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SQUARE–FREE FACTORIZATION OF MIXED TRIGONOMETRIC–POLYNOMIALS

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Abstract. This paper proposes a procedure to square-free factorization of mixed trigonometric-polynomials and some examples are presented to show the effectiveness of the algorithm.

1. Introduction

In recent years, a class of transcendental functions named mixed trigonometric-polynomial (denoted as MTP simply) and defined by the formula $F(x) = \sum a_i x^{p_i} \sin^{q_i}(x) \cos^{r_i}(x)$ attract more and more scholars' attention [1, 3, 12, 10, 11, 7, 2, 13, 19, 4, 16, 8, 20, 18, 17, 9], which frequently occur in applications in physics, numerical analysis and engineering, where $a_i \in \mathbb{R}$, $p_i, q_i, r_i \in \mathbb{N} \cup \{0\}$. There have been numerous research studies on the MTP inequalities and the real root isolation for the MTPs already, while the problem of the general positivity of a real function over an interval is an undecidable problem [7]. To tackle the problem of proving MTP inequalities, a so-called 'Natural Approach' introduced in [2, 13], approximates $\sin(x)$ and $\cos(x)$ by their Taylor polynomials and reduce the problem of proving an MTP inequality to the problem of proving some polynomial (or rational function) inequalities. Chen and Liu [19] provided a complete algorithm proving MTP inequalities and discussed systematically, for the first time, the termination of the algorithm. Notably, a recent breakthrough is the development and implementation of an algorithm capable of 'isolating' all the real roots of an MTP, and as a direct application, an algorithm proving MTP inequalities over any interval (bounded or not) with end-points in $\mathbb{Q} \cup \{+\infty, -\infty\}$ can then be derived [4]. In other word, the general positivity of MTPs is a decidable problem at the intervals with rational or infinite endpoints.

As the algorithms for the MPTs will work well if their roots are all simple and the algorithms in computer algebra often depend on the methods for the square-free decomposition of functions, the square-free factorization of MTP is an especially important topic.

Factorization of trigonometric functions is a classic field and the following methods or tools are commonly used. The first is the quotient ring $\mathbb{Q}[s,c]/\langle s^2 + c^2 - 1 \rangle$, but it is not a unique factorization domain and so, the factorization is not unique in general, furthermore, it is still needed to decide whether each factor has multiple roots [14].

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The second is using Tan-half angle substitutions, $\sin(x) = \frac{2t}{1+t^2}$, $\cos(x) = \frac{1-t^2}{1+t^2}$, to transform the trigonometric polynomial to a rational expression, where $t = \tan(\frac{x}{2})$ and $x \neq (2k+1)\pi$ for $k \in \mathbb{N}$ [14, 6, 15]. The third is reducing the trigonometric functions to polynomials in the complex field $\mathbb{Q}(I)[e,e^{-1}]$ by Euler Theorem [15], where e denotes e^{Ix}. This scheme requires $g(e) \in \mathbb{Q}(I)[e,e^{-1}]$ be even or odd function of e. Besides, the above methods can deal with the trigonometric polynomials with form $f(\sin(x),\cos(x))$ only, where $f(x,y) \in \mathbb{R}[x,y]$ or $\mathbb{Q}[x,y]$.

In this paper, we will present a procedure to decompose the mixed trigonometric-polynomials with the form $f(x,\sin(x),\cos(x))$, where $f \in \mathbb{A}[x,y,z]$, i.e. the monomials contain one variable and trigonometric functions applied to the same variable, and the domain is $(0,+\infty)$ or $(-\infty,+\infty)$ without excluding any special point. By Euler Formula, a mixed trigonometric-polynomial can be transformed into a polynomial in the complex field, and then the tools of the polynomials such as Sylvester resultant, can be employed to decide whether the correlative polynomials are square-free. We also developed a mechanism to guarantee that the MTPs corresponding to the above polynomials in complex field are real-valued. The method is simple but practical.

