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MODIFIED JAIN–PETHE–BASKAKOV–DURRMEYER OPERATORS AND THEIR QUANTITATIVE ESTIMATES

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Abstract. In this paper, we present a modification of Jain-Pethe-Baskakov-Durrmeyer operators and estimate their moments. Then, we establish the uniform convergence of the proposed family of operators. Further, we use modulus of continuity and K-functional to establish local approximation behavior of these operators. Also, we compute the rate of convergence of the operators through Lipschitz class functions. In the last section, we present some quantitative results for difference of the proposed operators with Jain-Pethe operators and Durrmeyer type variant of Jain-Pethe-Baskakov operators.

1. Introduction

Research work in the field of mathematical analysis is significantly contributed by the approximation via sequence of linear positive operators which has not only the applications in mathematics but also in the field of engineering and physics. Weierstrass approximation theorem lead the foundation of approximation theory and strongly propagated by a classy proof of the aforesaid theorem, given by S. N. Bernstein [18]. In 1930, an integral form of Bernstein operators was introduced by Kantorovich [16] and in 1957, Baskakov [17] investigated another class of operators in the linear space $C[0,\infty)$ known as Baskakov operators. Later, in order to approximate the Lebesgue integrable functions, Durrmeyer [19] introduced another integral type generalization of Bernstein operators. Recently, Kantorovich and Durrmeyer type generalization of several operators were presented and investigated for their approximation behavior; see [13, 14, 15].

In context of the present article, Jain-Pethe operators [11] are given by

$$J_m^{(\alpha)}(f;x) = \sum_{r=0}^{\infty} s_{m,r}^{[\alpha]}(x) f\left(\frac{r}{m}\right), \quad \text{for} \quad 0 \leqslant m\alpha \leqslant 1,$$

where

$$s_{m,r}^{[\alpha]}(x) = (1+m\alpha)^{\frac{-x}{\alpha}} \left(\alpha + \frac{1}{m}\right)^{-r} \frac{x^{(r,-\alpha)}}{r!}, \quad x \in [0,\infty),$$
$$x^{(r,-\alpha)} = x(x+\alpha)\dots(x+(r-1)\alpha), \quad x^{(0,-\alpha)} = 1,$$

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and $\alpha = (\alpha_m)_{m\in\mathbb{N}}$ is such that $0 \leqslant \alpha_m \leqslant \frac{1}{m}$.

By considering a new parameter $\lambda > 0$, a modification of Jain-Pethe operators was introduced by Abel and Ivan [10] as follows:

$$L_{m,\lambda}(f;x) = \sum_{r=0}^{\infty} g_{m,r,\lambda}(x) f\left(\frac{r}{m}\right),$$

where $g_{m,r,\lambda}(.)$ is the basis function defined as

$$g_{m,r,\lambda}(x) = \left(\frac{\lambda}{1+\lambda}\right)^{m\lambda x} \frac{(m\lambda x)_r}{r!} \frac{1}{(1+\lambda)^r}$$

and $(m)_r = \prod_{i=0}^{r-1}(m+i)$, $(m)_0 = 1$ denotes the Pochhammer symbol.

LEMMA 1. [9] *Moments of Jain-Pethe operators are given as follows:*

$$\begin{aligned} L_{m,\lambda}(1;x) &= 1, \\ L_{m,\lambda}(t;x) &= x, \\ L_{m,\lambda}(t^2;x) &= x^2 + \left(\frac{1+\lambda}{\lambda}\right)\frac{x}{m}, \\ L_{m,\lambda}(t^3;x) &= x^3 + \left(\frac{2(1+\lambda)+1}{\lambda}\right)\frac{x^2}{m} + \left(\frac{(1+\lambda)^2+1}{\lambda^2}\right)\frac{x}{m^2}. \end{aligned}$$

Inspired by the above studies, here, we introduce a modification of Jain-Pethe-Baskakov-Durrmeyer operators using the basis function considered by Abel and Ivan [10] and study their approximation properties.

2. Construction of operator

For $f \in C[0,\infty)$, we define modified form of Jain-Pethe-Baskakov-Durrmeyer operators, as follows:

$$J_m^{\lambda}(f;x) = (m-1)\sum_{r=0}^{\infty} g_{m,r,\lambda}(x) \int_0^{\infty} \begin{bmatrix} m+r-1 \\ r \end{bmatrix} \frac{t^r}{(1+t)^{m+r}} f(t)\,dt, \tag{1}$$

where $g_{m,r,\lambda}(.)$ is the basis function.

Now, we recall the definition of well-known β-function given as follows:

$$\beta(m,n) = \int_0^{\infty} \frac{x^m}{(1+x)^{m+n}}dx, \quad m,n > 0. \tag{2}$$

LEMMA 2. *For $l \in \mathbb{N}$ and*

$$I_l(m,r) = (m-1)\int_0^{\infty} \begin{bmatrix} m+r-1 \\ r \end{bmatrix} \frac{t^r}{(1+t)^{m+r}} t^l\,dt,$$

we have

$$I_l(m,r) = \frac{(r+1)(r+2)\ldots\ldots(r+l)}{(m-2)(m-3)\ldots(m-l-1)}, \quad l \geqslant 1$$

and $I_0(m,r) = 1$.

