiently large r with $r \in S_2$, where S_2 be a set with infinite logarithmic measure.

And for $j \neq m$, for that ε we have

$$T(r,A_j) \leqslant \exp_{p-1}\left\{(\log_{q-1} r)^{\rho_1+\varepsilon}\right\} \tag{40}$$

for sufficiently large r.

Again by the definition of $\lambda_{\frac{1}{A_m}}(p,q)$, we have for the above ε and for sufficiently large r

$$N(r,A_m) \leqslant \exp_{p-1}\left\{(\log_{q-1} r)^{\lambda+\varepsilon}\right\} \tag{41}$$

taking $\lambda_{\frac{1}{A_m}}(p,q) = \lambda$.

Now, using all the above relations (39)–(41) and chosing ε such that $0 < \varepsilon < \frac{1}{2}\min\{\rho-\rho_1, \rho-\lambda\}$, we have from (38)

$$\begin{aligned}\exp_{p-1}\left\{(\tau-\varepsilon)(\log_{q-1} r)^{\rho}\right\} &< O\left(\exp_{p-1}\left\{(\log_{q-1} r)^{\rho_1+\varepsilon}\right\}\right) + 3\left\{T(2r,f)\right\}^2 \\ &\quad + \exp_{p-1}\left\{(\log_{q-1} r)^{\lambda+\varepsilon}\right\} + O(1) \\ \Rightarrow 3\left\{T(2r,f)\right\}^2 &> O\left(\exp_{p-1}\left\{(\log_{q-1} r)^{\rho+\varepsilon}\right\}\right)\end{aligned}$$

for sufficiently large r and $r \in S_2 \backslash S_1$.

Which implies, $\rho_f(p,q) \geqslant \rho$. □

In the next theorem we consider non-homogeneous linear difference equation which may have more than one coefficient with the maximal (p,q)-order. For those type of equations we need to consider the (p,q)-type among the coefficients having maximal (p,q)-order.

THEOREM 6. *Let $A_j(z)(j=0,1,\ldots,k)$ and $F(z)$ be meromorphic functions. If there exits an $A_m(z)(0 \leqslant m \leqslant k)$ such that*

$$\lambda_{\frac{1}{A_m}}(p,q) < \rho_{A_m}(p,q) < \infty,$$

$$\max\{\rho_{A_j}(p,q) : j=0,1,\ldots,k,\ j \neq m\} \leqslant \rho_{A_m}(p,q)$$

and

$$\max\{\tau_{A_j}(p,q) : \rho_{A_j}(p,q) = \rho_{A_m}(p,q),\ j=0,1,\ldots,k,\ j \neq m\} < \tau_{A_m}(p,q) < \infty,$$

then the following cases arise:

i) If $\rho_F(p,q) < \rho_{A_m}(p,q)$, or $\rho_F(p,q) = \rho_{A_m}(p,q)$ and $\tau_F(p,q) \neq \tau_{A_m}(p,q)$, then every nonzero meromorphic solution $f(z)$ of (2) satisfies

$$\rho_f(p,q) \geqslant \rho_{A_m}(p,q).$$

ii) If $\rho_F(p,q) > \rho_{A_m}(p,q)$ then every nonzero meromorphic solution $f(z)$ of (2) satisfies $\rho_f(p,q) \geqslant \rho_F(p,q)$.

