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NEW CONVERGENCE DEFINITIONS FOR DOUBLE SEQUENCES IN g–METRIC SPACES

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Abstract. In this paper, we define g-convergence and g-Cauchy of double sequences in g-metric spaces. Also we prove that g-limit is unique and every g-convergent double sequence is a g-Cauchy sequence. Additionally g-statistical convergence of double sequences is introduced and the theorem giving the relationship between statistical convergence and strongly Cesáro summability in a g-metric space is demonstrated. Further, we put forward the notations of g-lacunary statistical convergence and g-strongly lacunary convergence of double sequences and we also present some inclusion theorems.

1. Introduction and preliminaries

In mathematical analysis, a distance function or metric is a generalization of the concept of physical distance. There are several ways to generalize the concept of distance function [15]. Because of very big and complex data sets, the description of the distance function needs to be generalized. Gähler [12] stated that a 2-metric is a generalization of the usual concept of a metric, but various researches demonstrated that there is no relation between these two functions. For example, Ha et al. [10] demonstrated that a 2-metric do not have to be a continuous function of its variables. These opinions led Bapure Dhage [6] in his Ph.D. thesis to investigate a new class of generalized metric space named D-metric space. Dhage [6] aimed to establish topological features in these spaces and these works formed the basis for those who studied in this field for a long time. In the studies [19, 20], the authors denoted that most of the claims concerning the basic topological features of D-metric spaces are incorrect, nullifying the validity of many results acquired in these spaces.

Among them, the notion of G-metric space that has been studied by Mustafa and Sims [18] is a different generalization of the ordinary metric. Metrics in this space are distance between three points. These properties supply if $G(x;y;z)$ is the perimeter of a triangle with vertices at x; y and z in $\mathbb{R}^2$, further getting α in the interior of the triangle denotes that $(G5)$ is best possible. G-metric function is a distance function that generalizes the notion of distance between 3 points. For more generalization, Choi et al. [5] put forward g-metric with degree n, that is a distance between $n+1$ points. Statistically convergent sequences with regards to the metrics on g-metric spaces was

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presented and significant properties of this statistical form of convergence were examined by Abazari [1].

A g-metric space and its properties are described by Choi et al. [5] as follows:

Let X be a nonempty set. A function $g: X^{t+1} \to \mathbb{R}^+$ is called a g-metric with order t on X if it satisfies the following conditions:

(g_1) $g(u_0, u_1, \cdots, u_t) = 0$ iff $u_0 = u_1 = \cdots = u_t$,

(g_2) $g(u_0, u_1, \cdots, u_t) = g\left(u_{\rho(0)}, u_{\rho(1)}, \cdots, u_{\rho(t)}\right)$ for any permutation ρ on $\{0, 1, \cdots, t\}$,

(g_3) $g(u_0, u_1, \cdots, u_t) \leqslant g(v_0, v_1, \cdots, v_t)$ for all $(u_0, \cdots, u_t), (v_0, \cdots, v_t) \in X^{t+1}$ with $\{u_i : i = 1, \cdots, t\} \subset \{v_i : i = 1, \cdots, t\}$,

(g_4) For all $u_0, u_1, \cdots, u_s, v_0, v_1, \cdots, v_m, w \in X$ with $s + m + 1 = t$

$$\begin{aligned} g(u_0, u_1, \cdots, u_s, v_0, v_1, \cdots, v_m) \leqslant\ & g(u_0, u_1, \cdots, u_s, w, \cdots, w) \\ & + g(v_0, v_1, \cdots, v_m, w, \cdots, w). \end{aligned}$$

The pair (X, g) is called a g-metric space.

THEOREM 1. ([5]) *Let (X, g) be a g-metric space with order s. In this context, the following properties are provided:*

(i) $g(\underbrace{u, \cdots, u}_{s \text{ times}}, v, \cdots, v) \leqslant g(\underbrace{u, \cdots, u}_{s \text{ times}}, w, \cdots, w) + g(\underbrace{w, \cdots, w}_{s \text{ times}}, v, \cdots, v)$,

(ii) $g(u, v, \cdots, v) \leqslant g(u, w, \cdots, w) + g(w, v, \cdots, v)$,

(iii) $g(\underbrace{u, \cdots, u}_{s \text{ times}}, v, \cdots, v) \leqslant sg(u, v, \cdots, v)$ *and*

$$g(\underbrace{u, \cdots, u}_{s \text{ times}}, v, \cdots, v) \leqslant (t + 1 - s) g(v, u, \cdots, u),$$

(iv) $g(u_0, u_1, \cdots, u_t) \leqslant \sum_{i=0}^{t} g(u_i, v, \cdots, v)$,

(v) $|g(v, u_1, u_2, \cdots, u_t) - g(w, u_1, u_2, \cdots, u_t)| \leqslant \max\{g(v, w, \cdots, w), g(w, v, \cdots, v)\}$,

(vi) $|g(\underbrace{u, \cdots, u}_{s \text{ times}}, v, \cdots, v) - g(\underbrace{u, \cdots, u}_{\tilde{s} \text{ times}}, v, \cdots, v)| \leqslant |s - \tilde{s}|\, g(u, v, \cdots, v)$,

(vii) $g(u, v, \cdots, v) \leqslant (1 + (s-1)(t+1-s)) g(\underbrace{u, \cdots, u}_{s \text{ times}}, v, \cdots, v)$.

