Journal of
Classical
Analysis
Volume 22, Number 2 (2023), 81–111
doi:10.7153/jca-2023-22-07

DISCRETE AND CONTINUOUS WELCH BOUNDS FOR BANACH SPACES WITH APPLICATIONS

K. MAHESH KRISHNA

Abstract. In 1974, Welch derived lower bounds (known as Welch bounds) on the maximum of modulus of inner products of distinct elements in a finite collection of unit vectors in a finite dimensional Hilbert space. Recently, continuous Welch bounds are derived for continuous Bessel family of unit vectors indexed over measure spaces in a finite dimensional Hilbert space. In this paper, we derive both discrete and continuous Welch bounds for finite dimensional Banach spaces which contain Welch bounds for finite dimensional Hilbert space case as a particular case. We formulate several problems for future research.

1. Introduction

Given a collection $\{\tau_j\}_{j=1}^n$ of unit vectors in $\mathbb{C}^d$, using Cauchy-Schwarz inequality we get

$$\max_{1\leqslant j,k\leqslant n, j\neq k} |\langle \tau_j, \tau_k\rangle|^2 \leqslant 1. \tag{1}$$

The natural question which comes immediately is whether there is any lower bound for the quantity max in (1). In his celebrated paper [66], L. Welch proved the following result in 1974.

THEOREM 1. [66] (Welch bounds) *Let $n \geqslant d$. If $\{\tau_j\}_{j=1}^n$ is any collection of unit vectors in $\mathbb{C}^d$, then*

$$\sum_{1\leqslant j,k\leqslant n} |\langle \tau_j, \tau_k\rangle|^{2m} \geqslant \frac{n^2}{\binom{d+m-1}{m}}, \quad \forall m \in \mathbb{N}.$$

Furthermore,

$$\max_{1\leqslant j,k\leqslant n, j\neq k} |\langle \tau_j, \tau_k\rangle|^{2m} \geqslant \frac{1}{n-1}\left[\frac{n}{\binom{d+m-1}{m}} - 1\right], \quad \forall m \in \mathbb{N}. \tag{2}$$

Mathematics subject classification (2020): 42C15.
Keywords and phrases: Banach space, Welch bound, approximate Schauder frame.

Paper JCA-22-07

REMARK 1. For $m=1$, (2) gives

$$\max_{1\leqslant j,k\leqslant n, j\neq k} |\langle \tau_j, \tau_k\rangle|^2 \geqslant \frac{n-d}{d(n-1)}. \tag{3}$$

Inequality (3) is called *first order Welch bound*. For $m \geqslant 2$, family of inequalities in (2) is called *higher order Welch bounds*.

There are several theoretical and practical applications of Theorem 1 such as in the study of root-mean-square (RMS) absolute cross relation of unit vectors [52], frame potential [4,6,10], correlations [51], codebooks [20], numerical search algorithms [67], quantum measurements [53], coding and communications [55, 62], code division multiple access (CDMA) systems [37,38], wireless systems [48], compressed sensing [59], 'game of Sloanes' [32], equiangular tight frames [56], etc.

Following are some of the important connections of Welch bounds to other active areas of research.

(I) Spherical t-designs have direct connection with Welch bounds (see Chapter 6 in [65]). Even though the existence of spherical t-designs is known (see [54]), their exact number is not known. Recently, in a ground breaking work, their asymptotic bounds are derived (see [7]).

(II) Welch bounds are very useful in the study of equiangular lines (see Chapter 12 in [65]). Existence of equiangular lines having a prescribed angle in a given dimension is not known (see [57]). An asymptotic bound is recently derived for equiangular lines (see [33]).

(III) Benedetto and Fickus (see [4]) were able to characterize finite unit norm frames for finite dimensional Hilbert spaces using frame potential which has connection with Welch bounds (see Chapter 6 in [65]). This characterization later led to the development of so called Fundamental Inequality for Finite Frames (see [10]).

(IV) In the context of compressive sensing, Welch bounds play an important role in the construction of matrices with small coherence which uses the inner product (see Chapter 5 in [24]).

In 2003, Waldron [63] derived Welch bounds for vectors which need not have unit norm. In 2016, Datta [18] derived Theorem 1 for fusion frames. In 2017, Waldron [64] improved Theorem 1 for real Hilbert spaces. In 2020, Christensen, Datta and Kim [13] derived first order Welch bound for dual pairs of frames. It is in the paper [17] where the following generalization of Theorem 1 has been done for continuous collections.