Proof. Let f be a meromorphic solution of (2). We divide (2) by $f(z+c_m)$ and we get

$$-A_m(z) = \sum_{j=0,j\neq m}^{k} \frac{A_j(z)f(z+c_j)}{f(z+c_m)} - \frac{F(z)}{f(z+c_m)}. \tag{42}$$

Now from Lemma 6 and Lemma 7, for any $\varepsilon > 0$ we get,

$$m\left(r, \frac{f(z+c_j)}{f(z+c_m)}\right) \leqslant o\left(\frac{\{T(r+3C,f)\}^{1+\varepsilon}}{r^{\delta}}\right),\ j=0,1,\ldots,k,\ j\neq m,$$

$$\text{and } m\left(r,\frac{1}{f(z+c_m)}\right) \leqslant T\left(r,\frac{1}{f(z+c_m)}\right)$$
$$= T\left(r, f(z+c_m)\right) + O(1) \leqslant (1+O(1))T\left(r+C,f\right).$$

where $C = \max_{0\leqslant \leqslant k}\{\left|c_j\right| : j=0,1,\ldots,k\}$, $r \notin S_1$ where S_1 is chosen as Lemma 7.

Using these from (42) we get,

$$\begin{aligned}
T(r,A_m) &= m(r,A_m) + N(r,A_m) \\
&\leqslant \sum_{j=0,j\neq m}^{k} m(r,A_j) + \sum_{j=0,j\neq m}^{k} m\left(r,\frac{f(z+c_j)}{f(z+c_m)}\right) + m(r,F) \\
&\quad + m\left(r,\frac{1}{f(z+c_m)}\right) + N(r,A_m) + O(1) \\
&\leqslant \sum_{j=0,j\neq m}^{k} T(r,A_j) + o\left(\frac{\{T(r+3C,f)\}^{1+\varepsilon}}{r^{\delta}}\right) + T(r,F) \\
&\quad + (1+O(1))T\left(r+C,f\right) + N(r,A_m) + O(1) \\
&\leqslant \sum_{j=0,j\neq m}^{k} T(r,A_j) + 3\left\{T(2r,f)\right\}^2 + T(r,F) + N(r,A_m) + O(1)
\end{aligned} \tag{43}$$

for $r \notin S_1$.

Now denote, $\rho = \rho_{A_m}(p,q)$, $\rho_1 = \max\{\rho_{A_j}(p,q) : j=0,1,\ldots,k,\ \rho_{A_j}(p,q) < \rho\}$, $\tau = \tau_{A_m}(p,q)$ and $\tau_1 = \max\{\tau_{A_j}(p,q) : j=0,1,\ldots,k,\ j\neq m,\ \rho_{A_j}(p,q) = \rho\}$.

Then clearly

$$\rho_1 < \rho \text{ and } \tau_1 < \tau$$

by the given hypothesis.

From Lemma 9 for that ε we have

$$T(r,A_m) > \exp_{p-1}\left\{(\tau-\varepsilon)(\log_{q-1} r)^{\rho}\right\} \tag{44}$$

for sufficiently large r with $r \in S_2$, where S_2 is a set with infinite logarithmic measure.

Again if for some j, $\rho_{A_j}(p,q) < \rho$, then for that ε we have

$$T(r,A_j) \leqslant \exp_{p-1}\left\{(\log_{q-1} r)^{\rho_1+\varepsilon}\right\} \tag{45}$$

for sufficiently large r.

And for those j, $j \neq m$ and $\rho_{A_j}(p,q) = \rho$, for that ε we have

$$T(r,A_j) \leqslant \exp_{p-1}\left\{(\tau_1+\varepsilon)(\log_{q-1} r)^{\rho}\right\} \tag{46}$$

for sufficiently large r.

Again by the definition of $\lambda_{\frac{1}{A_m}}(p,q)$, we have for the above ε and for sufficiently large r

$$N(r,A_m) \leqslant \exp_{p-1}\left\{(\log_{q-1} r)^{\lambda+\varepsilon}\right\} \tag{47}$$

taking $\lambda_{\frac{1}{A_m}}(p,q) = \lambda$.

Now for the proof of the first part, we take $\rho_F(p,q) < \rho$, then for that ε we have

$$T(r,F) \leqslant \exp_{p-1}\left\{(\log_{q-1} r)^{\rho_F(p,q)+\varepsilon}\right\} \tag{48}$$

for sufficiently large r.