Proof. Using definition of β-function, we have

$$\begin{aligned} I_l(m,r) &= (m-1)\int_0^\infty \begin{bmatrix} m+r-1 \\ r \end{bmatrix} \frac{t^r}{(1+t)^{m+r}} t^l \, dt \\ &= (m-1)\begin{bmatrix} m+r-1 \\ r \end{bmatrix} \beta(r+l+1, m-l-1) \\ &= \frac{(r+1)(r+2)\ldots(r+l)}{(m-2)(m-3)\ldots(m-l-1)}. \quad \square \end{aligned}$$

LEMMA 3. *For* $e_k = t^k$, $k = 0,1,2,3$, *the moments of the operators* (1) *are as follows:*

$$\begin{aligned} (i) \ \ & J_m^\lambda(e_0;x) = 1, \\ (ii) \ \ & J_m^\lambda(e_1;x) = \frac{mx-1}{m-2}, \\ (iii) \ \ & J_m^\lambda(e_2;x) = \frac{1}{(m-2)(m-3)}\left[m^2x^2 + \left(\frac{(1+\lambda)}{\lambda}\right)mx + 3mx + 2\right], \\ (iv) \ \ & J_m^\lambda(e_3;x) = \frac{1}{(m-2)(m-3)(m-4)}\left[m^3x^3 + \left(\frac{2(1+\lambda)+1}{\lambda} + 6\right)m^2x^2\right. \\ & \qquad \left. + \left(\frac{(1+\lambda)^2+1}{\lambda^2} + 6\frac{(1+\lambda)}{\lambda} + 11\right)mx + 6\right]. \end{aligned}$$

Proof. For $f(t) = t^k$, $k = 0,1,2,3$, the proposed operators can be expressed as:

$$J_m^\lambda(t^k;x) = \sum_{r=0}^{\infty} g_{m,r,\lambda}(x) I_k(m,r).$$

For $k = 0$, we have

$$\begin{aligned} J_m^\lambda(e_0;x) &= \sum_{r=0}^{\infty} g_{m,r,\lambda}(x) I_0(m,r) \\ &= L_{m,\lambda}(1;x) \\ &= 1. \end{aligned}$$

Using Lemma 1 & 2 together with the equation (2), we have

$$\begin{aligned}
J_m^{\lambda}(e_1;x) &= \sum_{r=0}^{\infty} g_{m,r,\lambda}(x) I_1(m,r) \\
&= \sum_{r=0}^{\infty} g_{m,r,\lambda}(x) \frac{(r+1)}{(m-2)} \\
&= \frac{1}{(m-2)} \sum_{r=0}^{\infty} g_{m,r,\lambda}(x)(r+1) \\
&= \frac{1}{(m-2)} (mL_{m,\lambda}(t;x) + L_{m,\lambda}(1;x)) \\
&= \frac{mx-1}{m-2}.
\end{aligned}$$

Following above for $e_2 = t^2$, we have

$$\begin{aligned}
J_m^{\lambda}(e_2;x) &= \sum_{r=0}^{\infty} g_{m,r,\lambda}(x) I_2(m,r) \\
&= \sum_{r=0}^{\infty} g_{m,r,\lambda}(x) \frac{(r+1)(r+2)}{(m-2)(m-3)} \\
&= \frac{1}{(m-2)(m-3)} \sum_{r=0}^{\infty} g_{m,r,\lambda}(x)(r^2+3r+2) \\
&= \frac{1}{(m-2)(m-3)} (m^2 L_{m,\lambda}(t^2;x) + 3mL_{m,\lambda}(t;x) + 2L_{m,\lambda}(1;x)) \\
&= \frac{1}{(m-2)(m-3)} \left[m^2x^2 + \left(\frac{(1+\lambda)}{\lambda} \right) mx + 3mx + 2 \right].
\end{aligned}$$

Similarly, for $e_3 = t^3$, we have

$$\begin{aligned}
J_m^{\lambda}(e_3;x) &= \sum_{r=0}^{\infty} g_{m,r,\lambda}(x) I_3(m,r) \\
&= \sum_{r=0}^{\infty} g_{m,r,\lambda}(x) \frac{(r+1)(r+2)(r+3)}{(m-2)(m-3)(m-4)} \\
&= \frac{1}{(m-2)(m-3)(m-4)} \sum_{r=0}^{\infty} g_{m,r,\lambda}(x)(r^3+6r^2+11r+6) \\
&= \frac{1}{(m-2)(m-3)(m-4)} \Big(m^3 L_{m,\lambda}(t^3;x) + 6m^2 L_{m,\lambda}(t^2;x) \\
&\quad + 11mL_{m,\lambda}(t;x) + 6L_{m,\lambda}(1;x) \Big) \\
&= \frac{1}{(m-2)(m-3)(m-4)} \left[m^3x^3 + \left(\frac{2(1+\lambda)+1}{\lambda} + 6 \right) m^2x^2 \right. \\
&\quad \left. + \left(\frac{(1+\lambda)^2+1}{\lambda^2} + 6\frac{(1+\lambda)}{\lambda} + 11 \right) mx + 6 \right]. \quad \square
\end{aligned}$$

