ON SOME SERIES INVOLVING HARMONIC
AND SKEW–HARMONIC NUMBERS

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Dedicated to Minh

Abstract. In this paper, we evaluate in closed form several different series involving the harmonic numbers and skew-harmonic numbers. We consider two classes of series involving these sequences. One class of series involves the product of the nth harmonic or skew-harmonic number and a tail. We provide the solution to two open problems concerning these harmonic series with tails from Alina Sîntămărian and Ovidiu Furdui's book Sharpening Mathematical Analysis Skills. The other class of series is the Hardy series, which involves a logarithm and the Euler-Mascheroni constant being subtracted from the nth harmonic number.

1. Introduction

The harmonic numbers are defined as $H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$. Its generalization is defined as $H_n^{(p)} = 1 + \frac{1}{2^p} + \ldots + \frac{1}{n^p}$. The generalized harmonic numbers are useful in evaluating some of our series in closed form. The alternating analogue of the harmonic numbers, skew-harmonic numbers, are defined to be $\overline{H}_n = 1 - \frac{1}{2} + \ldots + \frac{(-1)^{n-1}}{n}$. Leonhard Euler famously derived

$$\sum_{n=1}^{\infty} \frac{H_n}{n^k} = \left(1 + \frac{k}{2}\right)\zeta(k+1) - \frac{1}{2}\sum_{n=1}^{k-2}\zeta(k-n)\zeta(n+1) \quad k \in \{2, 3, \ldots\},$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function (see [1, p. 807], [26, 25.2]). The result can be found in [7] and proved in [23, pp. 47–49]. Other formulae for Euler sums have been derived, one of which we will use from [15, Theorem 7.2, p. 33]:

$$\sum_{n=1}^{\infty} (-1)^{n-1}\frac{H_n^{(p)}}{n^q} = \frac{(1 - (-1)^p)\zeta(p)\eta(q) + \eta(p+q)}{2}$$

$$+ \sum_{j+2k=p}\binom{q+j-1}{q-1}(-1)^{j+1}\eta(q+j)\eta(2k) \qquad (1.1)$$

$$+ (-1)^p\sum_{i+2k=q}\binom{p+i-1}{p-1}\zeta(p+i)\eta(2k), \quad p+q \text{ is odd}$$

where $\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$ denotes the Dirichlet eta function (see [1, p. 807]).

Series involving the Riemann zeta function can be found in [12] and [29]. Other infinite series evaluated in closed form incorporating the Riemann zeta function and harmonic numbers can be found in [4] and [7]. Special values of the zeta function can be found in [1, p. 807].

In this paper, we will be evaluating in closed form two classes of series involving harmonic and skew-harmonic numbers. One of such is a series with a tail. Sîntămărian and Furdui derive in [28, Problem 7.90 (a)] the following harmonic series with tail of $\zeta(2)$:

$$\sum_{n=1}^{\infty} H_n \left(\zeta(2) - 1 - \frac{1}{2^2} - \ldots - \frac{1}{n^2} - \frac{1}{n} \right) = -1$$

and pose the following related open problem in [28, p. 214], which we will calculate in this paper:

$$S_1 := \sum_{n=1}^{\infty} H_{2n} \left(\zeta(2) - 1 - \frac{1}{2^2} - \ldots - \frac{1}{n^2} - \frac{1}{n} \right).$$

Sîntămărian and Furdui derive in [28, Problem 3.81 (c)] another harmonic series with a tail of $\zeta(2)$:

$$\sum_{n=1}^{\infty} \frac{\overline{H}_n}{n} \left(\zeta(2) - 1 - \frac{1}{2^2} - \ldots - \frac{1}{n^2} \right) = \frac{53\pi^4}{1440} + \frac{\pi^2 \log^2(2)}{4} - \frac{\log^4(2)}{8}$$
$$- \frac{21\log(2)}{8} \zeta(3) - 3\operatorname{Li}_4\left(\frac{1}{2} \right)$$

where $\operatorname{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$ denotes the polylogarithm of order s (see [20, p. 189], [26, 25.12]). Note that $\operatorname{Li}_s(1) = \zeta(s)$ for $s > 1$. Sîntămărian and Furdui also pose in [28, p. 101] a similar harmonic series as an open problem with a tail of $\zeta(3)$,

$$S_2 := \sum_{n=1}^{\infty} \frac{\overline{H}_n}{n} \left(\zeta(3) - 1 - \frac{1}{2^3} - \ldots - \frac{1}{n^3} \right),$$

which we will calculate in this paper.

The other class of series we will evaluate in closed form is the Hardy series, which can be thought of as a series involving the Euler-Mascheroni constant γ (see [14, pp. 28–34]) and the logarithm subtracted from the nth harmonic number. Hardy derived the alternating Hardy series (see [10, p. 277])

$$\sum_{n=1}^{\infty} (-1)^n \left(H_n - \log(n) - \gamma \right) = \frac{\gamma - \log(\pi)}{2}.$$

We will provide a generalization of this sum as well as analogues of this generalization in section 3.

To evaluate these series, we require Abel's summation formula (see [28, p. 38]), which states that

$$\sum_{k=1}^{\infty} a_k b_k = \lim_{n \to \infty} A_n b_{n+1} + \sum_{k=1}^{\infty} A_k \left(b_k - b_{k+1} \right) \tag{1.2}$$

if $(a_n)_{\geqslant 1}$ and $(b_n)_{\geqslant 1}$ are two sequences of real numbers and $A_n := \sum_{k=1}^{n} a_n$.

2. Harmonic series with a tail

We will now evaluate some harmonic series with tails. In order to do so, we will make use of the integral representations of the harmonic and skew-harmonic numbers. Due to Euler (see [27]), we have

$$H_n = \int_0^1 \frac{1-x^n}{1-x}\, dx, \quad n \in \{1,2,\ldots\}. \tag{2.1}$$

Similarly, we also have the following integral representation for the skew-harmonic numbers (see [28, Problem 3.66, p. 95]):

$$\overline{H}_n = \int_0^1 \frac{1-(-x)^n}{1+x}\, dx, \quad n \in \{1,2,\ldots\}. \tag{2.2}$$

2.1. The calculation of S_1

To evaluate S_1, we will establish some lemmas.

LEMMA 1. *The following equality holds:*

$$\int_0^1 \frac{\operatorname{arctanh}(x)\log\left(1-x^2\right)}{x}\, dx = -\frac{7}{8}\zeta(3)$$

where $\operatorname{arctanh}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$ *is the inverse hyperbolic tangent function for* $x < 1$.