As a result, each g-metric space is topologically equal to a metric space derived from the ordinary metric. This allows many concepts and findings from metric spaces to be transferred into the g-metric context.

Let (X, g) be a g-metric space, $u \in X$ be a point and (u_k) be a sequence in X. Then, we have the followings:

(i) (u_k) is g-convergent to u, if for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $i_1, i_2, \cdots, i_t \geqslant n_0$, $g(u, u_{i_1}, \cdots, u_{i_t}) < \varepsilon$.

(ii) (u_k) is said to be g-Cauchy, if for all $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that

$$i_0, i_1, i_2, \cdots, i_t \geqslant n_0 \Rightarrow g\left(u_{i_0}, u_{i_1}, \cdots, u_{i_t}\right) < \varepsilon.$$

Statistical convergence, which is one of our other main subjects, is investigated by Fast [7] and Steinhaus [25], independently. In 1959, Schoenberg [24] presented several fundamental statistical convergence properties. After then, numerous research have been done on the idea of statistical convergence, which has been rapidly studied. Fridy [9] presented the properties of statistical convergence in a significant study. Quite recently, Savaş and Patterson [23] investigated the lacunary statistical analogue for double sequences.

The density of the subsets of the set $\mathbb{N}$ of natural numbers is base to the concept of statistical convergence. The natural density of a subset A of $\mathbb{N}$ is defined by

$$\delta(A) = \lim_n \frac{1}{n} |\{k \leqslant n : k \in A\}|$$

and the sequence (u_k) is said to be statistically convergent to a point u, if for every $\varepsilon > 0$

$$\lim_n \frac{1}{n} |\{k \leqslant n : |u_k - u| \geqslant \varepsilon\}| = 0.$$

If (u_k) is statistically convergent to u, it is denoted as $st - \lim u_k = u$. Also, the sequence (u_k) is said to be statistically Cauchy sequence, if for every $\varepsilon > 0$ and there exists a positive integer number N depending on ε such that,

$$\lim_n \frac{1}{n} |\{k \leqslant n : |u_k - u_N| \geqslant \varepsilon\}| = 0.$$

A double sequence $\theta_2 = \theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary sequence if there exist two increasing sequences of integers (k_r) and (l_s) such that

$$k_0 = 0, \quad h_r = k_r - k_{r-1} \to \infty \quad \text{and} \quad l_0 = 0, \quad h_s = l_s - l_{s-1} \to \infty, \quad r, s \to \infty.$$

We will utilize the following notation $k_{rs} := k_r l_s$, $h_{rs} := h_r h_s$ and θ_{rs} is identified by

$$I_{rs} := \{(k,l) : k_{r-1} < k \leqslant k_r \quad \text{and} \quad l_{s-1} < l \leqslant l_s\},$$

$$q_r := \frac{k_r}{k_{r-1}}, \quad q_s := \frac{l_s}{l_{s-1}} \quad \text{and} \quad q_{rs} := q_r q_s.$$

Throughout the paper, by $\theta_2 = \theta_{r,s} = \{(k_r, l_s)\}$ we will denote a double lacunary sequence of positive real numbers, respectively, unless otherwise stated.

In 2022, Abazari [1] extended on these ideas by proposing statistical convergent and statistical Cauchy analogs of these definitions, as well as their properties in g-metric spaces. Let $t \in \mathbb{N}, A \in \mathbb{N}^t$ and

$$A(n) = \{i_1, i_2, \cdots, i_t \leqslant n : (i_1, i_2, \cdots, i_t) \in A\},$$

then

$$\delta_t(A) = \lim_n \frac{t!}{n^t} |A(n)|$$

is called t-dimensional natural density of the set A.

DEFINITION 1. Let (u_n) be a sequence in a g-metric space (X,g).
(i) (u_n) is statistically convergent to u, if for all $\varepsilon > 0$,

$$\lim_n \frac{t!}{n^t} \left|\left\{(i_1,i_2,\cdots,i_t) \in \mathbb{N}^t : i_1,i_2,\cdots,i_t \leqslant n, g\left(u,u_{i_1},\cdots,u_{i_t}\right) \geqslant \varepsilon\right\}\right| = 0,$$

and is denoted by $gS - \lim u_n = u$ or $u_n \xrightarrow{gS} u$.
(ii) (u_n) is said to be statistically g-Cauchy, if for all $\varepsilon > 0$ and there exist $i_0 \in \mathbb{N}$ such that

$$\lim_n \frac{t!}{n^t} \left|\left\{(i_1,i_2,\cdots,i_t) \in \mathbb{N}^t : i_1,i_2,\cdots,i_t \leqslant n, g\left(u_{i_0},u_{i_1},\cdots,u_{i_t}\right) \geqslant \varepsilon\right\}\right| = 0.$$

Pringsheim's investigations in 1897 and 1900 include double sequences and the well-known Pringsheim idea of convergence for these sequences ([21], [22]). Finally, let's finish by providing the definition of double sequence that we employ in our research.

