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SOME CESÀRO–TYPE AND LACUNARY STATISTICAL CONVERGENCE IN A–METRIC SPACES

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Abstract. In this paper, we investigated the concepts of statistical convergence, Cesàro convergence, lacunary convergence, and lacunary statistical convergence in A-metric spaces. We also discussed the relationships between these concepts.

1. Introduction

In 1906, Fréchet [7] introduced the concept of metric space for the first time. In the following years, the concept of metric space attracted the attention of many researchers, and generalization studies of metric spaces were made. For those interested in studying the generalization of metric spaces, see the research papers in [5, 14, 17, 20, 21, 26]. The idea of A-metric spaces, which resulted from these investigations, was initially presented by Abbas et al. [2] in 2015 as a generalization of the S-metric space. They proved some coupled common fixed point theorems for mixed weakly monotone maps in partially ordered A-metric spaces.

The concept of statistical convergence was introduced in 1951 by Fast [6] and Steinhaus [27]. Afterwards, Shoenberg [25] introduced it in 1959 and also studied the concept as a summability method. Since then, the properties of statistical convergence have been studied by different mathematicians and applied in several area (see, [1, 3, 8, 9, 10, 13, 16, 19, 24, 29].) Connor [4] established a relationship between strong Cesàro summability and statistical convergence. Later, Fridy-Orhan [11, 12] introduced the concepts of lacunary statistical convergence and lacunary statistical summability, as well as explored their relationships with previously presented summability theory concepts. Recenty, Küçük and Gümüş [18] studied the concept of lacunary statistical convergence in G-metric spaces. Nuray [22] examined statistical convergence in 2-metric spaces and also examined the relationships between the concepts of Cesàro convergence, N_θ-convergence, and lacunary statistical convergence. Also Nuray [23] examined statistical convergence in partial metric spaces. Then, Gülle et al. [15] presented strong lacunary and strong q-lacunary summability, as well as lacunary statistical convergence in partial metric spaces. They also discussed the concept of lacunary

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statistical convergence in partial metric space and looked at certain relationships. Subsequently lacunary sequence have been examined in [15]. The papers in [15, 18, 22] motivated this study.

The primary goal of this paper is to introduce the concepts of Cesàro convergence, N_θ-convergence, and lacunary statistical convergence in A-metric spaces. We will also show how these notions are related to the concepts of statistical convergence and statistical Cauchy sequence defined in [28].

DEFINITION 1. [2] Let X be a nonempty set. A function $A : X^n \to [0,\infty)$ is called an A-metric on X if for any $x_i, a \in X, i = 1,2,\ldots,n$ the following conditions hold;

(A1) $A(x_1,x_2,\ldots,x_{n-1},x_n) \geqslant 0$,

(A2) $A(x_1,x_2,\ldots,x_{n-1},x_n) = 0 \Leftrightarrow x_1 = x_2 = \ldots = x_n$,

(A3) $A(x_1,x_2,\ldots,x_{n-1},x_n) \leqslant \sum_{i=1}^{n} A(\underbrace{x_i,x_i,\ldots,x_i}_{n-1},a)$.

Also the pair (X,A) is called an A-metric space.

EXAMPLE 1. [2] Let $X = \mathbb{R}$. Define a function $A : X^n \to [0,\infty)$ by

$$A(x_1,x_2,\ldots,x_{n-1},x_n) = \sum_{i=1}^{n} \sum_{i<j} |x_i - x_j|.$$

Then (X,A) is an A-metric.

LEMMA 1. [2] *Let (X,A) be an A-metric space. Then*

$$A(x,x,\ldots,x,y) = A(y,y,\ldots,y,x)$$

for all $x,y \in X$.

LEMMA 2. [2] *Let (X,A) be an A-metric space. For all $x,y \in X$ we get*

$$A(x,x,\ldots,x,y) \leqslant (n-1)A(x,x,\ldots,x,z) + A(y,y,\ldots,y,z) \text{ and}$$
$$A(x,x,\ldots,x,z) \leqslant (n-1)A(x,x,\ldots,x,y) + A(z,z,\ldots,z,y).$$

DEFINITION 2. [2] Let (X,A) be an A-metric space. A subset B of X is said to be bounded if there exists an $r > 0$ such that $A(y,y,\ldots,y,x) \leqslant r$ for every $x,y \in X$. Otherwise, X is unbounded.