LEMMA 4. *The central moments of the operators* (1) *are as follows:*

$$\begin{aligned}
J_m^{\lambda}(t-x;x) &= \frac{1}{m-2}(2x-1),\\
J_m^{\lambda}((t-x)^2;x) &= \frac{1}{(m-2)(m-3)}\left[(m+6)x^2+\left(\frac{1+\lambda}{\lambda}+\frac{5m-6}{m}\right)mx+2\right],\\
J_m^{\lambda}((t-x)^3;x) &= \frac{1}{(m-2)(m-3)(m-4)}\\
&\quad\times\left[2(5m+12)x^3+\left(\frac{12m}{\lambda}-13m^2+69m-36\right)x^2\right.\\
&\quad\left.+\left(\frac{(1+\lambda)^2+1}{\lambda^2}m+\frac{6m}{\lambda}+11m+24\right)x+6\right].
\end{aligned}$$

THEOREM 1. *Let*

$$E=\{\psi : x\in[0,\infty),\ \frac{\psi(x)}{1+x^2}\ \text{converges as } x\to\infty\},$$

and $C[0,\infty)$ be the linear space of all real valued continuous functions defined in $[0,\infty)$. Then, for $f\in E\cap C[0,\infty)$ and $\lambda>0$, the sequence of operators $\{J_m^{\lambda}(f;.)\}_{m\in\mathbb{N}}$ uniformly converges to f on every compact subset of $[0,\infty)$.

Proof. Considering Lemma 3, we have

$$\lim_{m\to\infty} J_m^{\lambda}(e_\kappa;x)=x^\kappa,\quad \kappa=0,1,2$$

uniformly on every compact subset of $[0,\infty)$ and then following [8, Theorem 3.1], we obtain the required result. □

3. Convergence behavior of the operators

Consider a subspace $C_B[0,\infty)$ of $C[0,\infty)$, as the space all real valued bounded continuous functions defined in $[0,\infty)$ equipped with supremum norm.

THEOREM 2. (Local approximation theorem) *For $f\in C_B[0,\infty)$, there exists a positive constant $M>0$ satisfying*

$$|J_m^{\lambda}(f;x)-f(x)|\leqslant M\omega_2(f,\delta_m(x))+\omega\left(f,\left|\frac{2x-1}{m-2}\right|\right),\quad \forall x\in[0,\infty),$$

where

$$\begin{aligned}
\delta_m^2(x) &= \frac{1}{(m-2)^2(m-3)}\left((m^2+8m-24)x^2+\left((6m^2-22m+24)\right.\right.\\
&\quad\left.\left.+\frac{m(m-2)}{\lambda}\right)x+(3m-7)\right).
\end{aligned}$$

Proof. To prove the result, we shall employ certain tools of quantitative approximation. For details, readers may refer to [3, Theorem 2.4].

For $f \in C_B[0,\infty)$, we define

$$\tilde{J}_m^{\lambda}(f;x) = J_m^{\lambda}(f;x) + f(x) - f\left(\frac{mx-1}{m-2}\right).$$

Using Lemma 3, we immediately get, $\tilde{J}_m^{\lambda}(t-x;x) = 0$.

For $g \in W^2 = \{\psi \in C_B[0,\infty) : \psi', \psi'' \in C_B[0,\infty)\}$, by applying Taylor's series formula, we have

$$\psi(t) = \psi(x) + (t-x)\psi'(x) + \int_x^t (t-u)\psi''(u)du.$$

On applying the operator $\{\tilde{J}_m^{\lambda}\}$ in the above equation, we have

$$\begin{aligned}
\tilde{J}_m^{\lambda}(\psi;x) &= \psi(x) + \tilde{J}_m^{\lambda}\left(\int_x^t (t-u)\psi''(u)du;x\right) \\
&= \psi(x) + J_m^{\lambda}\left(\int_x^t (t-u)\psi''(u)du;x\right) \\
&\quad - \int_x^{\frac{mx-1}{m-2}} \left(\frac{mx-1}{m-2} - u\right)\psi''(u)du.
\end{aligned}$$

Thus,

$$\begin{aligned}
|\tilde{J}_m^{\lambda}(\psi;x) - \psi(x)| &\leqslant J_m^{\lambda}\left(\int_x^t |t-u||\psi''(u)|du;x\right) + \int_x^{\frac{mx-1}{m-2}} \left|\frac{mx-1}{m-2} - u\right||\psi''(u)|du \\
&\leqslant J_m^{\lambda}((t-x)^2;x)\|\psi''\| + \left(\frac{mx-1}{m-2} - x\right)^2 \|\psi''\|.
\end{aligned}$$