Proof. We have

$$\int_0^1 \frac{\operatorname{arctanh}(x)\log\left(1-x^2\right)}{x}\, dx = \frac{1}{2}\int_0^1 \frac{\log\left(\frac{1+x}{1-x}\right)\log\left(1-x^2\right)}{x}\, dx$$

$$= \frac{1}{2}\int_0^1 \frac{\log^2(1+x)}{x}\, dx - \frac{1}{2}\int_0^1 \frac{\log^2(1-x)}{x}\, dx. \tag{2.3}$$

From [25, p. 134], we make use of the following identity:

$$\int_0^1 \frac{\log^q(1+x)}{x}\, dx = \frac{\log^{q+1}(2)}{q+1} + q!\zeta(q+1)$$

$$- q!\sum_{k=0}^q \frac{\log^{q-k}(2)}{(q-k)!}\operatorname{Li}_{k+1}\left(\frac{1}{2}\right), \quad q \in \{1,2,\ldots\}. \tag{2.4}$$

We get that

$$\int_0^1 \frac{\log^2(1+x)}{x}\, dx = \frac{\zeta(3)}{4} \tag{2.5}$$

using (2.4) with $q = 2$ and plugging in the following special values of the polylogarithms (see [20, pp. 6, 155]:

$$\operatorname{Li}_1\left(\frac{1}{2}\right) = \log(2), \quad \operatorname{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{\log^2(2)}{2},$$

$$\text{Li}_3\left(\frac{1}{2}\right) = \frac{7}{8}\zeta(3) + \frac{\log^3(2)}{6} - \frac{\pi^2}{12}\log(2).$$

To deal with the other integral, we simply integrate by substituting $x \mapsto 1 - e^{-x}$ and make use of the identity (see [13, p. 189])

$$\int_0^\infty \frac{t^{s-1}}{e^t - 1}\, dt = \zeta(s)\Gamma(s)$$

where $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt$ is the classical gamma function (see [26, 5.2, 5.4], [1, pp. 255–257]) to get that

$$\int_0^1 \frac{\log^2(1-x)}{x}\, dx = 2\zeta(3). \tag{2.6}$$

Plugging in (2.5) and (2.6) into (2.3), we get our desired result. $\square$

We note that the special value $\text{Li}_1(1/2) = \log(2)$ is obvious using the power series representation of the polylogarithm.

LEMMA 2. *The following equality holds:*

$$\int_0^1 \frac{\log\left(1 - x^2\right)}{x}\, dx = -\frac{\pi^2}{12}.$$

Proof. Integrate by substituting $x \mapsto \sqrt{x}$. From there, we can make use of the identities $-\int_0^z \frac{\log(1-x)}{x} = \text{Li}_2(z)$ and $\text{Li}_2(1) = \zeta(2) = \frac{\pi^2}{6}$ (see [20, pp. 1, 4]). $\square$

We can now begin evaluating S_1 in closed form.

THEOREM 2.1. *The following equality holds:*

$$\sum_{n=1}^\infty H_{2n}\left(\zeta(2) - 1 - \frac{1}{2^2} - \ldots - \frac{1}{n^2} - \frac{1}{n}\right) = \log(2) - \frac{7}{8}\zeta(3) - 1.$$

Proof. We begin by plugging (2.1) into S_1:

$$S_1 = \sum_{n=1}^\infty \int_0^1 \frac{1 - x^{2n}}{1 - x}\left(\zeta(2) - 1 - \frac{1}{2^2} - \ldots - \frac{1}{n^2} - \frac{1}{n}\right) dx$$

$$= -\sum_{n=1}^\infty \int_0^1 \frac{1 - x^{2n}}{1 - x}\left(\frac{1}{n} + H_n^{(2)} - \zeta(2)\right) dx$$

Since $\frac{1-x^{2n}}{1-x} \geqslant 0$ and $\frac{1}{n} + H_n^{(2)} - \zeta(2) \geqslant 0$[1] for integers $n \geqslant 1$ and $x \in [0,1)$, we may apply Tonelli's theorem for non-negative functions (see [32, Theorem 23.17, p. 558])

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[1]To verify this, let $c_n := \frac{1}{n} + H_n^{(2)} - \zeta(2)$. Notice that $c_{n+1} - c_n = -\frac{1}{n(n+1)^2} < 0$, $c_1 = 2 - \zeta(2) > 0$ and $\lim_{n\to\infty} c_n = 0$. Thus, $c_n \geqslant 0$ for integers $n \geqslant 1$.

and interchange the order of summation and integration. Making use of the identity (see [28, Problem 2.47, p. 39])

$$\sum_{n=1}^{\infty} \left(\zeta(2) - 1 - \frac{1}{2^2} - \dots - \frac{1}{n^2} - \frac{1}{n+k} \right) = H_{k+1} + \frac{k}{k+1} - \zeta(2), \quad k \in \{0, 1, 2, \dots\}$$

with $k = 0$, the generating function from [28, p. 65]

$$\sum_{n=1}^{\infty} \left(\zeta(2) - 1 - \frac{1}{2^2} - \dots - \frac{1}{n^2} \right) x^n = \frac{x\zeta(2) - \mathrm{Li}_2(x)}{1-x}, \quad x \in [-1, 1)$$

and the known power series for the function $-\log(1-x)$, we get

$$S_1 = \int_0^1 \frac{1}{1-x} \left(1 - \frac{\zeta(2) - \mathrm{Li}_2\left(x^2\right)}{1-x^2} - \log(1-x) - \log(1+x) \right) dx$$

$$= \lim_{t \to 1^-} \left[\frac{\log^2(1-t)}{2} - \log(1-t) - \frac{\zeta(2)}{2} \left(\mathrm{arctanh}(t) + \frac{t}{1-t} \right) \right.$$

$$\left. + \int_0^t \frac{\mathrm{Li}_2\left(x^2\right)}{(1-x)^2(1+x)} dx - \int_0^t \frac{\log(1+x)}{1-x} dx \right].$$

(2.7)

We now have two integrals

$$I_1 := \int_0^t \frac{\mathrm{Li}_2\left(x^2\right)}{(1-x)^2(1+x)} dx - \int_0^t \frac{\log(1+x)}{1-x} dx. \tag{2.8}$$

We can apply apply integration by parts with the derivative of the dilogarithm (see [20, p. 1]):

$$\int_0^t \frac{\mathrm{Li}_2\left(x^2\right)}{(1-x)^2(1+x)} dx = \frac{1}{2} \int_0^t \frac{\mathrm{Li}_2\left(x^2\right)}{(1-x)^2} dx + \frac{1}{2} \int_0^t \frac{\mathrm{Li}_2\left(x^2\right)}{1-x^2} dx$$

$$= \frac{\mathrm{Li}_2\left(t^2\right)}{2(1-t)} + \int_0^t \left(\frac{\log\left(1-x^2\right)}{x} + \frac{\log\left(1-x^2\right)}{1-x} \right) dx$$

$$+ \frac{\mathrm{arctanh}(t)\,\mathrm{Li}_2\left(t^2\right)}{2} + \int_0^t \frac{\mathrm{arctanh}(x)\log\left(1-x^2\right)}{x} dx.$$