DEFINITION 3. [2] Let (X,A) be an A-metric space. A sequence (x_k) in X is said to be convergent to x in X if for every $\varepsilon > 0$ there exists a natural k_0 such that $A(x_k,x_k,\ldots,x_k,x) < \varepsilon$ for every $k \geqslant k_0$.

DEFINITION 4. [2] Let (X,A) be an A-metric space. A sequence (x_k) in X is said to be a Cauchy sequence if for each $\varepsilon > 0$, there exists a $k_0 \in \mathbb{N}$ such that for all $k,m \geqslant k_0$ we have $A(x_k,x_k,\ldots,x_k,x_m) < \varepsilon$.

2. Main results

In this section, we present the definitions of Cesàro and statistical convergence, the statistical Cauchy sequence in A-metric spaces, and the relations between them. The definitions of statistical convergence and statistical Cauchy sequence in A-metric spaces were presented in [28].

DEFINITION 5. [28] Let (X,A) be an A-metric space. A sequence (x_k) in X is said to be statistically convergent to an element $x \in X$ if for every $\varepsilon > 0$

$$\lim_{t\to\infty} \frac{1}{t} |\{k \leqslant t : A(x_k, x_k, \ldots, x_k, x) \geqslant \varepsilon\}| = 0$$

or equivalently

$$\lim_{t\to\infty} \frac{1}{t} |\{k \leqslant t : A(x_k, x_k, \ldots, x_k, x) < \varepsilon\}| = 1$$

and is denoted by $x_k \xrightarrow{AS} x$. In this case, we can write $st - \lim_{k\to\infty} A(x_k, x_k, \ldots, x_k, x) = 0$.

The set S of statistically convergent sequences is defined as follows;

$$S = \left\{ (x_k) \subseteq X : \lim_{t\to\infty} \frac{1}{t} |\{k \leqslant t : A(x_k, x_k, \ldots, x_k, x) \geqslant \varepsilon\}| = 0, \text{ for some } x \right\}.$$

DEFINITION 6. [28] Let (X,A) be an A- metric space. A sequence (x_k) in X is said to be a statistically Cauchy sequence if for all $x \in X$ and for every $\varepsilon > 0$

$$\lim_{t\to\infty} \frac{1}{t} |\{k, m \leqslant t : A(x_k, x_k, \ldots, x_k, x_m) \geqslant \varepsilon\}| = 0$$

or equivalently

$$\lim_{t\to\infty} \frac{1}{t} |\{k, m \leqslant t : A(x_k, x_k, \ldots, x_k, x_m) < \varepsilon\}| = 1.$$

In this case, we can write $st - \lim_{k,m\to\infty} A(x_k, x_k, \ldots, x_k, x_m) = 0$.

DEFINITION 7. Let (X,A) be an A-metric space. A sequence (x_k) in X is said to be Cesàro convergent to an element $x \in X$ if

$$\lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{t} A(x_k, x_k, \ldots, x_k, x) = 0$$

and is denoted by $x_k \xrightarrow{A\sigma_1} x$.

The set σ_1^A of Cesàro convergent sequences is defined, as follows;

$$\sigma_1^A = \left\{ (x_k) \subseteq X : \lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{t} A(x_k, x_k, \ldots, x_k, x) = 0, \text{ for some } x \right\}.$$

THEOREM 1. *Let* (X,A) *be an* A*-metric space and* (x_k) *be a sequence in* X*. Then*

(i) *If* (x_k) *is Cesàro convergent to* x *then* (x_k) *is statistically convergent to* x*.*

(ii) *If* (X,A) *is bounded and* (x_k) *is statistically convergent to* x*, then* (x_k) *Cesàro convergent to* x*.*

Proof. (i) Let (X,A) be an A-metric space and (x_k) be Cesàro convergent to x. For $\varepsilon > 0$, we have

$$\begin{aligned}\frac{1}{t}\sum_{k=1}^{t} A(x_k,x_k,\ldots,x_k,x) &= \frac{1}{t}\sum_{\substack{k=1\\ A(x_k,x_k,\ldots,x_k,x)\geqslant\varepsilon}}^{t} A(x_k,x_k,\ldots,x_k,x)\\ &\quad+\frac{1}{t}\sum_{\substack{k=1\\ A(x_k,x_k,\ldots,x_k,x)<\varepsilon}}^{t} A(x_k,x_k,\ldots,x_k,x)\\ &\geqslant \frac{1}{t}\sum_{\substack{k=1\\ A(x_k,x_k,\ldots,x_k,x)\geqslant\varepsilon}}^{t} A(x_k,x_k,\ldots,x_k,x)\\ &\geqslant \frac{1}{t}\left|\{1\leqslant k\leqslant t : A(x_k,x_k,\ldots,x_k,x)\geqslant\varepsilon\}\right|\varepsilon.\end{aligned}$$

