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SHARP COEFFICIENT BOUNDS OF ANALYTIC FUNCTIONS SUBORDINATE TO SHELL–LIKE CURVES CONNECTED WITH k–FIBONACCI NUMBERS

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Abstract. Let us consider the function

$$\tilde{p}_k(z) = \frac{1+\tau_k^2 z^2}{1-k\tau_k z-\tau_k^2 z^2} := 1+\sum_{n=1}^{\infty} \tilde{p}_{k,n} z^n.$$

The coefficients of the function $\tilde{p}_k(z)$ are connected with k-Fibonacci numbers:

$$\tilde{p}_{k,n} = \left(F_{k,n-1}+F_{k,n+1}\right)\tau_k^n \qquad (n=1,2,\ldots).$$

If the function p of the form $p(z)=1+p_1 z+p_2 z^2+\cdots$ satisfies

$$p(z) \prec \tilde{p}_k(z),$$

then we have

$$\begin{aligned} |p_1| &\leqslant k|\tau_k| = \left|\tilde{p}_{k,1}\right|, \\ |p_2| &\leqslant \left(k^2+2\right)\tau_k^2 = \left|\tilde{p}_{k,2}\right|, \\ |p_3| &\leqslant \left(k^3+3k\right)|\tau_k|^3 = \left|\tilde{p}_{k,3}\right|. \end{aligned}$$

In this paper, we prove that

$$|p_n| \leqslant \left(F_{k,n-1}+F_{k,n+1}\right)|\tau_k|^n = \left|\tilde{p}_{k,n}\right| \qquad (n=1,2,\ldots).$$

1. Introduction

Let $\mathbb{R}=(-\infty,\infty)$ be the set of real numbers, $\mathbb{C}$ be the set of complex numbers and

$$\mathbb{N} := \{1,2,3,\ldots\} = \mathbb{N}_0 \setminus \{0\}$$

be the set of positive integers.

Assume that $\mathscr{H}$ is the class of analytic functions in the open unit disc

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

For two functions $f,g \in \mathscr{H}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

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if there exists a Schwarz function

$$\omega \in \Omega := \{\omega \in \mathscr{H} : \omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \ (z \in \mathbb{U})\},$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \mathbb{U}).$$

Indeed, it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Rightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

DEFINITION 1. [6] For any positive real number k, the k-Fibonacci sequence $\{F_{k,n}\}_{n\in\mathbb{N}_0}$ is defined recurrently by

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1} \qquad (n \in \mathbb{N})$$

with initial conditions

$$F_{k,0} = 0, \qquad F_{k,1} = 1.$$

Furthermore n^{th} k-Fibonacci number is given by

$$F_{k,n} = \frac{(k-\tau_k)^n - \tau_k^n}{\sqrt{k^2+4}}, \tag{1}$$

where

$$\tau_k = \frac{k - \sqrt{k^2+4}}{2}.$$

For $k = 1$, we obtain the classic Fibonacci sequence $\{F_n\}_{n\in\mathbb{N}_0}$:

$$F_0 = 0, \qquad F_1 = 1, \qquad \text{and} \quad F_{n+1} = F_n + F_{n-1} \quad (n \in \mathbb{N}).$$

THEOREM 1. [9] *Let $\{F_{k,n}\}_{n\in\mathbb{N}_0}$ be the sequence of k-Fibonacci numbers defined in Definition 1. If*

$$\tilde{p}_k(z) = \frac{1+\tau_k^2 z^2}{1 - k\tau_k z - \tau_k^2 z^2} := 1 + \sum_{n=1}^{\infty} \tilde{p}_{k,n} z^n, \tag{2}$$

then we have

$$\tilde{p}_{k,1} = k\tau_k, \quad \tilde{p}_{k,2} = \left(k^2+2\right)\tau_k^2, \quad \tilde{p}_{k,n} = \left(F_{k,n-1} + F_{k,n+1}\right)\tau_k^n \qquad (n \in \mathbb{N}). \tag{3}$$

The function (2) was applied for defining several classes of analytic functions, for more details we refer to [1, 3, 4, 5, 7, 8, 10, 11, 12, 13, 14].