(2.9)

Substituting (2.9) into (2.8), we get

$$I_1 = \frac{\mathrm{Li}_2\left(t^2\right)}{2(1-t)} + \int_0^t \frac{\log\left(1-x^2\right)}{x} dx + \int_0^t \frac{\log(1-x)}{1-x} dx$$

$$+ \frac{\mathrm{arctanh}(t)\,\mathrm{Li}_2\left(t^2\right)}{2} + \int_0^t \frac{\mathrm{arctanh}(x)\log\left(1-x^2\right)}{x} dx$$

$$= \frac{\mathrm{Li}_2\left(t^2\right)}{2(1-t)} - \frac{\log^2(1-t)}{2} + \int_0^t \frac{\log\left(1-x^2\right)}{x} dx + \frac{\mathrm{arctanh}(t)\,\mathrm{Li}_2\left(t^2\right)}{2}$$

$$+ \int_0^t \frac{\mathrm{arctanh}(x)\log\left(1-x^2\right)}{x} dx.$$

(2.10)

Plug (2.10) into (2.7):

$$S_1 = \lim_{t \to 1^-} \left[\frac{\operatorname{arctanh}(t)}{2} \left(\operatorname{Li}_2\left(t^2\right) - \zeta(2) \right) - \log(1-t) + \frac{\operatorname{Li}_2\left(t^2\right) - t\zeta(2)}{2(1-t)} \right.$$

$$\left. + \int_0^t \frac{\log\left(1-x^2\right)}{x} \, dx + \int_0^t \frac{\operatorname{arctanh}(x)\log\left(1-x^2\right)}{x} \, dx \right].$$

By Lemmas 1 and 2, we have

$$S_1 = -\frac{\pi^2}{12} - \frac{7}{8}\zeta(3) + \lim_{t \to 1^-} \frac{\operatorname{arctanh}(t)}{2} \left(\operatorname{Li}_2\left(t^2\right) - \zeta(2) \right)$$

$$+ \lim_{t \to 1^-} \frac{\operatorname{Li}_2\left(t^2\right) - t\zeta(2) - 2(1-t)\log(1-t)}{2(1-t)}. \tag{2.11}$$

By applying L'Hôpital's rule and making use of the fact that $\zeta(2) = \frac{\pi^2}{6}$ (see [1, p. 807]), we obtain

$$\lim_{t \to 1^-} \frac{\operatorname{Li}_2\left(t^2\right) - t\zeta(2) - 2(1-t)\log(1-t)}{2(1-t)} = \frac{\pi^2}{12} - 1 + \log(2). \tag{2.12}$$

It is known that the dilogarithm satisfies the functional equation (see [20, p. 5]):

$$\frac{\pi^2}{6} - \operatorname{Li}_2(x) = \operatorname{Li}_2(1-x) + \log(x)\log(1-x).$$

Using this identity and the special value $\operatorname{Li}_2(1) = \zeta(2) = \frac{\pi^2}{6}$, we get that

$$\lim_{t \to 1^-} \frac{\operatorname{arctanh}(t)}{2} \left(\operatorname{Li}_2\left(t^2\right) - \zeta(2) \right) = 0. \tag{2.13}$$

Plugging (2.12) and (2.13) into (2.11) and simplifying gives us the desired result. $\quad\square$

2.2. The calculation of S_2

In order to evaluate in closed form S_2, we establish some lemmas.

LEMMA 3. *The following equality holds:*

$$\int_0^1 \frac{\operatorname{Li}_4(-x)}{1+x} \, dx = \frac{17}{16}\zeta(5) - \frac{3}{8}\zeta(2)\zeta(3) - \frac{7}{8}\log(2)\zeta(4).$$

Proof. Begin by integrating by parts:

$$\int_0^1 \frac{\operatorname{Li}_4(-x)}{1+x} \, dx = \log(2)\operatorname{Li}_4(-1) - \int_0^1 \frac{\log(1+x)\operatorname{Li}_3(-x)}{x} \, dx$$

$$= -\frac{7}{8}\log(2)\zeta(4) - \int_0^1 \frac{\log(1+x)\operatorname{Li}_3(-x)}{x} \, dx.$$

We differentiated the polylogarithm using the identity $\mathrm{Li}_n(z) = \int_0^z \frac{\mathrm{Li}_{n-1}(t)}{t}\,dt$ (see [20, p. 189]). We also used the identities $\mathrm{Li}_n(-1) = -\eta(n)$, which can be verified via the power series in [20, p. 189], and $\eta(s) = (1 - 2^{1-s})\zeta(s)$ (see [1, 23.2.19, p. 807]). With the Cauchy product of $-\log(1-x)\mathrm{Li}_3(x)$ (see [25, 2.4.4, p. 101][2])

$$-\log(1-x)\mathrm{Li}_3(x) = 2\sum_{n=1}^\infty \frac{H_n}{n^3}x^n + \sum_{n=1}^\infty \frac{H_n^{(2)}}{n^2}x^n + \sum_{n=1}^\infty \frac{H_n^{(3)}}{n}x^n - 4\mathrm{Li}_4(x),$$

we may switch the order of summation and integration [3]. Integrating gives us

$$-\frac{7}{8}\log(2)\zeta(4) + 2\sum_{n=1}^\infty (-1)^n\frac{H_n}{n^4} + \sum_{n=1}^\infty (-1)^n\frac{H_n^{(2)}}{n^3} + \sum_{n=1}^\infty (-1)^n\frac{H_n^{(3)}}{n^2} - 4\mathrm{Li}_5(-1).$$

Integration of the polylogarithm was carried out using [20, 7.2, p. 189]. We can then evaluate the remaining series using formula (1.1) with the appropriate p and q values. We once again make use of the polylogarithm's relationship to the Dirichlet eta function in terms of the Riemann zeta function using [1, 23.2.19, p. 807]. The values of these series are also given in [30, pp. 310, 311]. Simplifying leaves us with the desired result. $\square$

We made use of the identity $\mathrm{Li}_s(-1) = -\eta(s)$.