Let X be a nonempty set and $\mathbb{N}$ be the set of all nonnegative integers. The sequence (u_{jk}) is said to be double sequence such that the function u is defined as

$$u : \mathbb{N} \times \mathbb{N} \to X; \ \ (j,k) \to u_{jk}.$$

A double sequence (u_{jk}) is said to be in the Pringsheim's sense if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|u_{jk} - u| < \varepsilon$ whenever $j,k \geqslant n_0$. u is called the Pringsheim limit of (u_{jk}). Additionally, a double sequence (u_{jk}) is said to be Cauchy sequence if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|u_{pq} - u_{jk}| < \varepsilon$ for all $p \geqslant j \geqslant n_0$ and $q \geqslant k \geqslant n_0$. We refer to [2, 3, 13, 14, 16, 17, 26] and recent monograph [4] for relevant literature for double sequences.

The aim of the present paper is to investigate the new kind of convergence for sequences in g-metric spaces. The following is how the paper is structured. The literature review is covered in Section 1 of the introduction. The key findings are then demonstrated in Section 2. That is, we intend to investigate the concepts of statistical convergence and lacunary statistical convergence for double sequences in g-metric spaces and to develop essential features of these concepts.

2. Main results

In this section, we first define g-convergence and g-Cauchy sequence for double sequences in g-metric spaces. The idea of g-statistical convergence in double sequences is then introduced. Finally, we prove the theorem that explains the connection between statistical convergence and strongly Cesáro summability after stating the classically known theorems in double sequences.

DEFINITION 2. (X,g) be a g-metric space and (u_{jk}) be a double sequence in X.
(i) (u_{jk}) is said to be g-convergent in the Pringsheim's sense if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$g\left(u,u_{j_1k_1},\cdots,u_{j_tk_t}\right) < \varepsilon$$

whenever $j_1,\cdots,j_t \geqslant n_0$ and $k_1,\cdots,k_t \geqslant n_0$. u is called the Pringsheim limit of $\left(u_{jk}\right)$.

(ii) $\left(u_{jk}\right)$ is said to be g-Cauchy sequence if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$g\left(u_{j_0k_0}, u_{j_1k_1}, \cdots, u_{j_tk_t}\right) < \varepsilon$$

for all $j_0 \geqslant j_1 \geqslant \cdots \geqslant j_t \geqslant n_0$ and $k_0 \geqslant k_1 \geqslant \cdots \geqslant k_t \geqslant n_0$.

PROPOSITION 1. *The following are true.*

(i) The limit of a g-convergent double sequence in a g-metric space is unique.

(ii) In g-metric space every g-convergent double sequence is a g-Cauchy sequence.

Proof. (i) Let (X,g) be a g-metric space and $\left(u_{jk}\right)$ be a double sequence in X. Suppose that $u,v \in X$ are the g-limits of $\left(u_{jk}\right)$. There are $n_1,n_2 \in \mathbb{N}$ such that

$$g\left(u, u_{j_1k_1}, \cdots, u_{j_tk_t}\right) < \frac{\varepsilon}{t+1} \quad \text{for all} \quad j_1,\cdots,j_t \geqslant n_1 \quad \text{and} \quad k_1,\cdots,k_t \geqslant n_1,$$
$$g\left(v, u_{j_1k_1}, \cdots, u_{j_tk_t}\right) < \frac{\varepsilon}{t+1} \quad \text{for all} \quad j_1,\cdots,j_t \geqslant n_2 \quad \text{and} \quad k_1,\cdots,k_t \geqslant n_2.$$

Set $N = \max\{n_1,n_2\}$. If $n_0 \geqslant N$, then we get

$$\begin{aligned} g(u,v,\cdots,v) &\leqslant g\left(u, u_{n_0n_0}, \cdots, u_{n_0n_0}\right) + g\left(u_{n_0n_0}, v, \cdots, v\right) \\ &\leqslant g\left(u, u_{n_0n_0}, \cdots, u_{n_0n_0}\right) + tg\left(v, u_{n_0n_0}, \cdots, u_{n_0n_0}\right) \\ &< \frac{\varepsilon}{t+1} + t\frac{\varepsilon}{t+1} = \varepsilon \end{aligned}$$

Since ε is arbitrary $g(u,v,\cdots,v) = 0$. Therefore, $u = v$.

(ii) Let (X,g) be a g-metric space and $\left(u_{jk}\right)$ is g-convergent to u. There exists $n_0 \in \mathbb{N}$ such that

$$g\left(u, u_{j_1k_1}, \cdots, u_{j_tk_t}\right) < \frac{\varepsilon}{t+1} \quad \text{for all} \quad j_1,\cdots,j_t \geqslant n_0 \quad \text{and} \quad k_1,\cdots,k_t \geqslant n_0.$$

Then we have

$$\begin{aligned} g\left(u_{j_0k_0}, u_{j_1k_1}, \cdots, u_{j_tk_t}\right) &\leqslant \sum_{i=0}^{t} g\left(u_{j_ik_i}, u, \cdots, u\right) \\ &< \sum_{i=0}^{t} \frac{\varepsilon}{t+1} = \varepsilon. \end{aligned}$$

Thus, $\left(u_{jk}\right)$ is a g-Cauchy sequence in X. □

DEFINITION 3. Let $t \in \mathbb{N}$, $K \subseteq \mathbb{N}^t \times \mathbb{N}^t$ set of positive integers and

$$K(n,m) = \{((j_1,\cdots,j_t),(k_1,\cdots,k_t)) \in K : j_1,\cdots,j_t \leqslant n, k_1,\cdots,k_t \leqslant m\},$$

then

$$\delta_t(K) = \lim_{n,m} \frac{t!}{(nm)^t} |K(n,m)|$$

is called t-dimensional natural density of the set K.