LEMMA 1. [14] *If* $p(z)=1+p_1z+p_2z^2+\cdots$ $(z\in\mathbb{U})$ *and*

$$p(z)\prec\tilde{p}_k(z)=\frac{1+\tau_k^2z^2}{1-k\tau_kz-\tau_k^2z^2},\qquad \tau_k=\frac{k-\sqrt{k^2+4}}{2},$$

then we have

$$|p_1|\leqslant k|\tau_k|\qquad \text{and}\qquad |p_2|\leqslant\left(k^2+2\right)\tau_k^2.$$

The above estimates are sharp.

LEMMA 2. [8] *If* $p(z)=1+p_1z+p_2z^2+\cdots$ $(z\in\mathbb{U})$ *and*

$$p(z)\prec\tilde{p}_k(z)=\frac{1+\tau_k^2z^2}{1-k\tau_kz-\tau_k^2z^2},\qquad \tau_k=\frac{k-\sqrt{k^2+4}}{2},$$

then we have

$$|p_3|\leqslant\left(k^3+3k\right)|\tau_k|^3.$$

The result is sharp.

Güney et al. [8] also stated the following conjecture:

CONJECTURE. If $p(z)=1+p_1z+p_2z^2+\cdots$ $(z\in\mathbb{U})$ and

$$p(z)\prec\tilde{p}_k(z)=\frac{1+\tau_k^2z^2}{1-k\tau_kz-\tau_k^2z^2},\qquad \tau_k=\frac{k-\sqrt{k^2+4}}{2},$$

then we have

$$|p_n|\leqslant\left(F_{k,n-1}+F_{k,n+1}\right)|\tau_k|^n\qquad(n\in\mathbb{N}).$$

The main purpose of this paper is to prove that this conjecture is true.

2. Main results

THEOREM 2. *If* $p(z)=1+p_1z+p_2z^2+\cdots$ *and*

$$p(z)\prec\tilde{p}_k(z)=\frac{1+\tau_k^2z^2}{1-k\tau_kz-\tau_k^2z^2}=1+\sum_{n=1}^{\infty}(F_{k,n-1}+F_{k,n+1})\tau_k^nz^n,\tag{4}$$

then we have

$$|p_n|\leqslant\left(F_{k,n-1}+F_{k,n+1}\right)|\tau_k|^n\qquad(n\in\mathbb{N}).\tag{5}$$

The result is sharp.

Proof. If p satisfies (4), then by the principle of subordination, there exists a Schwarz function $\omega\in\Omega$ such that

$$p(z)=\frac{1+a\omega^2(z)}{1-b\omega(z)-a\omega^2(z)}\qquad(z\in\mathbb{U}),\tag{6}$$

where $a=\tau_k^2$, $b=k\tau_k$. This gives

$$p(z)-1=a\omega^2(z)(1+p(z))+b\omega(z)p(z).$$

Therefore, we can write

$$\sum_{m=1}^{n} p_m z^m+\sum_{m=n+1}^{\infty} c_m z^m=a\omega^2(z)\left(1+\sum_{m=0}^{n-2} p_m z^m\right)+b\omega(z)\left(\sum_{m=0}^{n-1} p_m z^m\right) \quad (p_0=1). \tag{7}$$

for some c_m, $m=n+1,n+2,\ldots$. Then, we have

$$\sum_{m=1}^{n} p_m z^m+\sum_{m=n+1}^{\infty} c_m z^m=\sum_{m=0}^{n-1}\left(a\omega^2(z)q_{m-1}z^{m-1}+b\omega(z)p_m z^m\right), \tag{8}$$

where $q_{-1}=0$, $q_0=2$ and $q_m=p_m$ for $m\in\mathbb{N}$. The left hand side of (8) is a function F, analytic in $\mathbb{U}$. From (8), we have

$$\begin{aligned}
|F(z)|^2&=\left|\sum_{m=0}^{n-1}\left(a\omega^2(z)q_{m-1}z^{m-1}+b\omega(z)p_m z^m\right)\right|^2\\
&\leqslant\left|\sum_{m=0}^{n-1}\left(a\omega(z)q_{m-1}z^{m-1}+bp_m z^m\right)\right|^2\\
&\leqslant\left|\sum_{m=0}^{n-1}a\omega(z)q_{m-1}z^{m-1}\right|^2+\left|\sum_{m=0}^{n-1}bp_m z^m\right|^2+2\left|\sum_{m=0}^{n-1}a\omega(z)q_{m-1}z^{m-1}\sum_{m=0}^{n-1}bp_m z^m\right|\\
&\leqslant\left|\sum_{m=0}^{n-1}aq_{m-1}z^{m-1}\right|^2+\left|\sum_{m=0}^{n-1}bp_m z^m\right|^2+2\left|\sum_{m=0}^{n-1}aq_{m-1}z^{m-1}\right|\left|\sum_{m=0}^{n-1}bp_m z^m\right|.
\end{aligned} \tag{9}$$