The following lemmas will require the use of the following result due to Vălean (see [31, Lemma 6, p. 4]):

$$\int_0^1 \frac{\log^2(1+x)\mathrm{Li}_2(-x)}{x}\,dx = \frac{2}{15}\log^5(2) - \frac{2}{3}\log^3(2)\zeta(2) + \frac{7}{4}\log^2(2)\zeta(3)$$

$$- \frac{1}{8}\zeta(2)\zeta(3) - \frac{125}{32}\zeta(5) + 4\log(2)\mathrm{Li}_4\left(\frac{1}{2}\right) \quad (2.14)$$

$$+ 4\mathrm{Li}_5\left(\frac{1}{2}\right).$$

LEMMA 4. *The following equality holds:*

$$\int_0^1 \frac{\log(1+x)\mathrm{Li}_3(-x)}{1+x}\,dx = \frac{1}{16}\zeta(2)\zeta(3) + \frac{125}{64}\zeta(5) - \frac{\log^5(2)}{15} + \frac{1}{3}\log^3(2)\zeta(2)$$

$$- \frac{5}{4}\log^2(2)\zeta(3) - 2\log(2)\mathrm{Li}_4\left(\frac{1}{2}\right) - 2\mathrm{Li}_5\left(\frac{1}{2}\right).$$

Proof. We begin by integrating by parts:

$$\int_0^1 \frac{\log(1+x)\mathrm{Li}_3(-x)}{1+x}\,dx = \frac{\log^2(2)\mathrm{Li}_3(-1)}{2} - \frac{1}{2}\int_0^1 \frac{\log^2(1+x)\mathrm{Li}_2(-x)}{x}\,dx$$

$$= -\frac{3}{8}\log^2(2)\zeta(3) - \frac{1}{2}\int_0^1 \frac{\log^2(1+x)\mathrm{Li}_2(-x)}{x}\,dx.$$

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[2]The author of second edition of this book has made some typos in the proof of the Cauchy product.

[3]Abel's Theorem (see [8, Theorem 45, p. 163]) tells us that the power series converges uniformly on $x \in [0,1]$, thus justifying the interchange of the sum and integral.

We immediately obtain the desired result with (2.14). □

LEMMA 5. *The following equality holds:*

$$\int_0^1 \frac{\text{Li}_2^2(-x)}{1+x}\,dx = \frac{4}{15}\log^5(2) - \frac{4}{3}\log^3(2)\zeta(2) + \frac{7}{2}\log^2(2)\zeta(3) + \frac{5}{8}\log(2)\zeta(4)$$
$$- \frac{125}{16}\zeta(5) - \frac{1}{4}\zeta(2)\zeta(3) + 8\log(2)\text{Li}_4\left(\frac{1}{2}\right) + 8\text{Li}_5\left(\frac{1}{2}\right).$$

Proof. We begin by integrating by parts:

$$\int_0^1 \frac{\text{Li}_2^2(-x)}{1+x}\,dx = \log(2)\text{Li}_2^2(-1) + 2\int_0^1 \frac{\log^2(1+x)\text{Li}_2(-x)}{x}\,dx$$
$$= \log(2)\cdot\frac{\zeta^2(2)}{4} + 2\int_0^1 \frac{\log^2(1+x)\text{Li}_2(-x)}{x}\,dx.$$

Making use of the fact that $\zeta^2(2) = \frac{5}{2}\zeta(4)$, since $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(4) = \frac{\pi^4}{90}$ (see [1, p. 807]), and plugging in (2.14), we get the desired result. □

We can now begin to evaluate S_2 in closed form.

THEOREM 2.2. *The following equality holds:*

$$\sum_{n=1}^{\infty} \frac{\overline{H}_n}{n}\left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3}\right) = \frac{193}{64}\zeta(5) - \frac{5}{16}\zeta(2)\zeta(3) - \frac{\log^5(2)}{15}$$
$$+ \frac{1}{3}\log^3(2)\zeta(2) - \frac{15}{16}\log(2)\zeta(4)$$
$$- 2\log(2)\text{Li}_4\left(\frac{1}{2}\right) - 2\text{Li}_5\left(\frac{1}{2}\right).$$

Proof. Plugging in the integral representation of the skew-harmonic number (2.2), we have

$$S_2 = \sum_{n=1}^{\infty} \frac{1}{n}\int_0^1 \frac{1-(-x)^n}{1+x}\left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3}\right)dx.$$

Since $\frac{1-(-x)^n}{1+x} \geqslant 0$ and $\zeta(3) - H_n^{(3)} > 0$ for integers $n \geqslant 1$ and real numbers $0 \leqslant x \leqslant 1$, we may use Tonelli's theorem for non-negative functions (see [32, Theorem 23.17, p. 558]) to justify interchanging the order of summation and integration. We make use of the generating function

$$\sum_{n=1}^{\infty}\left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3}\right)\frac{x^n}{n}$$
$$= \begin{cases} \log(1-x)\left[\text{Li}_3(x) - \zeta(3)\right] - \text{Li}_4(x) + \frac{\text{Li}_2^2(x)}{2} & \text{if } x \in [-1,1) \\ \frac{\zeta(4)}{4} & \text{if } x = 1 \end{cases}$$

(see [28, Problem 7.87 (b), p. 212]), leaving us with

$$S_2 = \frac{\zeta(4)}{4}\log(2) - \int_0^1 \frac{\log(1+x)\operatorname{Li}_3(-x)}{1+x}\,dx + \zeta(3)\int_0^1 \frac{\log(1+x)}{1+x}\,dx$$
$$+ \int_0^1 \frac{\operatorname{Li}_4(-x)}{1+x}\,dx - \frac{1}{2}\int_0^1 \frac{\operatorname{Li}_2^2(-x)}{1+x}\,dx.$$

Using $\int_0^1 \frac{\log(1+x)}{1+x}\,dx = \frac{1}{2}\log^2(2)$, plugging in Lemmas 3, 4 and 5, and simplifying gives us the desired result. $\square$

2.3. Harmonic series with tail e^y

We will derive a harmonic series with tail e^y, similar to the following series (see [9, p. 10]):

$$\sum_{n=1}^{\infty} H_n\left(e^y - 1 - \frac{y}{1!} - \frac{y^2}{2!} - \ldots - \frac{y^n}{n!}\right) = e^y\left(\operatorname{Ein}(y) - y + 1\right) - 1, \quad y \in \mathbb{R}$$

where Ein denotes the complementary exponential integral (see [26, 6.2.3], in [1, p. 228]). We will derive a similar result involving the skew-harmonic numbers.