On applying Lemma 4, we get

$$\begin{aligned}
|\tilde{J}_m^{\lambda}(\psi;x) - \psi(x)| &\leqslant \left(\frac{1}{(m-2)(m-3)}\left[(m+6)x + \left(\frac{1+\lambda}{\lambda} + \frac{5m-6}{m}\right)mx + 2\right]\right. \\
&\quad \left. + \left(\frac{mx-1}{m-2} - x\right)^2\right)\|\psi''\| \\
&= \frac{1}{(m-2)^2(m-3)}\left((m^2+8m-24)x^2 + \left((6m^2-22m+24)\right.\right. \\
&\quad \left.\left. + \frac{m(m-2)}{\lambda}\right)x + (3m-7)\right)\|\psi''\| \\
&= \delta_m^2(x)\|\psi''\|.
\end{aligned}$$

Also, $|\tilde{J}_m^{\lambda}(f;x)| \leqslant 3\|f\|$. We observe that, for $f \in C_B[0,\infty)$ and $\psi \in W^2$, we get

$$\begin{aligned}
|J_m^{\lambda}(f;x) - f(x)| &= \left|\tilde{J}_m^{\lambda}(f;x) - f(x) + f\left(\frac{mx-1}{m-2}\right) - f(x)\right| \\
&\leqslant |\tilde{J}_m^{\lambda}(f-\psi;x)| + |\tilde{J}_m^{\lambda}(\psi;x) - \psi(x)| \\
&\quad + |\psi(x) - f(x)| + \left|f\left(\frac{mx-1}{m-2}\right) - f(x)\right| \\
&\leqslant 4\|f-\psi\| + \delta_m^2(x)\|\psi''\| + \omega\left(f, \left|\frac{2x-1}{m-2}\right|\right) \\
&\leqslant 4\left(\|f-\psi\| + \delta_m^2(x)\|\psi''\|\right) + \omega\left(f, \left|\frac{2x-1}{m-2}\right|\right).
\end{aligned}$$

For all $\psi \in W^2$, on the right hand side, taking the infimum, we obtain

$$|J_m^{\lambda}(f;x) - f(x)| \leqslant 4K_2\left(f, \delta_m^2(x)\right) + \omega\left(f, \left|\frac{2x-1}{m-2}\right|\right).$$

Finally, we get

$$|J_m^{\lambda}(f;x) - f(x)| \leqslant M\omega_2\left(f, \delta_m(x)\right) + \omega\left(f, \left|\frac{2x-1}{m-2}\right|\right),$$

which is the required result. $\square$

The Lipschitz type subspace [12] of the space $C[0,\infty)$ is defined as

$$Lip_M^*(r) = \left\{ f \in C[0,\infty) : |f(t) - f(x)| \leqslant M_f \frac{|t-x|^r}{(t+x)^{\frac{r}{2}}};\ t,x \geqslant 0 \right\},$$

here M_f is a positive constant which depends on the function f and $0 < r \leqslant 1$.

THEOREM 3. *If $f \in Lip_M^*(r)$ and $0 < r \leqslant 1$, then for $x > 0$,*

$$|J_m^{\lambda}(f(t);x) - f(x)| \leqslant M_f \left(\frac{\delta_m(x)}{\sqrt{x}}\right)^r.$$

Proof. For $f \in Lip_M^*(r)$,

$$\begin{aligned}
&|J_m^{\lambda}(f(t);x) - f(x)| \\
&\leqslant (m-1)\sum_{\kappa=0}^{\infty} g_{m,\kappa,\lambda}(x) \int_0^{\infty} \begin{bmatrix} m+\kappa-1 \\ \kappa \end{bmatrix} \frac{t^{\kappa}}{(1+t)^{m+\kappa}} |f(t) - f(x)| dt \\
&\leqslant (m-1)\sum_{\kappa=0}^{\infty} g_{m,\kappa,\lambda}(x) \int_0^{\infty} \begin{bmatrix} m+\kappa-1 \\ \kappa \end{bmatrix} \frac{t^{\kappa}}{(1+t)^{m+\kappa}} M_f \frac{|t-x|^r}{(t+x)^{\frac{r}{2}}} dt.
\end{aligned}$$

Applying the integral form of Holder's inequality with $\alpha = \frac{2}{2-r}$, $\beta = \frac{2}{r}$, we get

$$\begin{aligned}
&|J_m^\lambda(f(t);x) - f(x)| \\
&\leqslant (m-1)M_f \sum_{\kappa=0}^{\infty} g_{m,\kappa,\lambda}(x) \left(\int_0^\infty \begin{bmatrix} m+\kappa-1 \\ \kappa \end{bmatrix} \frac{t^\kappa}{(1+t)^{m+\kappa}} dt \right)^{\frac{(2-r)}{2}} \\
&\times \left(\int_0^\infty \begin{bmatrix} m+\kappa-1 \\ \kappa \end{bmatrix} \frac{t^\kappa}{(1+t)^{m+\kappa}} \frac{|t-x|^2}{(t+x)} dt \right)^{\frac{r}{2}}.
\end{aligned}$$