From Hölder's inequality, we have

$$\begin{aligned}
&\int_0^{2\pi}\left|\sum_{m=0}^{n-1}aq_{m-1}z^{m-1}\right|\left|\sum_{m=0}^{n-1}bp_m z^m\right|\mathrm{d}\phi\\
&\leqslant\sqrt{\int_0^{2\pi}\left|\sum_{m=0}^{n-1}aq_{m-1}z^{m-1}\right|^2\mathrm{d}\phi\ \int_0^{2\pi}\left|\sum_{m=0}^{n-1}bp_m z^m\right|^2\mathrm{d}\phi}\,,
\end{aligned}$$

where $z=re^{i\phi}$. Therefore, integrating the both sides of equality (9) around $z=re^{i\phi}$ and taking limit $r\to 1^-$ we obtain

$$\begin{aligned}
\sum_{m=1}^{n}|p_m|^2+\sum_{m=n+1}^{\infty}|c_m|^2&\leqslant\sum_{m=0}^{n-1}|aq_{m-1}|^2+\sum_{m=0}^{n-1}|bp_m|^2+2\sqrt{\sum_{m=0}^{n-1}|aq_{m-1}|^2\sum_{m=0}^{n-1}|bp_m|^2}\\
&=\left(\sqrt{\sum_{m=0}^{n-1}|aq_{m-1}|^2}+\sqrt{\sum_{m=0}^{n-1}|bp_m|^2}\right)^2\\
&\leqslant\sum_{m=0}^{n-1}\left(|aq_{m-1}|+|bp_m|\right)^2,
\end{aligned}$$

applying in the last step Minkowski's inequality

$$\sqrt{\left|\sum_{k=1}^{n} x_k^2\right|}+\sqrt{\left|\sum_{k=1}^{n} y_k^2\right|} \leqslant \sqrt{\sum_{k=1}^{n}\left(|x_k|+|y_k|\right)^2}.$$

Hence we find

$$|p_n|^2+\sum_{m=1}^{n-1}|p_m|^2 \leqslant \sum_{m=0}^{n-1}\left\{|a|^2|q_{m-1}|^2+|b|^2|p_m|^2+2|abq_{m-1}p_m|\right\}, \tag{10}$$

where $q_{-1}=0$, $q_0=2$ and $q_m=p_m$ for $m\in\mathbb{N}$, so (10) gives

$$|p_1|^2 \leqslant b^2, \tag{11}$$

$$|p_2|^2 \leqslant \left(b^2-1\right)|p_1|^2+4|ab||p_1|+4a^2+b^2, \tag{12}$$

and

$$\begin{aligned}|p_n|^2 \leqslant\ & \left(b^2-1\right)|p_1|^2+4|ab||p_1|+4a^2+b^2\\ &+\sum_{m=2}^{n-1}\left\{a^2|p_{m-1}|^2+\left(b^2-1\right)|p_m|^2+2|ab||p_{m-1}p_m|\right\}, \quad (n\geqslant 3).\end{aligned} \tag{13}$$

We note that the inequality (11) implies

$$|p_1| \leqslant k|\tau_k| = \left|\tilde{p}_{k,1}\right| = (F_{k,0}+F_{k,2})|\tau_k|. \tag{14}$$

Now consider the inequality (12). Because $a=\tau_k^2$ and $b=k\tau_k$, after a standard calculations, we obtain that $\left(b^2-1\right)|p_1|^2+4|ab||p_1|+4a^2+b^2$ increases with respect to $|p_1|$, where by (14) $|p_1|\in[0,k|\tau_k|]$. Since then

$$\begin{aligned}\left(b^2-1\right)|p_1|^2+4|ab||p_1|+4a^2+b^2 &\leqslant \left[\left(b^2-1\right)|p_1|^2+4|ab||p_1|+4a^2+b^2\right]_{|p_1|=k|\tau_k|}\\ &=\tau_k^4(2+k^2)^2=(F_{k,1}+F_{k,3})^2\tau_k^4,\end{aligned} \tag{15}$$

and (12) gives

$$|p_2| \leqslant (k^2+2)\tau_k^2 = \left|\tilde{p}_{k,2}\right| = (F_{k,1}+F_{k,3})\tau_k^2. \tag{16}$$