THEOREM 2.3. *For all y, we have*

$$\sum_{n=1}^{\infty} \overline{H}_n\left(e^y - 1 - \frac{y}{1!} - \frac{y^2}{2!} - \ldots - \frac{y^n}{n!}\right) = ye^y\left[\operatorname{Ein}(2y) - \operatorname{Ein}(y)\right] - \cosh(y) + 1.$$

Proof. We can easily check that the series evaluates to 0 for $y = 0$. For now let us assume $y \neq 0$. We begin by inserting the integral representation of the skew-harmonic numbers 2.2 and interchange the sum and integral:

$$\sum_{n=1}^{\infty} \overline{H}_n\left(e^y - 1 - \frac{y}{1!} - \frac{y^2}{2!} - \ldots - \frac{y^n}{n!}\right)$$
$$= \sum_{n=1}^{\infty} \int_0^1 \frac{1-(-x)^n}{1+x}\left(e^y - 1 - \frac{y}{1!} - \frac{y^2}{2!} - \ldots - \frac{y^n}{n!}\right)dx$$

It is known via the Lagrange error bound that

$$\left|e^y - 1 - \frac{y}{1!} - \frac{y^2}{2!} - \ldots - \frac{y^n}{n!}\right| \leqslant \frac{e^t|y|^{n+1}}{(n+1)!}$$

for some arbitrary positive t such that $|y| < t$. This inequality was also stated in [9]. Since

$$\sum_{n=0}^{\infty}\left(e^y - 1 - \frac{y}{1!} - \frac{y^2}{2!} - \ldots - \frac{y^n}{n!}\right)x^n = \begin{cases} \frac{e^y - e^{xy}}{1-x} & \text{if } x \neq 1 \\ ye^y & \text{if } x = 1 \end{cases}$$

(see [16, p. 154]) and

$$\left| \sum_{n=1}^{N} (1 - (-x)^n) \left(e^y - 1 - \frac{y}{1!} - \frac{y^2}{2!} - \cdots - \frac{y^n}{n!} \right) \right|$$

$$\leqslant \sum_{n=1}^{N} |1 - (-x)^n| \left| e^y - 1 - \frac{y}{1!} - \frac{y^2}{2!} - \cdots - \frac{y^n}{n!} \right|$$

$$\leqslant 2e^t \sum_{n=1}^{\infty} \frac{|y|^{n+1}}{(n+1)!}$$

$$\leqslant 2e^t (e^{|y|} - 1 - |y|)$$

for integers $n \geqslant 1$, real numbers x such that $0 \leqslant x \leqslant 1$, all real numbers y and some arbitrary positive t such that $|y| < t$, we may justify interchanging the sum and integral via Lebesgue's Dominated Convergence Theorem (see [32, pp. 188–189]). Thus, the series can be rewritten as

$$\int_0^1 \left(ye^y - \frac{e^y - e^{-xy}}{1+x} \right) \frac{dx}{1+x} = \int_0^1 \frac{-e^y + ye^y(1+x) + e^{-xy}}{(1+x)^2} dx$$

$$= -\frac{e^y}{2} + \int_0^1 \frac{ye^y(1+x) + e^{-xy}}{(1+x)^2} dx.$$

Integrating by parts gives us

$$-\frac{e^y}{2} + 1 - \frac{e^{-y}}{2} + y \int_0^1 \frac{e^y - e^{-xy}}{1+x} dx = 1 - \cosh(y) + y \int_0^1 \frac{e^y - e^{-xy}}{1+x} dx.$$

Since $y \neq 0$, we may integrate by substituting $x \mapsto x/y - 1$, which yields

$$1 - \cosh(y) + ye^y \int_y^{2y} \frac{1 - e^{-x}}{x} dx = ye^y \left[\mathrm{Ein}(2y) - \mathrm{Ein}(y) \right] - \cosh(y) + 1.$$

We used the fact that $\mathrm{Ein}(z) = \int_0^z \frac{1-e^{-t}}{t} dt$ (see [26, 6.2.3]). If we plug in $y = 0$ in the derived expression we also obtain a value of 0. Thus, the expression is valid for all real y. $\square$

COROLLARY 1. *The following equality holds:*

$$\sum_{n=1}^{\infty} \frac{\overline{H}_n\{n!e\}}{n!} = e \left[\mathrm{Ein}(2) - \gamma \right] - \cosh(1) - \delta + 1$$

where $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x and δ is the Euler-Gompertz constant (see [14, p. 423–426]).

Proof. Using the identity (see [17])

$$\{n!e\} = n! \left(e - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{n!} \right),$$

we have

$$\sum_{n=1}^{\infty} \frac{\overline{H}_n\{n!e\}}{n!} = \sum_{n=1}^{\infty} \overline{H}_n \left(e - 1 - \frac{1}{1!} - \frac{1}{2!} - \ldots - \frac{1}{n!} \right).$$

Using Theorem 2.3 with $y = 1$, we simplify to get

$$e\left[\mathrm{Ein}(2) - \mathrm{Ein}(1)\right] - \cosh(1) + 1 = e\left[\mathrm{Ein}(2) + \mathrm{Ei}(-1) - \gamma\right] - \cosh(1) + 1$$
$$= e\left[\mathrm{Ein}(2) - \gamma\right] + e\mathrm{Ei}(-1) - \cosh(1) + 1$$

where $\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t}\,dt$ denotes the exponential integral (see [26, 6.2.5]). We used the identity $\mathrm{Ei}(\pm x) = -\mathrm{Ein}(\mp x) + \log(x) + \gamma$ (see [26, 6.2.7]). Applying $-e\mathrm{Ei}(-1) = \delta$ (see [14, p. 303, 424]) leads us to the desired result. $\square$

3. Hardy series

THEOREM 3.1. *For $k \in \{1, 2, 3, \ldots\}$ and $x > 0$, we have*

$$\sum_{n=1}^{\infty} (-1)^n \left(H_n - \log \sqrt[k]{(n+x-1)^{\overline{k}}} - \gamma \right) = \frac{\gamma}{2} + \frac{1}{k} \log \left[\frac{\Gamma\left(\frac{x+k}{2}\right)}{\Gamma\left(\frac{x}{2}\right)} \right]$$

where $z^{\overline{k}} = z(z+1)\ldots(z+k-1)$ denotes the Pochhammer symbol (see [26, 5.2(iii)]).

Proof. We begin by applying Abel's summation formula (1.2) to our summation with $a_n = (-1)^n$ and $b_n = H_n - \log \sqrt[k]{(n+x-1)^{\overline{k}}} - \gamma$ to get

$$\lim_{n \to \infty} \frac{-1+(-1)^n}{2} \left(H_{n+1} - \log \sqrt[k]{(n+x)^{\overline{k}}} - \gamma \right)$$
$$+ \sum_{n=1}^{\infty} \frac{1-(-1)^n}{2} \left(\frac{1}{n+1} + \log \sqrt[k]{\frac{n+x-1}{n+x+k-1}} \right).$$

It is easy to check that the limit evaluates to 0. This leaves us with the infinite series:

$$\sum_{n=1,3,5,\ldots} \frac{1-(-1)^n}{2} \left(\frac{1}{n+1} + \log \sqrt[k]{\frac{n+x-1}{n+x+k-1}} \right)$$
$$= \sum_{n=1}^{\infty} \left(\frac{1}{2n} + \log \sqrt[k]{\frac{2n+x-2}{2n+x+k-2}} \right)$$
$$= \log \sqrt[k]{\prod_{n=1}^{\infty} \frac{2n+x-2}{2n+x+k-2} e^{k/(2n)}}.$$