Now, using all the above relations (44)–(48) and chosing ε such that $0 < \varepsilon < \frac{1}{2}\min\{\rho-\rho_1,\, \tau-\tau_1,\, \rho-\lambda,\, \rho-\rho_F(p,q)\}$, we have from (43)

$$\begin{aligned}
&\exp_{p-1}\left\{(\tau-\varepsilon)(\log_{q-1} r)^{\rho}\right\} \\
&< O\left(\exp_{p-1}\left\{(\log_{q-1} r)^{\rho_1+\varepsilon}\right\}\right) + O\left(\exp_{p-1}\left\{(\tau_1+\varepsilon)(\log_{q-1} r)^{\rho}\right\}\right) \\
&\quad +3\{T(2r,f)\}^2 + \exp_{p-1}\left\{(\log_{q-1} r)^{\rho_F(p,q)+\varepsilon}\right\} + \exp_{p-1}\left\{(\log_{q-1} r)^{\lambda+\varepsilon}\right\} + O(1) \\
&\Rightarrow 3\{T(2r,f)\}^2 > O\left(\exp_{p-1}\left\{(\log_{q-1} r)^{\rho+\varepsilon}\right\}\right)
\end{aligned}$$

for sufficiently large r and $r \in S_2 \backslash S_1$.

Which implies, $\rho_f(p,q) \geqslant \rho$.

Next we suppose that $\rho_F(p,q) = \rho$ and $\tau_F(p,q) < \tau$, then for the above ε we have

$$T(r,F) \leqslant \exp_{p-1}\left\{(\tau_F(p,q)+\varepsilon)(\log_{q-1} r)^{\rho}\right\} \tag{49}$$

for sufficiently large r.

Now, using relations (44)–(47) and (2), such that for choosen ε where

$$0 < \varepsilon < \frac{1}{2}\min\{\rho-\rho_1, \tau-\tau_1, \rho-\lambda, \tau-\tau_F(p,q)\},$$

we have from (43)

$$\begin{aligned}
&\exp_{p-1}\left\{(\tau-\varepsilon)(\log_{q-1} r)^{\rho}\right\} < O\left(\exp_{p-1}\left\{(\log_{q-1} r)^{\rho_1+\varepsilon}\right\}\right) \\
&+O\left(\exp_{p-1}\left\{(\tau_1+\varepsilon)(\log_{q-1} r)^{\rho}\right\}\right) + 3\{T(2r,f)\}^2 \\
&+\exp_{p-1}\left\{(\tau_F(p,q)+\varepsilon)(\log_{q-1} r)^{\rho}\right\} + \exp_{p-1}\left\{(\log_{q-1} r)^{\lambda+\varepsilon}\right\} + O(1) \\
&\Rightarrow 3\{T(2r,f)\}^2 > O\left(\exp_{p-1}\left\{(\log_{q-1} r)^{\rho+\varepsilon}\right\}\right),
\end{aligned}$$

for sufficiently large r and $r \in S_2 \backslash S_1$.

Which implies, $\rho_f(p,q) \geqslant \rho$.

For the last case of first part we take, $\rho_F(p,q) = \rho$ and $\tau_F(p,q) > \tau$, then by Lemma 9, and for the above ε we have

$$T(r,F) > \exp_{p-1}\left\{(\tau_F(p,q) - \varepsilon)(\log_{q-1} r)^{\rho}\right\} \tag{50}$$

for sufficiently large r and $r \in S_3$, where S_3 is chosen as Lemma 9.