The rest of the paper is organized as follows. Section 2 proposes the scheme of square-free factorization of MTP. Section 3 presents examples to show the effectiveness of the algorithm. We will conclude the paper in Section 4.

2. Square-free factorization of mixed trigonometric polynomial

In this section, we extend the coefficients of MTP to complex field. By Euler Formula, $\sin(x) = \frac{e^{Ix}-e^{-Ix}}{2I}$, $\cos(x) = \frac{e^{Ix}+e^{-Ix}}{2}$, an MTP can be transformed to an exponential polynomial in complex field $f(x,e^{Ix},e^{-Ix})$, where $I^2 = -1$.

Let $e^{Ix} = y$, then an MTP can be reduced to a Laurent polynomial in the form of $f[x,y,y^{-1}]$. Denote $\mathbb{LR} := \mathbb{C}[x,y,y^{-1}]$, then for any $P \in \mathbb{LR}$, there exists one and only one polynomial $Q \in \mathbb{C}[x,y]$ such that $P = Q/y^p$, Q and y are coprime, $p \in \mathbb{Z}$. We denote the numerator of a rational polynomial or expression RP by $numer(RP)$, then $numer(P) = Q$. If $factor(numer(P))$ is a factorization of $numer(P)$, we call $factor(numer(P))/y^p$ or $y^{-p} \times factor(numer(P))$ a factorization of P.

By Lindemann Theorem, we have that

LEMMA 2.1. *For $\forall x \in \mathbb{C}$, there exists at least one transcendental number between x, e^{Ix}.*

LEMMA 2.2. *If $f_1(x,y), f_2(x,y) \in \mathbb{A}[x,y]$ are co-prime, then $F_1(x) = f_1(x,e^{Ix})$ and $F_2(x) = f_2(x,e^{Ix})$ have no common roots other than 0.*

Proof. Suppose $F_1(x)$ and $F_2(x)$ have common root x_0 and $x_0 \neq 0$.

Let $g(x) = res(f_1(x,y), f_2(x,y), y)$, then $g \in \mathbb{A}[x]$ and $g(x_0) = 0$. As $f_1(x,y)$ and $f_2(x,y)$ are co-prime, then $g(x) \neq 0$, so x_0 is algebraic. Then $h(y) = f_1(x_0,y) \in \mathbb{A}[y]$, and $h(e^{Ix_0}) = f_1(x_0,e^{Ix_0}) = 0$, which implies e^{Ix_0} is algebraic and contradicts Lemma 2.1. $\square$

THEOREM 2.1. *If* $f(x,y) \in \mathbb{A}[x,y]$ *is irreducible, then* $F(x) = f(x,e^{Ix})$ *has no multiple roots other than* 0.

Proof. Let $f'(x,y) \in \mathbb{A}[x,y]$ such that $F'(x) = f'(x,e^{Ix})$, then $f'(x,y) = f'_x + I \times y \times f'_y$, so $degree(f',x) \leqslant degree(f,x)$, $degree(f',y) \leqslant degree(f,y)$. As $f(x,y)$ is irreducible, so $f(x,y)$ and $f'(x,y)$ are co-prime. Then $F(x)$ and $F'(x)$ have no common roots other than 0. We conclude that Theorem 2.1 holds. $\square$

COROLLARY 2.1. *If* $f(x,y) \in \mathbb{A}[x,y]$ *is square-free, then* $F(x) = f(x,e^{Ix})$ *has no multiple roots other than* 0.

We extend the operation of complex conjugation to $\mathbb{LR}$ as follows.

Given $P = \sum\limits_{j=1}^{n} a_j(x)y^{v_j} \in \mathbb{LR}$, where $a_1,\cdots,a_n \in \mathbb{C}[x]$, define its formal conjugate to be $con(P) = \sum_{j=1}^{n} \overline{a_j(x)}y^{-v_j}$, where $\overline{a_j(x)}$ is the standard conjugate function of $a_j(x)$, and specially, $con(a_j(x)) = \overline{a_j(x)}$.