It is known that

$$\Gamma(z+1) = e^{-\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n} \right)^{-1} e^{z/n}, \quad z \notin \{0, -1, -2, \ldots\},$$

which follows from Weierstrass' definition of the gamma function (see [13, p. 176]) and the functional equation $z\Gamma(z) = \Gamma(z+1)$ (see [1, p. 256]). Using this equality, we finally get

$$\log \sqrt[k]{\prod_{n=1}^{\infty} \frac{2n+x-2}{2n+x+k-2} e^{\frac{k}{2n}}} = \log \sqrt[k]{\frac{e^{-\gamma\left(\frac{x+k-2}{2}\right)} \prod_{n=1}^{\infty} \left(1+\frac{x+k-2}{2n}\right)^{-1} e^{\frac{x+k-2}{2n}}}{e^{-\gamma\left(\frac{x-2}{2}\right)} \prod_{n=1}^{\infty} \left(1+\frac{x-2}{2n}\right)^{-1} e^{\frac{x-2}{2n}}} \cdot e^{\frac{k\gamma}{2}}}$$

$$= \frac{\gamma}{2} + \frac{1}{k} \log\left[\frac{\Gamma\left(\frac{x+k}{2}\right)}{\Gamma\left(\frac{x}{2}\right)}\right]. \quad \square$$

COROLLARY 2. *The following equalities hold:*

(i) If $k \in \{1,2\ldots\}$, then

$$\sum_{n=1}^{\infty} (-1)^n \left(H_n - \log\sqrt[k]{n^k} - \gamma\right) = \frac{\gamma}{2} - \frac{\log(\pi)}{2k} + \frac{1}{k}\log\Gamma\left(\frac{k+1}{2}\right).$$

(ii) If $x > 0$, then

$$\sum_{n=1}^{\infty} (-1)^n \left(H_n - \log(n+x-1) - \gamma\right) = \frac{\gamma}{2} + \log\left[\frac{\Gamma\left(\frac{x+1}{2}\right)}{\Gamma\left(\frac{x}{2}\right)}\right].$$

Proof. We immediately get (i) by letting $x = 1$ in Theorem 3.1 and using the special value $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ (see [1, p. 255]) and simplifying. We immediately get (ii) by letting $k = 1$. $\square$

REMARK 3.1. Theorem 3.1 can be used to solve problems 2.52, 2.53 (a), 2.53 (b), and 2.54 from [28, pp. 42, 43] with the appropriate values of x and k.

THEOREM 3.2. *For $k \in \{1,2,\ldots\}$ and $x > 0$, we have*

$$\sum_{n=1}^{\infty} \left(H_n - \log\sqrt[k]{(n+x-1)^k} - \gamma + \frac{x-2}{n} + \frac{k}{2n}\right)$$

$$= \gamma\left(x + \frac{k}{2} - 1\right) + \frac{1 - \log(2\pi)}{2} + \frac{1}{k}\log\left[\frac{G(x+k)}{G(x)}\right]$$

where G denotes the Barnes G-function (see [5], [21]).

Proof. We know that the Pochhammer symbol $z^{\overline{k}} = z(z+1)\ldots(z+k-1)$ (see [1, p. 256], [26, 5.2(iii)]). Using this definition, we can turn the summand into a finite sum

and change the order of summation:

$$\sum_{n=1}^{\infty} \left(H_n - \log \sqrt[k]{(n+x-1)^{\overline{k}}} - \gamma + \frac{x-2}{n} + \frac{k}{2n} \right)$$

$$= \frac{1}{k} \sum_{n=1}^{\infty} \left(kH_n - \log\left[(n+x-1)^{\overline{k}} \right] - k\gamma + \frac{k(x-2)}{n} + \frac{k^2}{2n} \right)$$

$$= \frac{1}{k} \sum_{j=1}^{k} \sum_{n=1}^{\infty} \left(H_n - \log(n+x+j-2) - \gamma + \frac{x+j}{n} - \frac{5}{2n} \right). \tag{3.1}$$

We now have an infinite series, which we will further simplify. Using the identity

$$\sum_{n=1}^{\infty} \left(H_n - \log(n) - \gamma - \frac{1}{2n} \right) = \frac{\gamma + 1 - \log(2\pi)}{2},$$

(see [16, p. 145]) we can separate the infinite series in (3.1) into two convergent series:

$$\sum_{n=1}^{\infty} \left(H_n - \log(n) - \gamma - \frac{1}{2n} \right) - \sum_{n=1}^{\infty} \left(\log\left(1 + \frac{x+j-2}{n} \right) - \frac{x+j-2}{n} \right)$$

$$= \frac{\gamma + 1 - \log(2\pi)}{2} - \sum_{n=1}^{\infty} \left(\log\left(1 + \frac{x+j-2}{n} \right) - \frac{x+j-2}{n} \right).$$

It can be shown by applying the logarithm to Weierstrass' definition of the gamma function (see [13, p. 176]) and using the functional equation $\Gamma(z+1) = z\Gamma(z)$ (see [1, p. 256]) to get that

$$\log\Gamma(z+1) + \gamma z = -\sum_{n=1}^{\infty} \left[\log\left(1 + \frac{z}{n} \right) - \frac{z}{n} \right].$$

This series can also be found in [6, p. 204]. Using this identity, we get

$$\gamma\left(x+j - \frac{3}{2} \right) + \frac{1 - \log(2\pi)}{2} + \log\Gamma(x+j-1).$$

We now substitute this expression into (3.1) to obtain the following:

$$\frac{1}{k} \sum_{j=1}^{k} \left[\gamma\left(x+j - \frac{3}{2} \right) + \frac{1 - \log(2\pi)}{2} + \log\Gamma(x+j-1) \right]$$

$$= \gamma\left(x + \frac{k}{2} - 1 \right) + \frac{1 - \log(2\pi)}{2} + \frac{1}{k} \log\left[\prod_{j=1}^{k} \Gamma(x+j-1) \right].$$

It can be shown by induction that

$$\prod_{j=1}^{k} \Gamma(x+j-1) = \frac{\mathbf{G}(x+k)}{\mathbf{G}(x)}$$

using the functional equation $\mathbf{G}(z+1) = \mathbf{G}(z)\Gamma(z)$ (see [26, 5.17.1]). □

Corollary 3. *The following equalities hold:*

(i) If $k \in \{1, 2, \ldots\}$, then

$$\sum_{n=1}^{\infty} \left(H_n - \log \sqrt[k]{n^k} - \gamma + \frac{k-2}{2n} \right) = \frac{\gamma k + 1 - \log(2\pi)}{2} + \frac{\log G(k+1)}{k}.$$

(ii) If $x > 0$, then

$$\sum_{n=1}^{\infty} \left(H_n - \log(n+x-1) - \gamma + \frac{x}{n} - \frac{3}{2n} \right)$$
$$= \gamma x + \log \Gamma(x) + \frac{1 - \gamma - \log(2\pi)}{2}.$$