Hence we get

$$\lim_{t\to\infty}\frac{1}{t}\left|\{1\leqslant k\leqslant t : A(x_k,x_k,\ldots,x_k,x)\geqslant\varepsilon\}\right| = 0$$

that is, (x_k) is statistically convergent to x.

(ii) Suppose that (x_k) is bounded and statistically convergent to x. Since (X,A) is bounded, we say $A(x_k,x_k,\ldots,x_k,x)\leqslant K$ for all k. For $\varepsilon>0$, we have

$$\begin{aligned}\frac{1}{t}\sum_{k=1}^{t} A(x_k,x_k,\ldots,x_k,x) &= \frac{1}{t}\sum_{\substack{k=1\\ A(x_k,x_k,\ldots,x_k,x)\geqslant\varepsilon}}^{t} A(x_k,x_k,\ldots,x_k,x)\\ &\quad+\frac{1}{t}\sum_{\substack{k=1\\ A(x_k,x_k,\ldots,x_k,x)<\varepsilon}}^{t} A(x_k,x_k,\ldots,x_k,x)\\ &\leqslant \frac{1}{t}K\sum_{\substack{k=1\\ A(x_k,x_k,\ldots,x_k,x)\geqslant\varepsilon}}^{t} 1\\ &\quad+\frac{1}{t}\sum_{\substack{k=1\\ A(x_k,x_k,\ldots,x_k,x)<\varepsilon}}^{t} A(x_k,x_k,\ldots,x_k,x)\\ &\leqslant K\frac{1}{t}\left|\{1\leqslant k\leqslant t : A(x_k,x_k,\ldots,x_k,x)\geqslant\varepsilon_k\}\right| + \frac{1}{t}\sum_{k=1}^{t}\varepsilon.\end{aligned}$$

Hence, we get

$$\lim_{t\to\infty}\frac{1}{t}\sum_{k=1}^{t}A(x_k,x_k,\ldots,x_k,x)=0,$$

that is, (x_k) is Cesàro convergent to x. □

First, we recall the consept of lacunary sequences. A lacunary sequence [9] is an increasing integer sequence $\theta=(p_r)$ such that $p_0=0$ and $h_r=p_r-p_{r-1}\to\infty$ as $r\to\infty$. The intervals $(p_{r-1},p_r]$ determined by $\theta=(p_r)$ will be denoted by I_r.

DEFINITION 8. Let (X,A) be an A-metric space and $\theta=(p_r)$ be any lacunary sequence. A sequence (x_k) in X is said to be N_θ-convergent to an element $x\in X$ if for every $\varepsilon>0$,

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}A(x_k,x_k,\ldots,x_k,x)=0$$

and is denoted by $x_k\xrightarrow{AN_\theta}x$. The set of N_θ-convergent sequences will be denoted by $\mathscr{N}_\theta$.

DEFINITION 9. Let (X,A) be an A-metric space and $\theta=(p_r)$ be any lacunary sequence. A sequence (x_k) in X is said to be lacunary statistically convergent to an element $x\in X$ if for every $\varepsilon>0$,

$$\lim_{r\to\infty}\frac{1}{h_r}|\{k\in I_r:A(x_k,x_k,\ldots,x_k,x)\geqslant\varepsilon\}|=0$$

or equivalently

$$\lim_{r\to\infty}\frac{1}{h_r}|\{k\in I_r:A(x_k,x_k,\ldots,x_k,x)<\varepsilon\}|=1$$

and is denoted by $x_k\xrightarrow{AS_\theta}x$.