DEFINITION 4. Let (u_{nm}) be a double sequence in a g-metric space (X,g).
(i) (u_{nm}) is g-statistically convergent to u, if for all $\varepsilon > 0$,

$$\lim_{n,m} \frac{t!}{(nm)^t} \left| \left\{ j_1, \cdots, j_t \leqslant n, k_1, \cdots, k_t \leqslant m : g\left(u, u_{j_1k_1}, \cdots, u_{j_tk_t}\right) \geqslant \varepsilon \right\} \right| = 0$$

and is denoted by $gS_2 - \lim_{n,m} u_{nm} = u$ or $u_{nm} \xrightarrow{gS_2} u$. We indicate the set of all g-statistically convergent double sequences by gS_2.

(ii) (u_{nm}) is a statistically g-Cauchy sequence if for all $\varepsilon > 0$, there exists $j_0, k_0 \in \mathbb{N}$ such that

$$\lim_{n,m} \frac{t!}{(nm)^t} \left| \left\{ j_1, \cdots, j_t \leqslant n, k_1, \cdots, k_t \leqslant m : g\left(u_{j_0k_0}, u_{j_1k_1}, \cdots, u_{j_tk_t}\right) \geqslant \varepsilon \right\} \right| = 0.$$

THEOREM 2. *Every g-convergent double sequence in g-metric spaces is g-statistically convergent.*

Proof. Let (u_{nm}) be a double sequence converging to u in g-metric space (X,g). For $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $j_1, \cdots, j_t \geqslant n_0$ and $k_1, \cdots, k_t \geqslant n_0$,

$$g\left(u, u_{j_1k_1}, \cdots, u_{j_tk_t}\right) < \varepsilon.$$

Set

$$\begin{aligned} K(n,m) = \{ & ((j_1, \cdots, j_t), (k_1, \cdots, k_t)) \in \mathbb{N}^t \times \mathbb{N}^t : \\ & j_1, \cdots, j_t \leqslant n, k_1, \cdots, k_t \leqslant m, g\left(x, u_{j_1k_1}, \cdots, u_{j_tk_t}\right) < \varepsilon \}, \end{aligned}$$

Then

$$|K(n,m)| \geqslant \binom{nm - n_0^2}{t}$$

and

$$\lim_{n,m} \frac{t! \, |K(n,m)|}{(nm)^t} \geqslant \lim_{n,m} \frac{t!}{(nm)^t} \binom{nm - n_0^2}{t} = 1$$

so $gS_2 - \lim_{n,m} u_{nm} = u$. $\square$

The converse of Theorem 2 does not generally hold, i.e., if (u_{nm}) is g-statistically convergent, then (u_{nm}) need not be convergent.

EXAMPLE 1. Let $X = \mathbb{R}$ and g be the metric as follows;

$$g : \mathbb{R}^3 \to \mathbb{R}^+$$
$$g(u,v,w) = \max\{|u-v|, |u-w|, |v-w|\}$$

and a double sequence (u_{nm}) in X be defined as,

$$u_{nm} = \begin{cases} nm, & \text{if } n \text{ and } m \text{ are squares,} \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that (u_{nm}) is g-statistically convergent while it is not convergent normally.

THEOREM 3. *In g-metric space g-statistical limit is unique.*

Proof. Let (u_{nm}) be a double sequence in g-metric space (X,g) such that $u_{nm} \xrightarrow{gS_2} u$ and $u_{nm} \xrightarrow{gS_2} v$.

For arbitrary $\varepsilon > 0$,

$$K(\varepsilon) = \left\{((j_1,\cdots,j_t),(k_1,\cdots,k_t)) \in \mathbb{N}^t \times \mathbb{N}^t : g\left(u,u_{j_1k_1},\cdots,u_{j_tk_t}\right) \geqslant \frac{\varepsilon}{2t}\right\},$$
$$L(\varepsilon) = \left\{((j_1,\cdots,j_t),(k_1,\cdots,k_t)) \in \mathbb{N}^t \times \mathbb{N}^t : g\left(v,u_{j_1k_1},\cdots,u_{j_tk_t}\right) \geqslant \frac{\varepsilon}{2t}\right\}.$$

Since $u_{nm} \xrightarrow{gS_2} u$ and $u_{nm} \xrightarrow{gS_2} v$, we have $\delta_t(K(\varepsilon)) = 0$ and $\delta_t(L(\varepsilon)) = 0$. Let $M(\varepsilon) = K(\varepsilon) \cup L(\varepsilon)$, then $\delta_t(M(\varepsilon)) = 0$, hence $\delta_t(M^c(\varepsilon)) = 1$.