Proof. (i) immediately follows from setting $x = 1$ in Theorem 3.2. (ii) follows from setting $k = 1$ in Theorem 3.2. □

Theorem 3.3. *For $x > 0$, we have*

$$\sum_{n=1}^{\infty} (-1)^n n \left(H_n - \log(n+x-1) - \gamma + \frac{x}{n} - \frac{3}{2n} \right)$$
$$= \frac{\gamma}{4} - \frac{x - \log(2) - 1}{2} - 2\log\left[\frac{G\left(\frac{x+1}{2}\right)}{G\left(\frac{x}{2}\right)} \right] + \log \Gamma\left(\frac{x}{2}\right).$$

Proof. We begin by separating the infinite series:

$$\sum_{n=1}^{\infty} (-1)^n n \left(H_n - \log(n+x-1) - \gamma + \frac{x}{n} - \frac{3}{2n} \right)$$
$$= \sum_{n=1}^{\infty} (-1)^n n \left(H_n - \log(n) - \gamma - \frac{1}{2n} \right) \qquad (3.2)$$
$$- \sum_{n=1}^{\infty} (-1)^n n \left(\log\left(\frac{n+x-1}{n} \right) - \frac{x-1}{n} \right).$$

It is known from [16, p. 145] that

$$\sum_{n=1}^{\infty} (-1)^n n \left(H_n - \log(n) - \gamma - \frac{1}{2n} \right) = \frac{\gamma+1}{4} + \frac{7}{12}\log(2) - 3\log(A) \qquad (3.3)$$

where A is the Glaisher-Kinkelin constant (see [14, pp. 135–138]). We are now left with another infinite series which we will evaluate by expressing the summand as an integral and interchanging the order of summation and integration [4]:

$$\sum_{n=1}^{\infty} (-1)^n n \left(\log\left(1 + \frac{x-1}{n} \right) - \frac{x-1}{n} \right) = \int_1^x \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(t-1)}{n+t-1} \, dt.$$

---

[4] The interchange of the sum and integral is justified via Monotone Convergence Theorem (see [32, Theorem 8.5, p. 162]) with $0 \leqslant \sum_{n=1}^{2N} \frac{(-1)^{n-1}}{n+t-1} \leqslant \sum_{n=1}^{2(N+1)} \frac{(-1)^{n-1}}{n+t-1}$ for real numbers $t > 0$ and integers $N \geqslant 1$.

From [15, p. 32] we know that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+z} = \frac{1}{2} \psi \left(\frac{z+1}{2} \right) - \frac{1}{2} \psi \left(\frac{z}{2} \right)$$

where ψ denotes the digamma function (see [1, pp. 258, 259]). Plugging in this identity leaves us with an integral, which we will evaluate via integration by parts and expressing the antiderivative in terms of the polygamma functions of negative order (see [3]):

$$\frac{1}{2} \int_1^x (t-1) \left(\psi \left(\frac{t+1}{2} \right) - \psi \left(\frac{t}{2} \right) \right) dt$$

$$= (x-1) \log \left[\frac{\Gamma \left(\frac{x+1}{2} \right)}{\Gamma \left(\frac{x}{2} \right)} \right] - \int_1^x \left(\log \Gamma \left(\frac{t+1}{2} \right) - \log \Gamma \left(\frac{t}{2} \right) \right) dt.$$

We integrated the digamma function using the identity $\psi(z) = d[\log(\Gamma(z))]/dz$ (see [1, 6.3.1, p. 258]). Since the antiderivative of $\log \Gamma(z)$ is $\psi^{(-2)}(z)$ (see [3, p. 196]), we are left with

$$(x-1) \log \left[\frac{\Gamma \left(\frac{x+1}{2} \right)}{\Gamma \left(\frac{x}{2} \right)} \right] - 2 \left[\psi^{(-2)} \left(\frac{x+1}{2} \right) - \psi^{(-2)} \left(\frac{x}{2} \right) - \psi^{(-2)}(1) + \psi^{(-2)} \left(\frac{1}{2} \right) \right].$$

Now we can further simplify by determining the special values of $\psi^{(-2)} \left(\frac{1}{2} \right)$ and $\psi^{(-2)}(1)$. It is known from [3, p. 196] that

$$\psi^{(-2)}(z) = \frac{z(1-z)}{2} + \frac{z}{2} \log(2\pi) - \zeta'(-1) + \zeta'(-1, z) \tag{3.4}$$

where $\zeta'(s, a)$ denotes the partial derivative of the Hurwitz zeta function $\zeta(s, a) = \sum_{n=0}^{\infty} (n+a)^{-s}$ with respect to s. It is also known from [3, p. 194] that

$$\zeta'(-1) = \frac{1}{12} - \log(A)$$

and from [22, p. 205] that

$$\zeta' \left(-2k+1, \frac{1}{2} \right) = -\frac{B_{2k} \log(2)}{4^k k} - \frac{(2^{2k-1} - 1) \zeta'(-2k+1)}{2^{2k-1}}$$

where B_n is the nth Bernoulli number (see [1, pp. 804–806], [18]) and k is a positive integer. From these equalities, we deduce the following:

$$\psi^{(-2)} \left(\frac{1}{2} \right) = \frac{3 \log(A)}{2} + \frac{5 \log(2)}{24} + \frac{\log(\pi)}{4}, \quad \psi^{(-2)}(1) = \frac{\log(2\pi)}{2}.$$

With these special values, we now have

$$\frac{\log(\pi)}{2} - 3 \log(A) + \frac{7}{12} \log(2) + (x-1) \log \left[\frac{\Gamma \left(\frac{x+1}{2} \right)}{\Gamma \left(\frac{x}{2} \right)} \right]$$

$$- 2 \left[\psi^{(-2)} \left(\frac{x+1}{2} \right) - \psi^{(-2)} \left(\frac{x}{2} \right) \right]. \tag{3.5}$$

Using (3.4) and the identity from [3, p. 197]

$$\zeta'(-1) - \zeta'(-1,z) = \log \mathbf{G}(z+1) - z \log \Gamma(z)$$

and recombining (3.3) and (3.5) with (3.2), we get the desired result. □

REMARK 3.2. We could also have expressed the closed form of the series from Theorem 3.3 with (3.5). We can then use (3.4) and the formulae from [22] and [19] to determine special values. For example, if it were necessary to evaluate $\psi^{(-2)}(p)$ at $p = \frac{1}{4}, \frac{3}{4}$, one may use the following formulae from those papers:

$$\left.\begin{array}{c}\zeta'\left(-2k+1,\frac{1}{4}\right)\\\zeta'\left(-2k+1,\frac{3}{4}\right)\end{array}\right\} = \mp \frac{\left(4^k-1\right)B_{2k}\pi}{4^{k+1}k} + \frac{\left(4^{k-1}-1\right)B_{2k}\log(2)}{2^{4k-1}k}$$
$$\mp \frac{(-1)^k\psi^{(2k-1)}\left(\frac{1}{4}\right)}{4(8\pi)^{2k-1}} - \frac{\left(2^{2k-1}-1\right)\zeta'(-2k+1)}{2^{4k-1}}$$

and

$$\left.\begin{array}{c}\psi^{(2k-1)}\left(\frac{1}{4}\right)\\\psi^{(2k-1)}\left(\frac{3}{4}\right)\end{array}\right\} = \frac{4^{2k-1}}{2k}\left[\pi^{2n}\left(2^{2k}-1\right)|B_{2k}| \pm 2(2k)!\beta(2k)\right]$$

where $\psi^{(n)}$ denotes the polygamma function of nth order (see [1, p. 260], [26, 5.15]) and $\beta(s) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}$ denotes the Dirichlet beta function (see [1, pp. 807, 808]).