The set S_θ of lacunary statistically convergent sequences is defined, as follows;

$$S_\theta=\{(x_k)\subseteq X:\lim_{r\to\infty}\frac{1}{h_r}\{k\in I_r:A(x_k,x_k,\ldots,x_k,x)\geqslant\varepsilon\}=0,\text{ for some }x\}.$$

DEFINITION 10. Let (X,A) be an A-metric space and $\theta=(p_r)$ be any lacunary sequence. A sequence (x_k) in X is said to be lacunary statistically Cauchy sequence if there is a subsequence $(x_{k_r'})$ of (x_k) such that $k_r'\in I_r$ for each r, $\lim_r x_{k_r'}=x$, and for each $\varepsilon>0$

$$\lim_{r\to\infty}\frac{1}{h_r}|\{k\in I_r:A(x_k,x_k,\ldots,x_k,x_{k_r'})\geqslant\varepsilon\}|=0.$$

THEOREM 2. *Let* (x_k) *be a sequence in an* A*-metric space* (X,A) *and* $\theta = (p_r)$ *be any lacunary sequence. The sequence* (x_k) *is lacunary statistically convergent if and only if* (x_k) *is a lacunary statistically Cauchy sequence.*

Proof. Let $x_k \xrightarrow{AS_\theta} x$, and write $K_j = \left\{k \in \mathbb{N} : A(x_k,x_k,\ldots,x_k,x) < \frac{1}{j}\right\}$ for each $j \in \mathbb{N}$. Hence, for each j, $K_j \supseteq K_{j+1}$ and

$$\frac{|K_j \cap I_r|}{h_r} = 1 \qquad \text{as} \qquad r \to \infty. \tag{1}$$

Choose m_1 such that $r \geqslant m_1$ implies $\frac{|K_1 \cap I_r|}{h_r} > 0$, that is, $K_1 \cap I_r \neq \emptyset$. Next choose $m_2 > m_1$ so that $r \geqslant m_2$ implies $K_2 \cap I_r \neq \emptyset$. Then for each r satisfying $m_1 \leqslant r < m_2$, choose $k_r^{'} \in I_r$ such that $k_r^{'} \in I_r \cap K_1$, that is, $A(x_{k_r'},x_{k_r'},\ldots,x_{k_r'},x) < 1$. In general, choose $m_{p+1} > m_p$ such that $r > m_{p+1}$ implies $I_r \cap K_{p+1} \neq \emptyset$. Then for all r satisfying $m_p \leqslant r < m_{p+1}$ choose $k_r^{'} \in I_r \cap K_p$, that is,

$$A(x_{k_r'},\ldots,x_{k_r'},x) < \frac{1}{p}. \tag{2}$$

Hence, we get $k_r^{'} \in I_r$ for every r, and (2) implies that $\lim_{r\to\infty} A(x_{k_r'},x_{k_r'},\ldots,x_{k_r'},x) = 0$. Furthermore, from Lemma 2, we have for every $\varepsilon > 0$,

$$\frac{1}{h_r}\left|\left\{k \in I_r : A(x_k,x_k,\ldots,x_k,x_{k_r'}) \geqslant \varepsilon\right\}\right| \leqslant \frac{(n-1)}{h_r}\left|\left\{k \in I_r : A(x_k,x_k,\ldots,x_k,x) \geqslant \frac{\varepsilon}{n}\right\}\right| + \frac{1}{h_r}\left|\left\{k \in I_r : A(x_{k_r'},x_{k_r'},\ldots,x_{k_r'},x) \geqslant \frac{\varepsilon}{n}\right\}\right|.$$

Using the assumptions that $x_k \xrightarrow{AS_\theta} x$, and $\lim_{r\to\infty} A(x_{k_r'},x_{k_r'},\ldots,x_{k_r'},x) = 0$, (x_k) is a lacunary statistically Cauchy sequence.

Conversely, suppose that (x_k) is a lacunary statistically Cauchy sequence. For every $\varepsilon > 0$,

$$\left|\left\{k \in I_r : A(x_k,x_k,\ldots,x_k,x) \geqslant \varepsilon\right\}\right| \leqslant (n-1)\left|\left\{k \in I_r : A(x_k,x_k,\ldots,x_k,x_{k_r'}) \geqslant \frac{\varepsilon}{n}\right\}\right| + \left|\left\{k \in I_r : A(x_{k_r'},x_{k_r'},\ldots,x_{k_r'},x) \geqslant \frac{\varepsilon}{n}\right\}\right|,$$

from which it follows that $x_k \xrightarrow{AS_\theta} x$. $\square$

THEOREM 3. *Let* (x_k) *be a sequence in an* A*-metric space* (X,A) *and* $\theta = (p_r)$ *be any lacunary sequence. If* $x_k \xrightarrow{AN_\theta} x$ *then* $x_k \xrightarrow{AS_\theta} x$.