Suppose $((j_1,\cdots,j_t),(k_1,\cdots,k_t)) \in M^c(\varepsilon)$, then we have;

$$\begin{aligned} g(u,v,\cdots,v) &\leqslant g\left(u,u_{j_1k_1},\cdots,u_{j_1k_1}\right) + g\left(u_{j_1k_1},v,\cdots,v\right) \\ &\leqslant g\left(u,u_{j_1k_1},\cdots,u_{j_1k_1}\right) + tg\left(v,u_{j_1k_1},\cdots,u_{j_1k_1}\right) \\ &\leqslant g\left(u,u_{j_1k_1},\cdots u_{j_tk_t}\right) + tg\left(v,u_{j_1k_1},\cdots u_{j_tk_t}\right) \\ &\leqslant t\left[g\left(u,u_{j_1k_1},\cdots u_{j_tk_t}\right) + g\left(v,u_{j_1k_1},\cdots u_{j_tk_t}\right)\right] \\ &< t\left(\frac{\varepsilon}{2t} + \frac{\varepsilon}{2t}\right) = \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get $g(u,v,\cdots,v) = 0$, therefore $u = v$. □

THEOREM 4. *Every g-statistically convergent double sequence is statistically g-Cauchy.*

Proof. Let (u_{nm}) be a g-statistically convergent double sequence in g-metric space (X,g) and $\varepsilon > 0$. Then, we have

$$\lim_{n,m} \frac{t!}{(nm)^t} \left| \left\{ j_1,\cdots,j_t \leqslant n, k_1,\cdots,k_t \leqslant m : g\left(u,u_{j_1k_1},\cdots,u_{j_tk_t}\right) < \frac{\varepsilon}{t(t+1)} \right\} \right| = 1$$

By the monotonicity condition and the definition of g-metric and Parts (iv) and (vi) of Theorem 1, we get

$$\begin{aligned} g\left(u_{j_0k_0},u_{j_1k_1},\cdots u_{j_tk_t}\right) &\leqslant \sum_{i=0}^{t} g\left(u_{j_ik_i},u,\cdots,u\right) \\ &\leqslant t\sum_{i=0}^{t} g\left(u,u_{j_ik_i},\cdots,u_{j_ik_i}\right) \\ &\leqslant t\sum_{i=0}^{t} g\left(u,u_{j_1k_1},\cdots u_{j_tk_t}\right) \\ &< t(t+1)\frac{\varepsilon}{t(t+1)} = \varepsilon. \end{aligned}$$

So

$$\left\{ j_1, \cdots, j_t \leqslant n, k_1, \cdots, k_t \leqslant m : g\left(u, u_{j_1k_1}, \cdots, u_{j_tk_t}\right) < \frac{\varepsilon}{t(t+1)} \right\}$$
$$\subseteq \left\{ j_1, \cdots, j_t \leqslant n, k_1, \cdots, k_t \leqslant m : g\left(u_{j_0k_0}, u_{j_1k_1}, \cdots, u_{j_tk_t}\right) < \varepsilon \right\}.$$

Therefore

$$\lim_{n,m} \frac{t!}{(nm)^t} \left| \left\{ j_1, \cdots, j_t \leqslant n, k_1, \cdots, k_t \leqslant m : g\left(u_{j_0k_0}, u_{j_1k_1}, \cdots, u_{j_tk_t}\right) < \varepsilon \right\} \right| = 1.$$

Thus (u_{nm}) is a statistically g-Cauchy sequence in X. □

Statistical convergence is closely related to strong Cesàro summability.

DEFINITION 5. Let (X, g) be a g-metric space and (u_{nm}) be a double sequence in X. The sequence (u_{nm}) is said to be strongly $g[C,1,1]$-statistically convergent to u if

$$\lim_{n,m} \frac{t!}{(nm)^t} \sum_{j_1,\cdots,j_t=1}^{n} \sum_{k_1,\cdots,k_t=1}^{m} g\left(u, u_{j_1k_1}, \cdots, u_{j_tk_t}\right) = 0.$$

This is denoted by $g[C,1,1] - \lim u_{nm} = u$ or $u_{nm} \xrightarrow{g[C,1,1]} u$. The set of all strongly $g[C,1,1]$-statistically convergent double sequences is denoted by $g[C,1,1]$.

THEOREM 5. *Let (X, g) be a g-metric space and (u_{nm}) be a double sequence in this space. Then,*

(i) $u_{nm} \xrightarrow{g[C,1,1]} u$ implies $u_{nm} \xrightarrow{gS_2} u$.

(ii) If g is a bounded function, $u_{nm} \xrightarrow{gS_2} u$ implies $u_{nm} \xrightarrow{g[C,1,1]} u$.

Proof. (i) Suppose that $u_{nm} \xrightarrow{g[C,1,1]} u$ and $\varepsilon > 0$ be given. Then,

$$\begin{aligned}
&\frac{t!}{(nm)^t} \sum_{j_1,\cdots,j_t=1}^{n} \sum_{k_1,\cdots,k_t=1}^{m} g\left(u, u_{j_1k_1}, \cdots, u_{j_tk_t}\right) \\
&\geqslant \frac{t!}{(nm)^t} \sum_{\substack{j_1,\cdots,j_t=1 \\ g\left(u,u_{j_1k_1},\cdots,u_{j_tk_t}\right)\geqslant\varepsilon}}^{n} \sum_{\substack{k_1,\cdots,k_t=1 \\ g\left(u,u_{j_1k_1},\cdots,u_{j_tk_t}\right)\geqslant\varepsilon}}^{m} g\left(u, u_{j_1k_1}, \cdots, u_{j_tk_t}\right) \\
&\geqslant \varepsilon \frac{t!}{(nm)^t} \left| \left\{ j_1, \cdots, j_t \leqslant n, k_1, \cdots, k_t \leqslant m : g\left(u_{j_0k_0}, u_{j_1k_1}, \cdots, u_{j_tk_t}\right) \geqslant \varepsilon \right\} \right|.
\end{aligned}$$

Then we obtain $u_{nm} \xrightarrow{gS_2} u$.