Now, we will prove (5) by applying mathematical induction. In (14) and (16) it was proved that (5) holds for the case $n=1$ and $n=2$. Suppose that (5) holds for $3,4,\ldots,n-1$. Then from (13), we have

$$\begin{aligned}|p_n|^2 \leqslant\ & \left(b^2-1\right)|p_1|^2+4|ab||p_1|+4a^2+b^2\\ &+\sum_{m=2}^{n-1}a^2(F_{k,m-2}+F_{k,m})^2|\tau_k|^{2(m-1)}\\ &+\sum_{m=2}^{n-1}(b^2-1)(F_{k,m-1}+F_{k,m+1})^2|\tau_k|^{2m}\\ &+\sum_{m=2}^{n-1}2|ab|(F_{k,m-2}+F_{k,m})(F_{k,m-1}+F_{k,m+1})|\tau_k|^{2m-1}.\end{aligned}$$

Using (15) and $a = \tau_k^2$ and $b = k\tau_k$, we get

$$\begin{aligned}|p_n|^2 \leqslant &(F_{k,1}+F_{k,3})^2\tau_k^4 \\ &+\sum_{m=2}^{n-1}(F_{k,m-2}+F_{k,m})^2|\tau_k|^{2m+2} \\ &+\sum_{m=2}^{n-1}k^2(F_{k,m-1}+F_{k,m+1})^2|\tau_k|^{2m+2} \\ &+\sum_{m=2}^{n-1}2k(F_{k,m-2}+F_{k,m})(F_{k,m-1}+F_{k,m+1})|\tau_k|^{2m+2} \\ &-\sum_{m=2}^{n-1}(F_{k,m-1}+F_{k,m+1})^2|\tau_k|^{2m}.\end{aligned}$$

Therefore, applying the property $F_{k,s} = kF_{k,s-1} + F_{k,s-2}$ $(s \geqslant 2)$, we obtain

$$\begin{aligned}|p_n|^2 &\leqslant (F_{k,1}+F_{k,3})^2\tau_k^4 \\ &\quad+\sum_{m=2}^{n-1}\left\{(F_{k,m-2}+F_{k,m})+k(F_{k,m-1}+F_{k,m+1})\right\}^2|\tau_k|^{2m+2} \\ &\quad-\sum_{m=2}^{n-1}(F_{k,m-1}+F_{k,m+1})^2|\tau_k|^{2m} \\ &=\sum_{m=1}^{n-1}\left\{F_{k,m}+F_{k,m+2}\right\}^2|\tau_k|^{2m+2}-\sum_{m=2}^{n-1}(F_{k,m-1}+F_{k,m+1})^2|\tau_k|^{2m+2} \\ &=(F_{k,n-1}+F_{k,n+1})^2|\tau_k|^{2n}.\end{aligned}$$

This proves $|p_n| \leqslant \left(F_{k,n-1}+F_{k,n+1}\right)|\tau_k|^n$, and by induction (5) holds for all $n \in \mathbb{N}$.

The coefficients $\tilde{p}_{k,n}$ of the function $\tilde{p}_k$ satisfy

$$\left|\tilde{p}_{k,n}\right| = \left(F_{k,n-1}+F_{k,n+1}\right)|\tau_k|^n \quad (n \in \mathbb{N}),$$

by Theorem 1, so in this sense Theorem 2 is sharp. □

For $k = 1$, we obtain following consequence which proves [13, Conjecture 2.1].

COROLLARY 1. *If* $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ *and*

$$p(z) \prec \tilde{p}(z) = \frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2} = 1+\sum_{n=1}^{\infty}(F_{n-1}+F_{n+1})\tau^n z^n, \quad \tau = \frac{1-\sqrt{5}}{2},$$

then we have

$$|p_n| \leqslant (F_{n-1}+F_{n+1})|\tau|^n \qquad (n \in \mathbb{N}).$$

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