Proof. Let $\varepsilon > 0$ and $x_k \xrightarrow{AN_\theta} x$. Then, we get

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} A(x_k, x_k, \ldots, x_k, x) = 0.$$

Also we can write

$$\begin{aligned}\frac{1}{h_r} \sum_{k\in I_r} A(x_k, x_k, \ldots, x_k, x) &\geqslant \frac{1}{h_r} \sum_{\substack{k\in I_r \\ A(x_k, x_k, \ldots, x_k, x)\geqslant\varepsilon}} A(x_k, x_k, \ldots, x_k, x) \\ &\geqslant \varepsilon . \frac{1}{h_r} |\{k \in I_r : A(x_k, x_k, \ldots, x_k, x) \geqslant \varepsilon\}|\end{aligned}$$

which yields the result. □

THEOREM 4. *Let (x_k) be a sequence in an A-metric space (X, A) and $\theta = (p_r)$ be any lacunary sequence and A be a bounded function in X. If $x_k \xrightarrow{AS_\theta} x$, then $x_k \xrightarrow{AN_\theta} x$.*

Proof. Let $\varepsilon > 0$, A is bounded and (x_k) is S_θ-convergent to x. Since A is bounded, there exists a $K > 0$ such that $A(x_k, x_k, \ldots, x_k, x) \leqslant K$ for all $k \in \mathbb{N}$. Thus, for every $\varepsilon > 0$

$$\begin{aligned}\frac{1}{h_r} \sum_{k\in I_r} A(x_k, x_k, \ldots, x_k, x) &= \frac{1}{h_r} \sum_{\substack{k\in I_r \\ A(x_k, x_k, \ldots, x_k, x)\geqslant\varepsilon}} A(x_k, x_k, \ldots, x_k, x) \\ &\quad + \frac{1}{h_r} \sum_{\substack{k\in I_r \\ A(x_k, x_k, \ldots, x_k, x)<\varepsilon}} A(x_k, x_k, \ldots, x_k, x) \\ &\leqslant K . \frac{1}{h_r} |\{k \in I_r : A(x_k, x_k, \ldots, x_k, x) \geqslant \varepsilon\}| + \varepsilon\end{aligned}$$

considering that (x_k) is S_θ-convergent, we get the result. □

COROLLARY 1. *Let (x_k) be a sequence in an A-metric space (X, A) and $\theta = (p_r)$ be any lacunary sequence and A be a bounded function in X. Then $x_k \xrightarrow{AS_\theta} x$ if and only if $x_k \xrightarrow{AN_\theta} x$.*

Proof. This is an immediate consequence of Theorem 3 and Theorem 4. □

The following two lemmas and the next theorem gives the relation between Cesàro convergence and N_θ-convergence in A-metric spaces.

LEMMA 3. *Let (x_k) be a sequence in an A-metric space (X, A) and $\theta = (p_r)$ be any lacunary sequence. If $\liminf_r \frac{p_r}{p_{r-1}} > 1$, then $\sigma_1^A \subseteq \mathcal{N}_\theta$.*

Proof. Let $\liminf_r \frac{p_r}{p_{r-1}} > 1$. Then there exists a $\delta > 0$ such that $1+\delta \leqslant \frac{p_r}{p_{r-1}}$ for all $r \geqslant 1$. Suppose that $(x_k) \in \sigma_1^A$, hence we can write

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r} A(x_k,x_k,\dots,x_k,x) &= \frac{1}{h_r}\sum_{k=1}^{p_r} A(x_k,x_k,\dots,x_k,x) \\ &\quad -\frac{1}{h_r}\sum_{k=1}^{p_{r-1}} A(x_k,x_k,\dots,x_k,x) \\ &= \frac{p_r}{h_r}\left(\frac{1}{p_r}\sum_{k=1}^{p_r} A(x_k,x_k,\dots,x_k,x)\right) \\ &\quad -\frac{p_{r-1}}{h_r}\left(\frac{1}{p_{r-1}}\sum_{k=1}^{p_{r-1}} A(x_k,x_k,\dots,x_k,x)\right).\end{aligned} \tag{3}$$

Since $h_r = p_r - p_{r-1}$, we have

$$\frac{p_r}{h_r} \leqslant \frac{1+\delta}{\delta} \qquad \text{and} \qquad \frac{p_{r-1}}{h_r} \leqslant \frac{1}{\delta}. \tag{4}$$