For $P = \sum\limits_{j=1}^{n} a_j(x)y^{v_j} \in \mathbb{LR}$, if $v_j \in \mathbb{Q}$, P is called a generalized Laurent polynomial($\mathbb{GLR}$). For $P \in \mathbb{GLR}$ and $v \in \mathbb{Q}$, define $LRhom[v](P) = P(x,e^{Ivx}) = \sum\limits_{j=1}^{n} a_j(x)(e^{Ivx})^{v_j}$ $= \sum\limits_{j=1}^{n} a_j(x)(\cos(v_j vx) + I\sin(v_j vx))$, then $LRhom[v](P)$ is a mixed trigonometric-polynomial with coefficients in the complex field. $LRhom[1](P)$ is abbreviated as $LRhom(P)$.

For an MTP $F(x) = f(x,\sin(x),\cos(x))$, let $P(x,y) = f(x,\frac{y-y^{-1}}{2I},\frac{y+y^{-1}}{2}) \in \mathbb{LR}$, then $LRhom(P) = F(x)$ obviously.

LEMMA 2.3. *For* $P = \sum\limits_{j=1}^{n} a_j(x)y^{v_j} \in \mathbb{GLR}$, $LRhom[v](con(P)) = \overline{LRhom[v](P)}$ *for* $\forall v \in \mathbb{Q}$.

Proof.

$$
\begin{aligned}
LRhom[v](con(P)) &= LRhom[v]\left(\sum_{j=1}^{n} \overline{a_j(x)}y^{-v_j}\right) \\
&= LRhom[v]\left(\sum_{j=1}^{n} \overline{a_j(x)}(e^{Ivx})^{-v_j}\right) \\
&= \sum_{j=1}^{n} \overline{a_j(x)}\,(\cos(-v_j vx) + I\sin(-v_j vx)) \\
&= \sum_{j=1}^{n} \overline{a_j(x)}(\cos(v_j vx) - I\sin(v_j vx)) \\
&= \sum_{j=1}^{n} \overline{a_j(x)\,(\cos(v_j vx) + I\sin(v_j vx))} \\
&= \overline{LRhom[v](P)}. \quad \square
\end{aligned}
$$

By Lemma 2.3 we get that

LEMMA 2.4. *For $P \in \mathbb{GLR}$, if $P = con(P)$, then $LRhom[v](P)$ is real-valued for $\forall v \in \mathbb{Q}$.*

LEMMA 2.5. *If $P \in \mathbb{C}[x,y]$ is coprime with y, P is irreducible iff $numer(con(P))$ is irreducible.*

Proof. Suppose that there exist non-constant P_1 *and* $P_2 \in \mathbb{C}[x,y]$ such that $P = P_1 \times P_2$.

Then, $con(P) = con(P_1) \times con(P_2)$. Let $n = degree(P,y)$, $n_1 = degree(P_1,y)$, $n_2 = degree(P_2,y)$. It is obviously that $n = n_1 + n_2$, $n = degree(con(P), y^{-1})$, $n_1 = degree(con(P_1), y^{-1})$, $n_2 = degree(con(P_2), y^{-1})$. So, $numer(con(P)) = y^n con(P)$, $numer(con(P_1)) = y^{n_1} con(P_1)$, $numer(con(P_2)) = y^{n_2} con(P_2)$.

We get that $numer(con(P)) = numer(con(P_1)) \times numer(con(P_2))$.

Furthermore, we declare that $numer(con(P_1))$ is non-constant, otherwise, $\exists c \in \mathbb{C}$ and $p \in \mathbb{N}$ such that $con(P_1) = c \times y^{-p}$, then $P_1 = \overline{c} \times y^p$, which conflicts the fact that P is coprime with y.

So, we get that the reducibility of $numer(con(P))$ implies the reducibility of P.

In the similar way, we will get that the reducibility of P implies the reducibility of $numer(con(P))$. Thus the Lemma holds. □

For two polynomials P and Q, we say that $P \sim Q$ if there exists a nonzero constant c such that $P = c \times Q$.