Again by the definition of $\tau_{A_m}(p,q)$, we have for the above ε and for sufficiently large r

$$T(r,A_m) \leqslant \exp_{p-1}\left\{(\tau+\varepsilon)(\log_{q-1} r)^{\rho}\right\}. \tag{51}$$

Now from (2) and Lemma 6 it follows that

$$T(r,F) \leqslant \sum_{j=0, j\neq m}^{k} T(r,A_j) + T(r,A_m) + (k+2)T(2r,f) \tag{52}$$

for sufficiently large r

Now, using relations (44), (45) and (50)–(52) and choosing ε such that $0 < \varepsilon < \frac{1}{2}\min\{\rho-\rho_1, \tau-\tau_1, \tau_F(p,q)-\tau\}$, we have

$$\begin{aligned}
&\exp_{p-1}\left\{(\tau_F(p,q)-\varepsilon)(\log_{q-1} r)^{\rho}\right\}\\
<\ & O\left(\exp_{p-1}\left\{(\log_{q-1} r)^{\rho_1+\varepsilon}\right\}\right) + O\left(\exp_{p-1}\left\{(\tau_1+\varepsilon)(\log_{q-1} r)^{\rho}\right\}\right)\\
&+O\left(\exp_{p-1}\left\{(\tau+\varepsilon)(\log_{q-1} r)^{\rho}\right\}\right) + (k+2)T(2r,f)\\
\Rightarrow\ & (k+2)T(2r,f) > O\left(\exp_{p-1}\left\{(\log_{q-1} r)^{\rho+\varepsilon}\right\}\right),
\end{aligned}$$

for sufficiently large r and $r \in S_3 \backslash S_1$.

It follows that, $\rho_f(p,q) \geqslant \rho$.

For the second part of the theorem, we take $\rho_F(p,q) > \rho = \rho_{A_m}(p,q)$.

If possible we suppose that $\rho_f(p,q) < \rho_F(p,q)$, then from (2) we get

$$\rho_q^p A_k(z)f(z+c_k) + A_{k-1}(z)f(z+c_{k-1}) + \cdots + A_1(z)f(z+c_1) + A_0(z)f(z) < \rho_q^p(F(z))$$

which is a contradiction.

Hence we have $\rho_f(p,q) \geqslant \rho_F(p,q)$. $\square$

4. Examples

In this section we provide some examples which illustrate few of our main results. First consider an example corresponding to the Theorem 1.

EXAMPLE 1. Consider the difference equation

$$A_2(z)f(z+2\pi) + A_1(z)f(z+\pi) + A_0(z)f(z) = 0, \tag{53}$$

where

$$A_2(z) = \exp\left\{4\pi z + \pi^2\right\},\ A_1(z) = 1,$$

$$A_0(z) = -\exp\{-3\pi z\} - \exp\left\{-\frac{\pi z}{2} - \frac{\pi^2}{4}\right\}.$$

We have

$$\rho_{A_2}(2,2) = \rho_{A_0}(2,2) = 1 \quad \text{and} \quad \rho_{A_1}(2,2) = 0.$$

Therefore

$$1 = \max_{0 \leqslant j \leqslant 1} \left\{\rho_{A_j}(2,2)\right\} \leqslant \rho = 1.$$

Choose

$$D = \left\{z \in \mathbb{C} : z = re^{i\theta}, r \in [1, +\infty[, \frac{\pi}{6} \leqslant \theta \leqslant \frac{\pi}{3}\right\},$$

a complex set satisfying $\overline{\log dens}\{r = |z| : z \in D\} > 0$.

Thus we get sufficiently small δ $(0 < \delta < \rho = 1)$ for which

$$\begin{aligned} |A_2(z)| &= \left|\exp\left\{4\pi z + \pi^2\right\}\right| \\ &= \left|\exp\left\{4\pi r \cos\theta + \pi^2\right\}\right| \\ &\geqslant \left|\exp\left\{\pi r + \pi^2\right\}\right| \\ &\geqslant \exp_2\left\{(\log r)^{1-\delta}\right\}, \end{aligned}$$