By (3) and (4), we get

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r} A(x_k,x_k,\dots,x_k,x) \leqslant \frac{1+\delta}{\delta}&\left(\frac{1}{p_r}\sum_{k=1}^{p_r} A(x_k,x_k,\dots,x_k,x)\right) \\ -\frac{1}{\delta}&\left(\frac{1}{p_{r-1}}\sum_{k=1}^{p_{r-1}} A(x_k,x_k,\dots,x_k,x)\right).\end{aligned} \tag{5}$$

Since $x_k \xrightarrow{A\sigma_1} x$

$$\frac{1}{p_r}\sum_{k=1}^{p_r} A(x_k,x_k,\dots,x_k,x) \to 0 \quad \text{and} \quad \frac{1}{p_{r-1}}\sum_{k=1}^{p_{r-1}} A(x_k,x_k,\dots,x_k,x) \to 0,$$

then by (5) for $r \to \infty$ we get that $(x_k) \in \mathcal{N}_\theta$. □

LEMMA 4. *Let (x_k) be a sequence in an A-metric space (X,A) and $\theta = (p_r)$ be any lacunary sequence. If $\limsup_r \frac{p_r}{p_{r-1}} < \infty$ then $\mathcal{N}_\theta \subseteq \sigma_1^A$.*

Proof. Let $\limsup_r \frac{p_r}{p_{r-1}} < \infty$. Then there exists $K' > 0$ such that $\frac{p_r}{p_{r-1}} < K'$ for all $r \geqslant 1$. Let $(x_k) \in \mathcal{N}_\theta$ and $\varepsilon > 0$. Then we can find $R > 0$ and $K > 0$ such that $\sup_{i\geqslant R} \tau_i < \varepsilon$ and $\tau_i < K$ for all $i = 1,2,\dots$ where $\tau_i = \frac{1}{h_r}\sum_{k\in I_r} A(x_k,x_k,\dots,x_k,x)$. Let we choose an integer m such that $p_{r-1} < m \leqslant p_r$ for $r > R$. So, we get

$$\begin{aligned}
\frac{1}{m}\sum_{k=1}^{m} A(x_k,x_k,\ldots,x_k,x) &\leqslant \frac{1}{p_{r-1}}\sum_{k=1}^{p_r} A(x_k,x_k,\ldots,x_k,x)\\
&= \frac{1}{p_{r-1}}\sum_{I_1} A(x_k,x_k,\ldots,x_k,x) + \frac{1}{p_{r-1}}\sum_{I_2} A(x_k,x_k,\ldots,x_k,x)\\
&\quad + \ldots + \frac{1}{p_{r-1}}\sum_{I_{r-1}} A(x_k,x_k,\ldots,x_k,x)\\
&\quad + \frac{1}{p_{r-1}}\sum_{I_r} A(x_k,x_k,\ldots,x_k,x)\\
&= \frac{p_1}{p_{r-1}}\tau_1 + \frac{p_2-p_1}{p_{r-1}}\tau_2 + \ldots + \frac{p_R-p_{R-1}}{p_{r-1}}\tau_R\\
&\quad + \frac{p_{R+1}-p_R}{p_{r-1}}\tau_{R+1} + \ldots + \frac{p_r-p_{r-1}}{p_{r-1}}\tau_r\\
&\leqslant \left(\sup_{1\leqslant i\leqslant R} \tau_i\right)\frac{p_R}{p_{r-1}} + \left(\sup_{i\geqslant R+1} \tau_i\right)\frac{p_r-p_R}{p_{r-1}}\\
&< K\frac{p_R}{p_{r-1}} + \varepsilon K'.
\end{aligned}$$

Consequencely for $r \to \infty$, we get that $(x_k) \in \sigma_1^A$. □

Combining Lemma 3 and Lemma 4 we have following theorem.

THEOREM 5. *Let (x_k) be a sequence in an A-metric space (X,A) and $\theta = (p_r)$ be any lacunary sequence. If* $1 < \lim_r \inf \frac{p_r}{p_{r-1}} \leqslant \lim_r \sup \frac{p_r}{p_{r-1}} < \infty$. *Then* $\mathscr{N}_\theta = \sigma_1^A$.

Following theorems state the relationships between statistical convergence and lacunary statistical convergence in A-metric spaces.

THEOREM 6. *Let (x_k) be a sequence in an A-metric space (X,A) and $\theta = (p_r)$ be any lacunary sequence. Then the following statements hold:*

(i) *If* $\liminf_r \frac{p_r}{p_{r-1}} > 1$ *then* $S \subseteq S_\theta$.