THEOREM 2.2. *If $P = \sum_{j=1} a_j(x) y^{v_j} \in \mathbb{LR}$ such that $P = con(P)$, then*

1) There exist polynomials $T_1, \ldots, T_m \in \mathbb{C}[x,y]$, $c \in \mathbb{C}$ and $p \in \mathbb{Z}$ such that T_is are square-free and pairwisely coprime, $P = cy^p T_1^{r_1} \cdots T_m^{r_m}$ and $T_i \sim numer(con(T_i))$ for $i = 1, \ldots, m$.

2) For each $i = 1, \ldots, m$, $\exists c_i \in \mathbb{C}$ and $p_i \in \mathbb{Z}$ such that

$$f_i(x) = LRhom[v](T_i y^{-p_i/2} c_i^{-1/2})$$

is real-valued or pure imaginary, and f_i has no multiple root other than 0 for $i = 1, \ldots, m$, f_i and f_j have no common root other than 0 for $i \neq j$, where v is an arbitrary algebraic number ;

3) $LRhom[v](P) = c_0 f_1^{r_1} \cdots f_m^{r_m}$, where $c_0 = c(c_1)^{r_1/2} \ldots (c_m)^{r_m/2}$.

Proof. 1) Let P be factorized as $cy^p P_1^{r_1} \cdots P_n^{r_n}$, where $c \in \mathbb{C}$, $p \in \mathbb{Z}$ and $P_1, \ldots, P_n \in \mathbb{C}[x,y]$ are irreducible, and $P_i \neq y$ for $i = 1, \ldots, n$, $P_i \neq P_j$ for $i \neq j$. Let $con(P) = c_0 y^q Q_1^{r_1} \ldots Q_n^{r_n}$, where $Q_i = numer(con(P_i))$.

By Lemma 2.5, $Q_1, \ldots, Q_n$ are all irreducible, then $P = con(P)$ implies that for each i, $\exists j_i$ such that $P_i \sim Q_{j_i}$, and then the index $r_i = r_{j_i}$ whether $i = j_i$ or not. If $i \neq j_i$, $P_i \sim Q_{j_i} = numer(con(P_{j_i}))$ implies that there exist $c_i \in \mathbb{C}$ and $p_i \in \mathbb{Z}$ such that $P_i = c_i y^{p_i} con(P_{j_i})$, $con(P_i) = con(c_i y^{p_i} con(P_{j_i})) = \overline{c_i} y^{-p_i} P_{j_i}$, so $Q_i = numer(con(P_i)) = numer(\overline{c_i} y^{-p_i} P_{j_i}) = \overline{c_i} \times P_{j_i}$, i.e. $P_{j_i} \sim Q_i$. Then we get that $P_i \times P_{j_i} \sim Q_i \times Q_{j_i}$.

Now reduce $P = cy^pP_1^{r_1}\cdots P_n^{r_n}$ to $P = cy^pT_1^{r_1}\cdots T_m^{r_m}$, where if $P_i \sim Q_i$, P_i is one of T_is, if $P_i \sim Q_{j_i}$ and $i \neq j_i$, $P_i \times P_{j_i}$ is one of T_is. And T_is are square-free and pairwisely co-prime, furthermore, $T_i \sim numer(con(T_i))$ for $i = 1,\ldots,m$.

2) $T_i \sim numer(con(T_i))$ implies that $\exists c_i \in \mathbb{C}$ and $p_i \in \mathbb{Z}$ such that $T_i = c_iy^{p_i}con(T_i)$, then $T_i^2 = c_iy^{p_i}con(T_i)T_i$, so $T_i^2c_i^{-1}y^{-p_i} = con(T_i)T_i$. Let $Q_i = T_iy^{-p_i/2}c_i^{-1/2}$, then $Q_i^2 = con(T_i)T_i$, so that $Q_i^2 = con(Q_i^2)$.

Let $f_i(x) = LRhom[v](Q_i)$, then $f_i(x)^2$ is real-valued due to Lemma 2.4, i.e. $f_i(x)$ is real-valued or pure imaginary.