$$|A_1(z)| = 1 \leqslant \exp_2\left\{\frac{1}{2}.(\log r)^{1-\delta}\right\}$$

and

$$\begin{aligned} |A_0(z)| &= \left|-\exp\{-3\pi z\} - \exp\left\{-\frac{\pi z}{2} - \frac{\pi^2}{4}\right\}\right| \\ &= \exp\{-3\pi r \cos\theta\} + \exp\left\{-\frac{\pi r \cos\theta}{2} - \frac{\pi^2}{4}\right\} \\ &\leqslant \exp\left\{-\frac{\pi r}{2} - \frac{\pi^2}{4}\right\} \leqslant \exp_2\left\{\frac{1}{2}.(\log r)^{1-\delta}\right\} \end{aligned}$$

as $z \to \infty$ for $z \in D$.

The meromorphic function $f(z) = e^{-\left(\frac{z^2}{2}\right)} \tan z$ is a solution of (53).

Thus all conditions of Theorem 1 are satisfied with $a = 1$ and $b = \frac{1}{2}$ and we get

$$1 = \rho_f(2,2) \geqslant \rho_{A_2}(2,2) = 1.$$

EXAMPLE 2. For Theorem 4 (i), we consider the meromorphic function

$$f(z) = \frac{\cos z}{z}.$$

Then $f(z)$ satisfies the difference equation

$$A_2(z) f(z+2\pi) + A_1(z) f(z+\pi) + A_0(z) f(z) = F(z),$$

where

$$A_2(z) = (z+2\pi)\exp z, \quad A_1(z) = (z+\pi)\exp z,$$
$$A_0(z) = z\exp z^2 \quad \text{and} \quad F(z) = \cos z \exp z^2.$$

We have

$$\rho_{A_2}(2,2) = \rho_{A_1}(2,2) = \rho_{A_0}(2,2) = 1 = \rho_F(2,2)$$

and

$$\tau_{A_2}(2,2) = \tau_{A_1}(2,2) = 1, \quad \tau_{A_0}(2,2) = \tau_F(2,2) = 2.$$

Therefore

$$1 = \max\left\{\rho_{A_j}(2,2) : j = 1,2\right\} \leqslant \rho_{A_0}(2,2) = 1,$$
$$1 = \max\left\{\tau_{A_j}(2,2) : \rho_{A_j}(2,2) = \rho_{A_0}(2,2)\right\} < \tau_{A_0}(2,2) = 2,$$
$$\rho_{A_0}(2,2) = 1 = \rho_F(2,2) \quad \text{and} \quad \tau_F(2,2) = 2 \leqslant \tau_{A_0}(2,2) = 2.$$

Thus all conditions of Theorem 4 (i) are satisfied and we get

$$1 = \rho_f(2,2) \geqslant \rho_{A_0}(2,2) = 1.$$

For Theorem 5 we consider the following example:

EXAMPLE 3. Consider the difference equation

$$A_2(z)f(z+2) + A_1(z)f(z+1) + A_0(z)f(z) = 0, \tag{54}$$

where

$$A_2(z) = 1, \quad A_1(z) = 1, \quad A_0(z) = -\exp\{4z+4\} - \exp\{2z+1\}.$$

We have

$$\rho_{A_2}(2,2) = \rho_{A_1}(2,2) = 0, \ \ \rho_{A_0}(2,2) = 1$$

and

$$\lambda_{\frac{1}{A_2}}(2,2) = \lambda_{\frac{1}{A_1}}(2,2) = \lambda_{\frac{1}{A_0}}(2,2) = 0.$$

Therefore

$$0 = \lambda_{\frac{1}{A_0}}(2,2) < \rho_{A_0}(2,2) = 1$$

and

$$0 = \max\{\rho_{A_2}(2,2), \rho_{A_1}(2,2)\} < \rho_{A_0}(2,2) = 1.$$

The function $f(z) = e^{z^2}$ is a solution of (54).
Thus all conditions of Theorem 5 are satisfied and we get

$$1 = \rho_f(2,2) \geqslant \rho_{A_0}(2,2) = 1.$$

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