By letting x in Theorem 3.3 take on certain values, we may derive some exotic series.

COROLLARY 4. *The following equalities hold:*

(i) *If $x = 1/2$, we have*

$$\sum_{n=1}^{\infty}(-1)^n n\left(H_n - \log\left(n-\frac{1}{2}\right) - \gamma - \frac{1}{n}\right) = \frac{\gamma+1}{4} - \frac{G}{\pi} - \frac{1}{4}\log\left(\frac{\varpi^2}{\pi}\right)$$

where $G = \beta(2) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$ is Catalan's constant (see [14, pp. 53–57]) and ϖ is the lemniscate constant (see [14, pp. 420–422]).

(ii) *If $x = 2/3$, then*

$$\sum_{n=1}^{\infty}(-1)^n n\left(H_n - \log\left(n-\frac{1}{3}\right) - \gamma - \frac{5}{6n}\right)$$
$$= \frac{\gamma+1}{4} - \frac{5\kappa}{6\pi} - \log(A) + \frac{19}{36}\log(2) + \frac{\log(3)}{24} + \frac{1}{3}\log\left[\frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{3}\right)}\right].$$

where κ denotes Gieseking's constant (see [24]).

Proof. If we let $x = 1/2$ in Theorem 3.3 and use the special value

$$\log \mathbf{G}\left(\frac{3}{4}\right) = \log \mathbf{G}\left(\frac{1}{4}\right) + \frac{G}{2\pi} - \frac{\log(2)}{8} - \frac{\log(\pi)}{4} + \log\Gamma\left(\frac{1}{4}\right)$$

and the identity

$$\varpi = \frac{\Gamma^2\left(\frac{1}{4}\right)}{2\sqrt{2\pi}}$$

(see [11, p. 94], [14, p. 420]), we get the desired result in (i).

If we let $x = 2/3$, we get

$$\frac{\gamma}{4} + \frac{\log(2)}{2} + \frac{1}{6} + 2\log \mathbf{G}\left(\frac{1}{3}\right) - 2\log \mathbf{G}\left(\frac{5}{6}\right) + \log\Gamma\left(\frac{1}{3}\right). \tag{3.6}$$

We will make use of the following special values (see [2]):

$$\log \mathbf{G}\left(\frac{1}{3}\right) = \frac{\log(3)}{72} + \frac{\pi}{18\sqrt{3}} - \frac{2}{3}\log\Gamma\left(\frac{1}{3}\right) - \frac{4}{3}\log(A) - \frac{\psi^{(1)}\left(\frac{1}{3}\right)}{12\pi\sqrt{3}} + \frac{1}{9} \tag{3.7}$$

$$\log \mathbf{G}\left(\frac{5}{6}\right) = -\frac{\log(12)}{144} + \frac{\pi}{20\sqrt{3}} - \frac{1}{6}\log\Gamma\left(\frac{5}{6}\right) - \frac{5}{6}\log(A) - \frac{\psi^{(1)}\left(\frac{5}{6}\right)}{40\pi\sqrt{3}} + \frac{5}{72}. \tag{3.8}$$

Using the polygamma multiplication formula (see [26, 5.15.7])

$$\psi^{(n)}(mz) = \frac{1}{m^{n+1}} \sum_{k=0}^{m-1} \psi^{(n)}\left(z + \frac{k}{m}\right), \quad m, n \in \{1, 2, \ldots\}$$

with $m = 2$, $n = 1$, and $z = 1/3$ alongside the polygamma reflection formula (see [26, 5.15.6])

$$\psi^{(n)}(1-z) + (-1)^{n-1}\psi^{(n)}(z) = (-1)^n \pi \frac{d^n}{dz^n}\cot(\pi z) \tag{3.9}$$

with $z = 2/3$ and $n = 1$, we get that

$$\psi^{(1)}\left(\frac{5}{6}\right) = \frac{16\pi^2}{3} - 5\psi^{(1)}\left(\frac{1}{3}\right). \tag{3.10}$$

Substituting (3.10) into (3.8) and plugging (3.7) and (3.8) into (3.6), we have

$$\frac{\gamma+1}{4} + \frac{5\pi}{18\sqrt{3}} - \log(A) + \frac{19}{36}\log(2) + \frac{\log(3)}{24} + \frac{1}{3}\log\left[\frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{3}\right)}\right] - \frac{5\psi^{(1)}\left(\frac{1}{3}\right)}{12\pi\sqrt{3}}. \tag{3.11}$$

We use the identity

$$\kappa = \frac{9 - \psi^{(1)}\left(\frac{2}{3}\right) + \psi^{(1)}\left(\frac{4}{3}\right)}{4\sqrt{3}}$$

and substitute the $\psi^{(1)}(2/3) = 4\pi^2/3 - \psi^{(1)}(1/3)$, which comes from (3.9), and $\psi^{(1)}(4/3) = \psi^{(1)}(1/3) - 9$, which comes from the recursive relationship

$$\psi^{(n)}(z+1) = \psi^{(n)}(z) + (-1)^n n! z^{-n-1}$$

with $n = 1$ and $z = 1/3$ (see [26, 5.15.5]). This gives us the special value

$$\psi^{(1)}\left(\frac{1}{3}\right) = 2\sqrt{3}\kappa + \frac{2\pi^2}{3}.$$

Substituting this value into (3.11) yields the result in (ii). □

4. Further research

The interested reader may consider a generalization to the series S_2:

$$\sum_{n=1}^{\infty} \frac{\overline{H_n}}{n}\left(\zeta(k) - 1 - \frac{1}{2^k} - \ldots \frac{1}{n^k}\right), \quad n \in \{2, 3, \ldots\}.$$

One may also consider other analogues to the Hardy series. For example, one may study the series

$$\sum_{n=1}^{\infty} (-1)^n n^k (H_n - \log(n) - \gamma - \sigma_k(n)), \quad k \in \{0, 1, \ldots\}$$

where $\sigma_k(n)$ is the correctional term that ensures the convergence of the series.

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