It is clearly that f_i has no multiple root other than 0 by Theorem 2.1, f_i and f_j have no common root other than 0 for $i \neq j$ by Lemma 2.2.

3) $P = con(P)$ implies that $degree(P,y) = degree(P,y^{-1})$, denoted by q. It is trivial that $p = -q$ and $degree(Q = T_1^{r_1}\cdots T_m^{r_m},y) = 2q$. Since $T_i = c_iy^{p_i}con(T_i)$, $p_i = degree(T_i,y) = degree(con(T_i),y^{-1})$, hence $r_1p_1+\ldots+r_mp_m = degree(Q,y)$, i.e. $p = -q = -\frac{r_1p_1+\ldots+r_mp_m}{2}$.

So,

$$
\begin{aligned}
& c_0f_1^{r_1}\cdots f_m^{r_m} \\
&= c(c_1)^{r_1/2}\ldots(c_m)^{r_m/2}(LRhom[v](T_1y^{-p_1/2}c_1^{-1/2}))^{r_1}\ldots(LRhom[v](T_my^{-p_m/2}c_m^{-1/2}))^{r_m} \\
&= c\times\left(LRhom[v](T_1y^{-\frac{p_1}{2}})\right)^{r_1}\ldots\left(LRhom(T_my^{-\frac{p_m}{2}})\right)^{r_m} \\
&= c\times LRhom[v]\left(y^{-\frac{p_1}{2}r_1-\ldots-\frac{p_m}{2}r_m}T_1^{r_1}\ldots T_m^{r_m}\right) \\
&= LRhom[v](cy^pT_1^{r_1}\ldots T_m^{r_m}) \\
&= LRhom[v](P). \quad \square
\end{aligned}
$$

COROLLARY 2.2. *For each mixed trigonometric polynomial*

$$F(x) = f(x,\sin(x),\cos(x)),$$

where $f \in \mathbb{R}_{alg}[x,y,z]$, there exist $c \in \mathbb{R}_{alg}$ and real-valued mixed trigonometric-polynomials $\{f_i\}$ such that f_i has no multiple root other than 0 , f_i and f_j have no common root other than 0 for $i \neq j$, $F(x) = cf_1(x)^{r_1}\ldots f_n(x)^{r_n}$.

Proof. Let $P = f(x,\frac{y-y^{-1}}{2I},\frac{y+y^{-1}}{2})$, by Theorem 2.2, there exist $c \in \mathbb{C}$ and mixed trigonometric polynomials $\{f_i\}$ such that f_i has no multiple root other than 0, f_i and f_j have no common root other than 0 for $i \neq j$, $F(x) = cf_1(x)^{r_1}\ldots f_n(x)^{r_n}$, each f_i is real-valued or pure imaginary.

If all f_is are real-valued, then Corollary 2.2 holds. If f_i is pure imaginary, let $f_i' = f_i/I$ and $c' = cI^{r_i}$, then f_i' is real-valued and $F(x) = c'f_1(x)^{r_1}\ldots f_i'(x)^{r_i}\ldots f_n(x)^{r_n}$. Repeat the above operation to ensure that each f_i is real-valued.

Now, $F(x)$ and all $f_i(x)$s (or $f_i'(x)s$) are real-valued, so the constant $c(or\ c')$ must be a real number. That is to say the corollary holds. $\square$

Theorem 2.2 and Corollary 2.2 guarantee the correctness of the following Algorithm 2.1.

ALGORITHM 2.1. (Square-free Factorization of MTP)

INPUT: An MTP $F(x) = f(x, \sin(x), \cos(x)), f \in \mathbb{R}_{alg}[x, y, z]$;

OUTPUT: $F(x) = c \times F_1(x)^{n_1} \cdots F_m(x)^{n_m}$; where $c \in \mathbb{R}_{alg}$ and $F_i(x)$ is a real-valued MTP and which has no multiple roots other than 0 for $i = 1, \cdots, m$, $F_i(x)$ and $F_j(x)$ have no common roots other than 0 for $i \neq j$

1) $P \leftarrow f(x, \frac{y-y^{-1}}{2I}, \frac{y+y^{-1}}{2})$;

2) $P\_lst \leftarrow factor(P) = c_0 y^p P_1^{r_1} \cdots P_n^{r_n}$; where $c_0 \in \mathbb{C}$, $P_1, \cdots, P_n \in \mathbb{C}[x, y]$ are irreducible.