(ii) Let $\varepsilon > 0$, g be a bounded function and $u_{nm} \xrightarrow{gS_2} u$. From the boundedness of g, there exists a positive number N such that

$$g\left(u, u_{n_1m_1}, \cdots, u_{n_tm_t}\right) < N$$

for all $n_1,\cdots,n_t$ and $m_1,\cdots,m_t$. Then

$$\begin{aligned}
&\frac{t!}{(nm)^t}\sum_{j_1,\cdots,j_t=1}^{n}\sum_{k_1,\cdots,k_t=1}^{m} g\left(u,u_{j_1k_1},\cdots,u_{j_tk_t}\right)\\
&=\frac{t!}{(nm)^t}\sum_{\substack{j_1,\cdots,j_t=1\\ g\left(u,u_{j_1k_1},\cdots,u_{j_tk_t}\right)\geqslant\varepsilon}}^{n}\ \sum_{\substack{k_1,\cdots,k_t=1\\ g\left(u,u_{j_1k_1},\cdots,u_{j_tk_t}\right)\geqslant\varepsilon}}^{m} g\left(u,u_{j_1k_1},\cdots,u_{j_tk_t}\right)\\
&\quad+\frac{t!}{(nm)^t}\sum_{\substack{j_1,\cdots,j_t=1\\ g\left(u,u_{j_1k_1},\cdots,u_{j_tk_t}\right)<\varepsilon}}^{n}\ \sum_{\substack{k_1,\cdots,k_t=1\\ g\left(u,u_{j_1k_1},\cdots,u_{j_tk_t}\right)<\varepsilon}}^{m} g\left(u,u_{j_1k_1},\cdots,u_{j_tk_t}\right)\\
&=M\frac{t!}{(nm)^t}\left|\left\{j_1,\cdots,j_t\leqslant n,k_1,\cdots,k_t\leqslant m: g\left(u_{j_0k_0},u_{j_1k_1},\cdots,u_{j_tk_t}\right)\geqslant\varepsilon\right\}\right|+\varepsilon.
\end{aligned}$$

We obtain that $u_{nm}\xrightarrow{g[C,1,1]}u$. □

DEFINITION 6. Let (X,g) be a g-metric space and (u_{nm}) be a double sequence in X. The sequence (u_{nm}) is said to be g-lacunary statistically convergent to u, if for all $\varepsilon>0$,

$$\lim_{r,s\to\infty}\frac{t!}{(h_{rs})^t}\left|\left\{(j_w,k_w)\in I_{rs},\ 1\leqslant w\leqslant t: g\left(u,u_{j_1k_1},\cdots,u_{j_wk_w}\right)\geqslant\varepsilon\right\}\right|=0,$$

and is indicated by $gS_{\theta_2}-\lim_{n,m}u_{nm}=u$ or $u_{nm}\xrightarrow{gS_{\theta_2}}u$. We indicate the set of all g-lacunary statistically convergent double sequences by gS_{θ_2}.

DEFINITION 7. The sequence (u_{nm}) is called to be g-strongly lacunary convergent to u provided that

$$\lim_{r,s\to\infty}\frac{t!}{(h_{rs})^t}\sum_{(j_w,k_w)\in I_{rs},\ 1\leqslant w\leqslant t} g\left(u,u_{j_1k_1},\cdots,u_{j_wk_w}\right)=0$$

where $1\leqslant w\leqslant t$, and is indicated by $gN_{\theta_2}-\lim_{n,m}u_{nm}=u$ or $u_{nm}\xrightarrow{gN_{\theta_2}}u$. We indicate the set of all g-strongly lacunary convergent double sequences by gN_{θ_2}.

THEOREM 6. *Let (X,g) be a g-metric space and (u_{nm}) be a double sequence in X. Then, the following statements hold:*

(i) $u_{nm}\xrightarrow{gN_{\theta_2}}u$ implies $u_{nm}\xrightarrow{gS_{\theta_2}}u$.

(ii) If g is a bounded function, $u_{nm}\xrightarrow{gS_{\theta_2}}u$ implies $u_{nm}\xrightarrow{gN_{\theta_2}}u$.

Now, we give relations between g-statistical convergence and g-lacunary statistical convergence of double sequences in g metric space.

THEOREM 7. *For a double lacunary sequence* $\theta_2 = \theta_{r,s}$, $gS_2 - \lim_{n,m} u_{nm} = u$ *implies* $gS_{\theta_2} - \lim_{n,m} u_{nm} = u$ *iff* $\liminf_r q_r > 1$ *and* $\liminf_s q_s > 1$.

Proof. Assume first that $\liminf_r q_r > 1$ and $\liminf_s q_s > 1$. Then, there exists $\alpha > 0$ such that $q_r > 1+\alpha$ and $q_s > 1+\alpha$ for sufficiently large r, s which means that

$$\frac{h_r}{k_r} \geqslant \frac{\alpha}{1+\alpha} \quad \text{and} \quad \frac{h_s}{k_s} \geqslant \frac{\alpha}{1+\alpha}.$$