(ii) *If* $\limsup_r \frac{p_r}{p_{r-1}} < \infty$ *then* $S_\theta \subseteq S$.

(iii) *If* $1 < \lim_r \inf \frac{p_r}{p_{r-1}} \leqslant \lim_r \sup \frac{p_r}{p_{r-1}} < \infty$ *then* $S_\theta = S$.

Proof. We only prove (i). The others can be proved in a similar way used in proving Lemma 4 and Theorem 5.

Let $\liminf_r \frac{p_r}{p_{r-1}} > 1$. Then there exists a $\delta > 0$ such that $\frac{p_r}{p_{r-1}} \geqslant 1+\delta$ for sufficiently large r, which implies that

$$\frac{h_r}{p_r} \geqslant \frac{\delta}{1+\delta}.$$

Let $(x_k) \in S$ and $\varepsilon > 0$, we can write

$$\begin{aligned}\frac{1}{p_r}|\{k \leqslant p_r : A(x_k,x_k,\dots,x_k,x) \geqslant \varepsilon\}| &\geqslant \frac{1}{p_r}|\{k \in I_r : A(x_k,x_k,\dots,x_k,x) \geqslant \varepsilon\}| \\ &\geqslant \frac{\delta}{1+\delta}\Big(\frac{1}{h_r}|\{k \in I_r : A(x_k,x_k,\dots,x_k,x) \geqslant \varepsilon\}|\Big).\end{aligned}$$

Considering that $x_k \xrightarrow{AS} x$ then, we get $x_k \xrightarrow{AS_\theta} x$. Hence, $(x_k) \in S_\theta$. □

THEOREM 7. *Let (x_k) be a sequence in an A-metric space (X,A) and $\theta = (p_r)$ be any lacunary sequence. If $(x_k) \in S \subseteq S_\theta$, then $S_\theta - \lim(x_k) = S - \lim(x_k)$.*

Proof. Assume that $S - \lim(x_k) = x$ and $S_\theta - \lim(x_k) = y$ and $x \neq y$. Then

$$A(x_k,x_k,\dots,x_k,x) \neq 0.$$

From (A3), Lemma 1 and Lemma 2, we can write

$$A(x,x,\dots,x,y) \leqslant (n-1)A(x_k,x_k,\dots,x_k,x) + A(y,y,\dots,y,x_k) \tag{6}$$

we take $\varepsilon < \frac{1}{n}A(x,x,\dots,x,y)$ from inequality (6), we have

$$\lim_{t\to\infty}\frac{1}{t}|\{k \leqslant t : A(x_k,x_k,\dots,x_k,y) \geqslant \varepsilon\}| = 1.$$

Consider the ith term of the statistical limit expression $\lim_{t\to\infty}\frac{1}{t}|\{k \leqslant t : A(x_k,x_k,\dots,x_k,y) \geqslant \varepsilon\}|$:

$$\frac{1}{k_i}\Big|\Big\{k \in \bigcup_{r=1}^{i} I_r : A(x_k,x_k,\dots,x_k,y) \geqslant \varepsilon\Big\}\Big| \tag{7}$$

$$= \frac{1}{k_i}\sum_{r=1}^{i}|\{k \in I_r : A(x_k,x_k,\dots,x_k,y) \geqslant \varepsilon\}| \tag{8}$$

$$= \frac{1}{\sum_{r=1}^{i} h_r}\sum_{r=1}^{i} h_r \frac{1}{h_r}|\{k \in I_r : A(x_k,x_k,\dots,x_k,y) \geqslant \varepsilon\}| \to 0. \tag{9}$$

Since $\theta = (p_r)$ is a lacunary sequence, (7) is a regular weighted mean transformation of the sequence converging to zero, so that itself converges to zero as $i \to \infty$. Also since this is a subsequence of

$$\Big\{\frac{1}{t}|\{k \leqslant t : A(x_k,x_k,\dots,x_k,y) \geqslant \varepsilon\}|\Big\}_t,$$

it follows that

$$\lim_{t\to\infty}\left\{\frac{1}{t}\left|\{k\leqslant t: A(x_k,x_k,\ldots,x_k,y)\geqslant\varepsilon\}\right|\right\}_t\neq 1,$$

and this is a contradiction. In this case we can not take $x\neq y$. □

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