Again, applying summation form of Holder's inequality for $\alpha = \frac{2}{2-r}$, $\beta = \frac{2}{r}$, we obtain

$$\begin{aligned}
&|J_m^\lambda(f(t);x) - f(x)| \\
&\leqslant M_f \left((m-1) \sum_{\kappa=0}^{\infty} g_{m,\kappa,\lambda}(x) \int_0^\infty \begin{bmatrix} m+\kappa-1 \\ \kappa \end{bmatrix} \frac{t^\kappa}{(1+t)^{m+\kappa}} dt \right)^{\frac{(2-r)}{2}} \\
&\times \left((m-1) \sum_{r=0}^{\infty} g_{m,\kappa,\lambda}(x) \int_0^\infty \begin{bmatrix} m+\kappa-1 \\ \kappa \end{bmatrix} \frac{t^\kappa}{(1+t)^{m+\kappa}} \frac{|t-x|^2}{(t+x)} dt \right)^{\frac{r}{2}} \\
&\leqslant \frac{M_f}{(\sqrt{x})^r} \left(J_m^\lambda((t-x)^2;x) \right)^{\frac{r}{2}}.
\end{aligned}$$

By setting $\delta_m^2(x) = J_m^\lambda((t-x)^2;x)$, we obtain the desired result. □

4. Quantitative estimates on difference of operators

In recent years, Acu and Rasa [1], have established some results for the difference of operators by involving the operators constructed on the same basis but having different functional. Later, Aral et al. [2] and V. Gupta [6, 4] also contributed some interesting theorems on polynomial differences in order to answer the problem raised by A. Lupas [5].

Recently, Gupta and Acu [7] established some results and estimated some numerical computations on the difference of operators having different basis functions. Following the results proved by Gupta and Acu in [7], here, in this section, some quantitative results have been presented for the difference of the proposed operators with the operators having different basis function.

Let $C(I)$ be the space of all real valued continuous functions defined on an interval $I \subset \mathbb{R}$ and $C_B(I) = \{f \in C(I) : ||f|| = \sup_{x \in I} |f(x)| < \infty\}$.

Further, we consider a positive linear functional $H : C(I) \to \mathbb{R}$ with $H(e_0) = 1$ and we set

$$\begin{aligned}
b^H &= H(e_1), \\
\mu_r^H &= H(e_1 - b^H e_0)^r, \quad r \in \mathbb{N}.
\end{aligned}$$

THEOREM 4. ([7, Theorem 2.1]) *Let*

$$U_m = \sum_{r=0}^{\infty} p_{m,r}(x) H_{m,r}(\psi) \quad \textit{and} \quad V_m = \sum_{r=0}^{\infty} s_{m,r}(x) G_{m,r}(\psi)$$

be positive linear operators having their basis as $p_{m,r}(x)$ *and* $s_{m,r}(x)$ *respectively. If* $\psi \in D(I)$ *with* $\psi'' \in C_B(I)$*, then*

$$|(U_m - V_m)(\psi;x)| \leqslant \phi(x)||\psi''|| + 2\omega_1(\psi;\delta_1(x)) + 2\omega_1(\psi;\delta_2(x)),$$

where

$$\begin{aligned}
\phi(x) &= \frac{1}{2}\sum_{r=0}^{\infty}\left(p_{m,r}(x)\mu_2^{H_{m,r}} + s_{m,r}(x)\mu_2^{G_{m,r}}\right),\\
\delta_1^2(x) &= \sum_{r=0}^{\infty} p_{m,r}(x)\left(b^{H_{m,r}} - x\right)^2,\\
\delta_2^2(x) &= \sum_{r=0}^{\infty} s_{m,r}(x)\left(b^{G_{m,r}} - x\right)^2.
\end{aligned}$$

Using the above assumptions, the proposed operators $\{J_m^c\}$ can be expressed as

$$J_m^\lambda(f;x) = \sum_{r=0}^{\infty} g_{m,r,\lambda}(x)H_{m,r}(f),$$

here

$$H_{m,r}(f) = (m-1)\int_0^\infty \begin{bmatrix} m+r-1 \\ r \end{bmatrix} \frac{t^r}{(1+t)^{m+r}} f(t)dt.$$

REMARK 1. By using Lemma 2, we have

$$H_{m,r}(e_i) = \frac{(r+1)(r+2)\ldots\ldots(r+i)}{(m-2)(m-3)\ldots(m-i-1)}, \quad i = 1,2,\ldots .$$

and $H_{m,r}(e_0) = 1$. Following the above equation, we get

$$b^{H_{m,r}} = \frac{(r+1)}{(m-2)}$$

and

$$\begin{aligned}
\mu_2^{H_{m,r}} &= H_{m,r}(e_1 - b^{H_{m,r}}e_0)^2\\
&= H_{m,r}(e_2) - (H_{m,r}(e_1))^2\\
&= \frac{(r+1)(r+2)}{(m-2)(m-3)} - \frac{(r+1)^2}{(m-2)^2}\\
&= \frac{r^2 + mr + m - 1}{(m-2)^2(m-3)}.
\end{aligned}$$

4.1. Difference of the proposed operators with classical Jain-Pethe operators

For $0 \leqslant \alpha m \leqslant 1$ and $f:[0,\infty) \to \mathbb{R}$, Jain-Pethe operators [11] can be expressed as

$$J_m^{(\alpha)}(f;x) = \sum_{r=0}^{\infty} s_{m,r}^{[\alpha]}(x) T_{m,r}(f),$$

here $T_{m,r}(f) = f\left(\frac{r}{m}\right)$.