3) $T \leftarrow \{P^{r_P} | P \in \{P_1, \cdots, P_n\}$ and $P \sim numer(P)\} \cup \{(P \times Q)^{r_P} | P, Q \in \{P_1, \cdots, P_n\}$ and $P \sim numer(Q)\}$, where r_P is the corresponding index of P in P_lst. Suppose that $T = \{T_1^{r_1}, \ldots, T_m^{r_m}\}$ and $T_i = c_i \times y^{p_i} \times con(T_i)$ for $i = 1, \ldots, m$.

4) $F \leftarrow 1$ and $c \leftarrow c_0 \times \prod_{i=1}^{m} (c_i)^{1/2}$;

5) For i form 1 to $nops(T)$

5.1) $f_i \leftarrow LRhom(T_i \times c_i^{-1/2} \times y^{-p_i/2})$

5.2) If f_i is purely imaginary, then $f_i \leftarrow f_i/I$ and $c \leftarrow c \times I^{r_i}$

5.3) $F \leftarrow F \times f_i^{r_i}$;

6) return $c \times F$.

3. Examples

We present some examples to show the effectiveness of the algorithm in this section.

EXAMPLE 1. Decide whether $f(x) = \frac{2}{3}x + x\cos(x) - \sin(x)$ has multiple roots.

Let $\cos(x) = \frac{e^{Ix}+e^{-Ix}}{2}$, $\sin(x) = \frac{e^{Ix}-e^{-Ix}}{2I}$, $y = e^{Ix}$, then $f = \frac{2}{3}x + \frac{x(y+y^{-1})}{2} - \frac{y-y^{-1}}{2I}$, and $factor(f) = \frac{1}{6}\frac{4xy+3xy^2+3x+3Iy^2-3I}{y}$. $4xy + 3xy^2 + 3x + 3Iy^2 - 3I$ is irreducible, which implies that $f(x)$ has no multiple roots other than 0.

$f(0) = 0$ and $f'(0) = \frac{2}{3}$, i.e. 0 is a simple root of $f(x)$. We get that $f(x)$ has no multiple roots.

EXAMPLE 2. Factorize $f(x) = 1 - \sin^3(x)$.

Let $\sin(x) = \frac{e^{Ix}-e^{-Ix}}{2I}$, $y = e^{Ix}$.

Then $f(x) = g(y) = 1 - \frac{1}{8}I(y - \frac{1}{y})^3 = \frac{-\frac{1}{8}I(y^4+2Iy^3-6y^2-2Iy+1)(y-I)^2}{y^3} = c_0 \times P_1 \times P_2^2$, where $c_0 = -\frac{I}{8}$, $P_1 = y^4 + 2Iy^3 - 6y^2 - 2Iy + 1$, and $P_2 = y - I$.

We get that $y = I$ is a multiple root of $g(x)$, and then $\{2k\pi + \frac{\pi}{2}, k \in \mathbb{Z}\}$ are multiple roots of $f(x)$.

As $con(P_1) = \frac{1}{y^4} - \frac{2I}{y^3} - \frac{6}{y^2} + \frac{2I}{y} + 1$, so $P_1 = c_1 y^4 con(P_1)$, where $c_1 = 1$.