If $gS_2 - \lim_{n,m} u_{nm} = u$, then for all $\varepsilon > 0$ and for sufficiently large r, s we get

$$\begin{aligned}
&\frac{t!}{(k_{rs})^t} \left|\left\{ j_1,\cdots,j_t \leqslant k_r, k_1,\cdots,k_t \leqslant l_s : g\left(u,u_{j_1k_1},\cdots,u_{j_tk_t}\right) \geqslant \varepsilon \right\}\right| \\
&\geqslant \frac{t!}{(k_{rs})^t} \left|\left\{ (j_w,k_w) \in I_{rs},\, 1 \leqslant w \leqslant t : g\left(u,u_{j_1k_1},\cdots,u_{j_wk_w}\right) \geqslant \varepsilon \right\}\right| \\
&\geqslant \left(\frac{h_{rs}}{k_{rs}}\right)^t \frac{t!}{(h_{rs})^t} \left|\left\{ (j_w,k_w) \in I_{rs},\, 1 \leqslant w \leqslant t : g\left(u,u_{j_1k_1},\cdots,u_{j_wk_w}\right) \geqslant \varepsilon \right\}\right| \\
&\geqslant \left(\frac{\alpha}{1+\alpha}\right)^t \frac{t!}{(h_{rs})^t} \left|\left\{ (j_w,k_w) \in I_{rs},\, 1 \leqslant w \leqslant t : g\left(u,u_{j_1k_1},\cdots,u_{j_wk_w}\right) \geqslant \varepsilon \right\}\right|.
\end{aligned}$$

As a result $gS_{\theta_2} - \lim_{n,m} u_{nm} = u$.

Suppose $\liminf_r q_r = 1$ and $\liminf_s q_s = 1$, and suppose without loss of generality that $\liminf_r q_r = 1$; means there is an ordinary subsequence $\left\{k_{\sigma_j}\right\}$ of the lacunary sequence θ_r such that $\frac{k_{\sigma_j}}{k_{\sigma_{j-1}}} < 1 + \frac{1}{j}$ and $\frac{k_{\sigma_{j}-1}}{k_{\sigma_{j-1}}} > j$ where $\sigma_j \geqslant \sigma_{j-1} + 2$. Let us establish u as follows:

$$u_{nm} := \begin{cases} 1, \text{ if } n \in I_{\sigma_j} \text{ and } m \in \mathbb{N} \\ 0, \text{ if not.} \end{cases}$$

So by ([8], p. 510), the rows are not in gN_{θ_r} however each row is such that u is in $g[C,1]$; therefore u is in gS_{θ_r} by part (C) of Theorem 6. Also each row is in gS such that $gS \nsubseteq gS_{\theta_r}$. Since the double lacunary sequences $\theta_{r,s}$, $gS_2 \nsubseteq gS_{\theta_{r,s}}$. □

THEOREM 8. *For a double lacunary sequence* $\theta_2 = \theta_{r,s}$, $gS_{\theta_2} - \lim_{n,m} u_{nm} = u$ *implies* $gS_2 - \lim_{n,m} u_{nm} = u$ *iff* $\limsup_r q_r < \infty$, $\limsup_s q_s < \infty$.

Proof. Assume $\limsup_r q_r < \infty$, $\limsup_s q_s < \infty$. Then there exists $Q > 0$ such that $q_r < Q$ and $q_s < Q$ for all $r,s \geqslant 1$. Assume that $gS_{\theta_2} - \lim_{n,m} u_{nm} = u$ and

$$N_{r,s} := \left|\left\{ (j_w,k_w) \in I_{rs},\, 1 \leqslant w \leqslant t : g\left(u,u_{j_1k_1},\cdots,u_{j_wk_w}\right) \geqslant \varepsilon \right\}\right|.$$

By the definition of $gS_{\theta_2} - \lim_{n,m} u_{nm} = u$ given $\varepsilon > 0$ there exists $r_0 \in \mathbb{N}$ such that $\frac{N_{r,s}}{h_{rs}} < \varepsilon$ for all $r,s \geqslant r_0$. Take

$$U = \max\left\{ N_{r,s} : 1 \leqslant r \leqslant r_0,\, 1 \leqslant s \leqslant r_0 \right\}.$$

Assume n,m be such that $k_{r-1} < m \leqslant k_r$ and $l_{s-1} < n \leqslant l_s$. As a result, we obtain the following:

$$\begin{aligned}
&\frac{t!}{(nm)^t}\left|\left\{j_1,\cdots,j_t \leqslant n,\, k_1,\cdots,k_t \leqslant m : g\left(u,u_{j_1k_1},\cdots,u_{j_wk_w}\right) \geqslant \varepsilon\right\}\right| \\
&\leqslant \frac{t!}{(k_{r-1}l_{s-1})^t}\left|\left\{j_1,\cdots,j_t \leqslant k_r,\, k_1,\cdots,k_t \leqslant l_s : g\left(u,u_{j_1k_1},\cdots,u_{j_wk_w}\right) \geqslant \varepsilon\right\}\right| \\
&= \frac{t!}{(k_{r-1}l_{s-1})^t}\left\{\sum_{j_1,\cdots,j_w=1}^{r}\sum_{k_1,\cdots,k_w=1}^{s} N_{j_1,\cdots,j_w,k_1,\cdots,k_w}\right\} \\
&\leqslant \frac{Ur_0^2t!}{(k_{r-1}l_{s-1})^t} + \frac{t!}{(k_{r-1}l_{s-1})^t}\left\{\sum_{j_1,\cdots,j_w=1}^{r}\sum_{k_1,\cdots,k_w=1}^{s} N_{j_1,\cdots,j_w,k_1,\cdots,k_w}\right\} \\
&\leqslant \frac{Ur_0^2t!}{(k_{r-1}l_{s-1})^t} + \frac{t!}{(k_{r-1}l_{s-1})^t}\left\{\sum_{j_1,\cdots,j_w=r_0+1}^{r}\sum_{k_1,\cdots,k_w=r_0+1}^{s} \frac{N_{j_1,\cdots,j_w,k_1,\cdots,k_w}h_{j_1,\cdots,j_w,k_1,\cdots,k_w}}{h_{j_1,\cdots,j_w,k_1,\cdots,k_w}}\right\} \\
&\leqslant \frac{Ur_0^2t!}{(k_{r-1}l_{s-1})^t} + \frac{t!}{(k_{r-1}l_{s-1})^t}\left(\sup_{j_1,\cdots,j_w,k_1,\cdots,k_w \geqslant r_0,r_0} \frac{N_{j_1,\cdots,j_w,k_1,\cdots,k_w}}{h_{j_1,\cdots,j_w,k_1,\cdots,k_w}}\right) \\
&\quad \times \left\{\sum_{j_1,\cdots,j_w=r_0+1}^{r}\sum_{k_1,\cdots,k_w=r_0+1}^{s} h_{j_1,\cdots,j_w,k_1,\cdots,k_w}\right\} \\
&\leqslant \frac{Ur_0^2t!}{(k_{r-1}l_{s-1})^t} + \varepsilon\left\{\sum_{j_1,\cdots,j_w=r_0+1}^{r}\sum_{k_1,\cdots,k_w=r_0+1}^{s} h_{j_1,\cdots,j_w,k_1,\cdots,k_w}\right\} \\
&\leqslant \frac{Ur_0^2t!}{(k_{r-1}l_{s-1})^t} + \varepsilon H^2.
\end{aligned}$$

The converse of this theorem follows similarly to that of Theorem 6. This completes the proof of this theorem. □

Theorem 7 and Theorem 8 imply the following:

THEOREM 9. *Let $\theta_{r,s}$ be a lacunary double sequence. If*

$$1 < \liminf_r q_r \leqslant \limsup_r q_r < \infty \text{ and } 1 < \liminf_s q_s \leqslant \limsup_s q_s < \infty,$$

then $gS_2 = gS_{\theta_2}$.

THEOREM 10. *If $u = (u_{nm}) \in gS_2 \cap gS_{\theta_2}$, then*

$$gS_{\theta_2} - \lim_{n,m\to\infty} u_{nm} = gS_2 - \lim_{n,m} u_{nm}.$$

Proof. Assume $gS_2 - \lim_{n,m\to\infty} u_{nm} = u_0$ and $gS_{\theta_2} - \lim_{n,m\to\infty} u_{nm} = u_1$ such that $u_0 \neq u_1$. Let

$$\lim_{n,m\to\infty} \frac{t!}{(nm)^t}\left|\left\{j_1,\cdots,j_t \leqslant n, k_1,\cdots,k_t \leqslant m : g\left(u_1,u_{j_1k_1},\cdots,u_{j_tk_t}\right) \geqslant \varepsilon\right\}\right| = 1$$

for $\varepsilon < \frac{1}{2}|u_0 - u_1|$. Let us now consider the $k_p l_v$-th term of the following expression:

$$\begin{aligned}
&\frac{t!}{(nm)^t}\left|\left\{j_1,\cdots,j_t \leqslant n, k_1,\cdots,k_t \leqslant m : g\left(u_1,u_{j_1k_1},\cdots,u_{j_tk_t}\right) \geqslant \varepsilon\right\}\right| \\
&= \frac{t!}{(k_pl_v)^t}\left|\left\{(j_t,k_t) \in \bigcup_{r,s=1,1}^{p,v} I_{r,s} : g\left(u_1,u_{j_1k_1},\cdots,u_{j_tk_t}\right) \geqslant \varepsilon\right\}\right| \\
&\leqslant \frac{1}{\sum_{r,s}^{p,v} h_{r,s}} \sum_{r,s=1,1}^{p,v} h_{r,s}t_{r,s} \\
&= \frac{t!}{(k_pl_v)^t} \sum_{r,s=1,1}^{p,v} \left|\left\{((j_t,k_t) \in I_{r,s} : g\left(u_1,u_{j_1k_1},\cdots,u_{j_tk_t}\right) \geqslant \varepsilon\right\}\right|
\end{aligned}$$

where

$$t_{r,s} = \frac{1}{h_{rs}}\left|\left\{(j_t,k_t) \in I_{r,s} : g\left(u_1,u_{j_1k_1},\cdots,u_{j_tk_t}\right) \geqslant \varepsilon\right\}\right|$$

is a Pringsheim null sequence, since $gS_{\theta_2} - \lim_{n,m\to\infty} u_{nm} = u_1$. Since $\theta_{r,s}$ is a double lacunary sequence, the last equation satisfies all conditions for a four-dimensional matrix transformation to map Pringsheim null sequence into Pringsheim null sequence ([11]), and therefore it also tends to zero in the Pringsheim sense. In addition, it is also a double sequence of

$$\frac{t!}{(nm)^t}\left|\left\{j_1,\cdots,j_t \leqslant n, k_1,\cdots,k_t \leqslant m : g\left(u_1,u_{j_1k_1},\cdots,u_{j_tk_t}\right) \geqslant \varepsilon\right\}\right|$$

which does not tends to 1 in the Pringsheim sense. This contradiction implies that $u_0 = u_1$. $\square$

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