LEMMA 5. ([8, Lemma 2.1]) *Some initial Moments for Jain-Pethe operators are given as*

$$\begin{aligned}
J_m^{(\alpha)}(1;x) &= 1,\\
J_m^{(\alpha)}(t;x) &= x,\\
J_m^{(\alpha)}(t^2;x) &= x^2 + \left(\alpha + \frac{1}{m}\right)x,\\
J_m^{(\alpha)}(t^3;x) &= x^4 + 6\left(\alpha + \frac{1}{m}\right)x^2 + \left(11\alpha^2 + \frac{18\alpha}{m} + \frac{7}{m^2}\right)x^2\\
&\quad + \left(6\alpha^3 + \frac{12\alpha^2}{m} + \frac{7\alpha}{m^2} + \frac{1}{m^3}\right)x.
\end{aligned}$$

Now, we have following quantitative result for the difference of the proposed operators with Jain-Pethe operators.

PROPOSITION 1. *Let $\psi, \psi', \psi'' \in C_B[0,\infty)$ and $x \in [0,\infty)$. Then, for $m \in \mathbb{N}$,*

$$|(J_m^{\lambda} - J_m^{(\alpha)}(\psi;x)| \leqslant \phi_1(x)||\psi''|| + 2\omega_1(\psi;\delta_1(x)) + 2\omega_1(\psi;\delta_2(x)),$$

where

$$\begin{aligned}
\phi_1(x) &= \frac{1}{2(m-2)^2(m-3)}\left(m^2x^2 + \left(\frac{(1+\lambda)}{\lambda} + m\right)mx + m - 1\right),\\
\delta_1^2(x) &= \frac{m^2}{(m-2)^2}\left(x^2 + \left(\frac{(1+\lambda)}{\lambda}\right)\frac{x}{m} + 2[1-(m-2)x]\frac{x}{m}\right.\\
&\quad \left. + \frac{1-2(m-2)x + (m-2)^2x^2}{m^2}\right)
\end{aligned}$$

and

$$\delta_2^2(x) = \left(\alpha + \frac{1}{m}\right)x.$$

Proof. Here, we observe that Theorem 4 validates the Proposition. It remains only to compute the values of $\phi_1(x)$, $\delta_1(x)$ and $\delta_2(x)$ as investigated in Theorem 4.

Using Remark 1, we have

$$\begin{aligned}\phi_1(x) &= \frac{1}{2}\sum_{r=0}^{\infty}\left(g_{m,r,\lambda}(x)\mu_2^{F_{m,r}} + s_{m,r}^{[\alpha]}(x)\mu_2^{T_{m,r}}\right)\\ &= \frac{1}{2}\sum_{r=0}^{\infty}\left[g_{m,r,\lambda}(x)\left(\frac{r^2+mr+m-1}{(m-2)^2(m-3)}\right) + s_{m,r}^{[\alpha]}(x)T_{m,r}(e_1-b^{T_{m,r}}e_0)^2\right]\\ &= \frac{1}{2}\sum_{r=0}^{\infty}g_{m,r,\lambda}(x)\left(\frac{r^2+mr+m-1}{(m-2)^2(m-3)}\right).\end{aligned}$$

Now, applying the Lemma 1, we get

$$\begin{aligned}\phi_1(x) &= \frac{1}{2(m-2)^2(m-3)}\left(m^2L_{m,\lambda}(e_2;x)+m^2L_{m,\lambda}(e_1;x)+(m-1)\right)\\ &= \frac{1}{2(m-2)^2(m-3)}\left(m^2x^2+\left(\frac{(1+\lambda)}{\lambda}+m\right)mx+m-1\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\delta_1^2(x) &= \sum_{r=0}^{\infty}g_{m,r,\lambda}(x)\left(b^{F_{m,r}}-x\right)^2\\ &= \sum_{r=0}^{\infty}g_{m,r,\lambda}(x)\left(F_{m,r}(e_1)-x\right)^2\\ &= \sum_{r=0}^{\infty}g_{m,r,\lambda}(x)\left(\frac{(r+1)}{(m-2)}-x\right)^2\\ &= \sum_{r=0}^{\infty}g_{m,r,\lambda}(x)\left(\frac{(r+1)^2}{(m-2)^2}-2x\frac{(r+1)}{(m-2)}+x^2\right)\\ &= \frac{m^2}{(m-2)^2}\sum_{r=0}^{\infty}g_{m,r,\lambda}(x)\Big(r^2+2[1-(m-2)x]r\\ &\quad +1-2(m-2)x+(m-2)^2x^2\Big)\\ &= \frac{1}{(m-2)^2}\Big(m^2L_{m,\lambda}(e_2;x)+2m[1-(m-2)x]L_{m,\lambda}(e_1;x)\\ &\quad +1-2(m-2)x+(m-2)^2x^2\Big)\\ &= \frac{m^2}{(m-2)^2}\Big(x^2+\left(\frac{(1+\lambda)}{\lambda}\right)\frac{x}{m}+2[1-(m-2)x]\frac{x}{m}\\ &\quad +\frac{1-2(m-2)x+(m-2)^2x^2}{m^2}\Big).\end{aligned}$$