Let

$$
\begin{aligned}
g_1 &= LRhom(P_1/(y^4)^{1/2}) = LRhom((y^4+2Iy^3-6y^2-2Iy+1)/y^2)\\
&= LRhom(y^2+2Iy-6-2Iy^{-1}+y^{-2})\\
&= e^{2Ix}+2e^{I\frac{\pi}{2}}e^{Ix}-6-2e^{I\frac{\pi}{2}}e^{-Ix}+e^{-2Ix}\\
&= \cos(2x)+I\sin(2x)+2\cos(\frac{\pi}{2}+x)+2I\sin(\frac{\pi}{2}+x)-6-2\cos(\frac{\pi}{2}-x)\\
&-2I\sin(\frac{\pi}{2}-x)+\cos(-2x)+I\sin(-2x)\\
&= \cos(2x)+I\sin(2x)-2\sin(x)+2I\cos(x)-6-2\sin(x)\\
&\quad -2I\cos(x)+\cos(2x)-I\sin(2x)\\
&= 2\cos(2x)-4\sin(x)-6
\end{aligned}
$$

As $con(P_2)=\frac{1}{y}+I$, then $P_2=c_2ycon(P_2)$, where $c_2=-I=e^{-I\frac{\pi}{2}}$.
Let

$$
\begin{aligned}
g_2 &= LRhom(P_2/(-Iy)^{\frac{1}{2}}) = LRhom((y-I)/(-Iy)^{\frac{1}{2}})\\
&= (e^{Ix}-e^{I\frac{\pi}{2}})/e^{-I\frac{\pi}{4}+I\frac{x}{2}}\\
&= e^{I\frac{x}{2}+I\frac{\pi}{4}}-e^{I\frac{3\pi}{4}-I\frac{x}{2}}\\
&= \cos\left(\frac{x}{2}+\frac{\pi}{4}\right)+I\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)-\left(\cos\left(\frac{3\pi}{4}-\frac{x}{2}\right)+I\sin\left(\frac{3\pi}{4}-\frac{x}{2}\right)\right)\\
&= \cos\left(\frac{x}{2}+\frac{\pi}{4}\right)+I\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)+\cos\left(\frac{x}{2}+\frac{\pi}{4}\right)-I\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)\\
&= 2\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)
\end{aligned}
$$

Let $c=c_0\times c_1^{1/2}\times(c_2^{1/2})^2=-\frac{I}{8}(-I)=-1/8$.

Then $f(x)=c\times g_1\times g_2^2=-\frac{1}{8}(2\cos(2x)-4\sin(x)-6)\times\cos^2(\frac{\pi}{4}+\frac{x}{2})=-\frac{1}{2}(\cos(2x)-2\sin(x)-3)\times(\cos(\frac{x}{2})-\sin(\frac{x}{2}))^2=-\frac{1}{2}f_1\times f_2^2$, where $f_1=\cos(2x)-2\sin(x)-3, f_2=\cos(\frac{x}{2})-\sin(\frac{x}{2})$, f_1 and f_2 have no common real roots and both of them have no multiple roots.

EXAMPLE 3. (Adapted form [17]) Compute the greatest common divisor(GCD) of $a=\sin(x)(1+\cos(x))$ and $b=-\cos^2(x)+\sin(x)\cos(x)+\sin(x)+1$.

we find that that a and b have exactly the following factorizations in classical manner: $a=\sin(x)(\cos(x)+1)$ and $b=(\cos(x)+\sin(x)+1)\sin(x)=(-\cos(x)+\sin(x)+1)(1+\cos(x))$. Now, $\sin(x)$ and $1+\cos(x)$ divide both a and b, but $\sin(x)(1+\cos(x)$ is not a common divisor of a and b.

By Algorithm 2.1, we get that $a=4\sin(\frac{x}{2})\cos^3(\frac{x}{2})$ and $b=4\cos^2(\frac{x}{2})\sin(\frac{x}{2})(\cos(\frac{x}{2})+\sin(\frac{x}{2}))$, so, $GCD(a,b)=\sin(\frac{x}{2})\cos^2(\frac{x}{2})$. Here, we omit the details to conclude that the common divisor $\sin(\frac{x}{2})\cos^2(\frac{x}{2})$ is the greatest.

Conclusion

In this paper, we present an algorithm to decompose the mixed trigonometric-polynomials without multiple roots. Furthermore, this algorithm can serve as a fundamental method for performing other algebraic operations on trigonometric polynomials, such as simplification, division, and computing the greatest common divisor.

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