and

$$
\begin{aligned}
\delta_2^2(x) &= \sum_{r=0}^{\infty} s_{m,r}^{[\alpha]}(x)\left(b^{T_{m,r}}-x\right)^2 \\
&= \sum_{r=0}^{\infty} s_{m,r}^{[\alpha]}(x)\left(T_{m,r}(e_1)-x\right)^2 \\
&= \sum_{r=0}^{\infty} s_{m,r}^{[\alpha]}(x)\left(\frac{r}{m}-x\right)^2 \\
&= \sum_{r=0}^{\infty} s_{m,r}^{[\alpha]}(x)\left(\frac{r^2}{m^2}-2x\frac{r}{m}+x^2\right) \\
&= \left(J_m^{(\alpha)}(e_2;x)-2xJ_m^{(\alpha)}(e_1;x)+x^2\right) \\
&= \left(x^2+\left(\alpha+\frac{1}{m}\right)x-2x^2+x^2\right) \\
&= \left(\alpha+\frac{1}{m}\right)x. \quad \square
\end{aligned}
$$

4.2. Difference with Jain-Pethe-Durrmeyer-Baskakov operators

For $f:[0,\infty)\to\mathbb{R}$, Jain-Pethe-Durrmeyer-Baskakov operators defined by Dev and Pratap [8], can be expressed as

$$
P_{m,\alpha}(f;x) = \sum_{r=0}^{\infty} s_{m,r}^{[\alpha]}(x)F_{m,r}(f),
$$

where

$$
\begin{aligned}
s_{m,r}^{[\alpha]}(x) &= (1+m\alpha)^{\frac{-x}{\alpha}}\left(\alpha+\frac{1}{m}\right)^{-r}\frac{x^{(r,-\alpha)}}{r!}, \quad x\in[0,\infty), \\
x^{(r,-\alpha)} &= x(x+\alpha)\dots(x+(r-1)\alpha), \quad x^{(0,-\alpha)}=1
\end{aligned}
$$

and $\alpha=(\alpha_m)_{m\in\mathbb{N}}$ is such that $0\leqslant\alpha_m\leqslant\frac{1}{m}$.

LEMMA 6. ([8, Lemma 2.1]) *Some initial Moments for Jain-Pethe-Durrmeyer-Baskakov operators are given as*

$$
\begin{aligned}
P_{m,\alpha}(1;x) &= 1, \\
P_{m,\alpha}(t;x) &= \frac{(m-3)!}{(m-2)!}(mx+1!), \\
P_{m,\alpha}(t^2;x) &= \frac{(m-4)!}{(m-2)!}((mx)^2+(\alpha m+4)(mx)+2!), \\
P_{m,\alpha}(t^3;x) &= \frac{(m-5)!}{(m-2)!}((mx)^3+(3\alpha m+2)(mx)^2 \\
&\quad +(2\alpha^2m^2+9\alpha m+19)(mx)+3!).
\end{aligned}
$$

Now, we give the quantitative estimates for the difference of the proposed operators with Jain-Pethe-Durrmeyer-Baskakov operators.

PROPOSITION 2. *Let* $\psi,\psi',\psi''\in C_B[0,\infty)$ *and* $x\in[0,\infty)$. *Then for* $m\in\mathbb{N}$, *we have*

$$|(J_m^{\lambda}-P_{m,\alpha})(\psi;x)|\leqslant \phi_1(x)||\psi''||+2\omega_1(\psi;\delta_1(x))+2\omega_1(\psi;\delta_2(x)),$$

where

$$\begin{aligned}\phi_1(x)&=\frac{m^2}{2(m-2)^2(m-3)}\left(2\left[x^2+x+\frac{(m-1)}{m^2}\right]+\left(\alpha+\frac{1}{m}\right)x+\left(\frac{1+\lambda}{\lambda}\right)\frac{x}{m}\right),\\ \delta_1^2(x)&=\frac{m^2}{(m-2)^2}\left(x^2+\left(\frac{1+\lambda}{\lambda}\right)\frac{x}{m}+2[1-(m-2)x]\frac{x}{m}\right.\\ &\quad\left.+\frac{1-2(m-2)x+(m-2)^2x^2}{m^2}\right)\end{aligned}$$

and

$$\begin{aligned}\delta_2^2(x)&=\frac{m^2}{(m-2)^2}\left(x^2+\left(\alpha+\frac{1}{m}\right)x+2[1-(m-2)x]\frac{x}{m}\right.\\ &\quad\left.+\frac{1-2(m-2)x+(m-2)^2x^2}{m^2}\right).\end{aligned}$$

Proof. Here, we observe that Theorem 4 validates the Proposition. It remains only to compute the values of $\phi_1(x)$, $\delta_1(x)$ and $\delta_2(x)$ as investigated in Theorem 4. Using Remark 1, we have

$$\begin{aligned}\phi_1(x)&=\frac{1}{2}\sum_{r=0}^{\infty}\left(g_{m,r,\lambda}(x)\mu_2^{F_{m,r}}+s_{m,r}^{[\alpha]}(x)\mu_2^{F_{m,r}}\right)\\ &=\frac{1}{2}\sum_{r=0}^{\infty}(s_{m,r}^{[\alpha]}(x)+g_{m,r,\lambda}(x))\left(\frac{r^2+mr+m-1}{(m-2)^2(m-3)}\right).\end{aligned}$$

Now, using the moments of Jain-Pethe operators and Lemma 1 in above, we get

$$\begin{aligned}\phi_1(x)&=\frac{1}{2(m-2)^2(m-3)}\left(m^2J_m^{(\alpha)}(e_2;x)+m^2J_m^{(\alpha)}(e_1;x)\right.\\ &\quad\left.+(m-1)+m^2L_{m,\lambda}(e_2;x)+m^2L_{m,\lambda}(e_1;x)+(m-1)\right)\\ &=\frac{m^2}{2(m-2)^2(m-3)}\left(x^2+\left(\alpha+\frac{1}{m}\right)x+x+\frac{(m-1)}{m^2}\right.\\ &\quad\left.+x^2+\left(\frac{1+\lambda}{\lambda}\right)\frac{x}{m}+x+\frac{(m-1)}{m^2}\right)\\ &=\frac{m^2}{2(m-2)^2(m-3)}\left(2\left[x^2+x+\frac{(m-1)}{m^2}\right]+\left(\alpha+\frac{1}{m}\right)x+\left(\frac{1+\lambda}{\lambda}\right)\frac{x}{m}\right).\end{aligned}$$

Similarly,

$$
\begin{aligned}
\delta_1^2(x) &= \sum_{r=0}^{\infty} g_{m,r,\lambda}(x)\left(b^{F_{m,r}} - x\right)^2 \\
&= \sum_{r=0}^{\infty} g_{m,r,\lambda}(x)\left(F_{m,r}(e_1) - x\right)^2 \\
&= \sum_{r=0}^{\infty} g_{m,r,\lambda}(x)\left(\frac{(r+1)}{(m-2)} - x\right)^2 \\
&= \sum_{r=0}^{\infty} g_{m,r,\lambda}(x)\left(\frac{(r+1)^2}{(m-2)^2} - 2x\frac{(r+1)}{(m-2)} + x^2\right) \\
&= \frac{1}{(m-2)^2}\sum_{r=0}^{\infty} g_{m,r,\lambda}(x)\Bigg(r^2 + 2[1-(m-2)x]r \\
&\quad + 1 - 2(m-2)x + (m-2)^2x^2\Bigg) \\
&= \frac{1}{(m-2)^2}\Bigg(m^2 L_{m,\lambda}(e_2;x) + 2m[1-(m-2)x] \\
&\quad \times L_{m,\lambda}(e_1;x) + 1 - 2(m-2)x + (m-2)^2x^2\Bigg) \\
&= \frac{m^2}{(m-2)^2}\Bigg(x^2 + \left(\frac{1+\lambda}{\lambda}\right)\frac{x}{m} + 2[1-(m-2)x]\frac{x}{m} \\
&\quad + \frac{1-2(m-2)x+(m-2)^2x^2}{m^2}\Bigg)
\end{aligned}
$$

and

$$
\begin{aligned}
\delta_2^2(x) &= \sum_{r=0}^{\infty} s_{m,r}^{[\alpha]}(x)\left(b^{F_{m,r}} - x\right)^2 \\
&= \sum_{r=0}^{\infty} s_{m,r}^{[\alpha]}(x)\left(F_{m,r}(e_1) - x\right)^2 \\
&= \sum_{r=0}^{\infty} s_{m,r}^{[\alpha]}(x)\left(\frac{(r+1)}{(m-2)} - x\right)^2 \\
&= \sum_{r=0}^{\infty} s_{m,r}^{[\alpha]}(x)\left(\frac{(r+1)^2}{(m-2)^2} - 2x\frac{(r+1)}{(m-2)} + x^2\right) \\
&= \frac{1}{(m-2)^2}\sum_{r=0}^{\infty} s_{m,r}^{[\alpha}(x)\Bigg(r^2 + 2[1-(m-2)x]r \\
&\quad + 1 - 2(m-2)x + (m-2)^2x^2\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{(m-2)^2}\Big(m^2 J_m^{(\alpha)}(e_2;x) + 2m[1-(m-2)x]J_m^{(\alpha)}(e_1;x) \\
&\quad + 1 - 2(m-2)x + (m-2)^2x^2\Big) \\
&= \frac{m^2}{(m-2)^2}\left(x^2 + \left(\alpha + \frac{1}{m}\right)x + 2[1-(m-2)x]\frac{x}{m}\right. \\
&\quad \left. + \frac{1-2(m-2)x+(m-2)^2x^2}{m^2}\right). \quad \square
\end{aligned}
$$

5. Conclusion

In this work, we have presented the construction of the generalized Jain-Pethe-Baskakov-Durrmeyer operators and established their uniform convergence. Further, the convergence properties of the operators have been studied by means of modulus of continuity, K-functional and Lipschitz class functions. Also, we have discussed some quantitative results for difference of the generalized operators with the family of Jain-Pethe operators.

Conflict of interest. The authors declare that they have no conflict of interest.

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