THEOREM 2. [17] *Let $\mathbb{CP}^{n-1}$ be the complex projective space and μ be a normalized measure on $\mathbb{CP}^{n-1}$. If $\{\tau_\alpha\}_{\alpha\in\mathbb{CP}^{n-1}}$ is a continuous frame for a d-dimensional subspace $\mathscr{H}$ of a Hilbert space $\mathscr{H}_0$, then*

$$\int_{\mathbb{CP}^{n-1}\times\mathbb{CP}^{n-1}} |\langle \tau_\alpha, \tau_\beta\rangle|^{2m}\, d(\mu\times\mu)(\alpha,\beta) \geqslant \frac{1}{\binom{d+m-1}{m}}, \quad \forall m \in \mathbb{N}.$$

Theorem 2 has been recently generalized in full generality for finite-dimensional Hilbert spaces and σ-finite measure spaces by the author in [40].

THEOREM 3. [40] *Let* (Ω,μ) *be a measure space and* $\{\tau_\alpha\}_{\alpha\in\Omega}$ *be a normalized continuous Bessel family for* $\mathscr{H}$ *of dimension* d. *If the diagonal* $\Delta:=\{(\alpha,\alpha):\alpha\in\Omega\}$ *is measurable in the measure space* $\Omega\times\Omega$, *then*

$$\int_{\Omega\times\Omega}|\langle\tau_\alpha,\tau_\beta\rangle|^{2m}\,d(\mu\times\mu)(\alpha,\beta)\geqslant\frac{\mu(\Omega)^2}{\binom{d+m-1}{m}},\quad\forall m\in\mathbb{N}.$$

Furthermore, we have the higher order continuous Welch bounds

$$\sup_{\alpha,\beta\in\Omega,\alpha\neq\beta}|\langle\tau_\alpha,\tau_\beta\rangle|^{2m}\geqslant\frac{1}{(\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)}\left[\frac{\mu(\Omega)^2}{\binom{d+m-1}{m}}-(\mu\times\mu)(\Delta)\right],\quad\forall m\geqslant 2$$

and the first order continuous Welch bound

$$\sup_{\alpha,\beta\in\Omega,\alpha\neq\beta}|\langle\tau_\alpha,\tau_\beta\rangle|^{2}\geqslant\frac{1}{(\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)}\left[\frac{\mu(\Omega)^2}{d}-(\mu\times\mu)(\Delta)\right].$$

Recently, Theorem 1 and Theorem 3 have been proved in the context of Hilbert C*-modules by the author in [39].

In this paper we derive both discrete and continuous Welch bounds for Banach spaces. We pose several open problems for further research. Following are important motivations for this paper.

(I) Starting from the theory of Schauder bases, biorthogonal systems are studied in Banach spaces but not systems which are not biorthogonal (see [29]).

(II) Recently, Chávez-Domínguez, Freeman and Kornelson constructed a very interesting collection of vectors and functionals on a finite dimensional Banach space and showed that a new way of thinking is required even in finite dimensional Banach spaces than finite dimensional Hilbert spaces (see Proposition 2.5 and its proof in [12]).

2. Discrete Welch bounds for Banach spaces

Throughout the paper, $\mathscr{X}$ denotes a finite dimensional Banach space and $\mathscr{X}^*$ denotes its dual. $I_{\mathscr{X}}$ denotes the identity operator on $\mathscr{X}$. We use $\mathbb{K}$ to denote $\mathbb{R}$ or $\mathbb{C}$.

DEFINITION 1. Let $\mathscr{X}$ be a finite dimensional Banach space. Given a collection $\{\tau_j\}_{j=1}^n$ in $\mathscr{X}$ and a collection $\{f_j\}_{j=1}^n$ in $\mathscr{X}^*$,

(i) The *frame operator* is defined as

$$S_{f,\tau} : \mathscr{X} \ni x \mapsto S_{f,\tau}x := \sum_{j=1}^{n} f_j(x)\tau_j \in \mathscr{X}.$$

(ii) The *analysis operator* is defined as

$$\theta_f : \mathscr{X} \ni x \mapsto \theta_f x := (f_j(x))_{j=1}^{n} \in \mathbb{K}^n.$$

(iii) The *synthesis operator* is defined as

$$\theta_\tau : \mathbb{K}^n \ni (a_j)_{j=1}^{n} \mapsto \theta_\tau (a_j)_{j=1}^{n} := \sum_{j=1}^{n} a_j\tau_j \in \mathscr{X}.$$

Using direct computation we see that the frame operator factorizes as $S_{f,\tau} = \theta_\tau\theta_f$.

DEFINITION 2. [25, 60] Let $\{\tau_j\}_{j=1}^{n}$ be a collection in $\mathscr{X}$ and $\{f_j\}_{j=1}^{n}$ be a collection in $\mathscr{X}^*$. The pair $(\{f_j\}_{j=1}^{n}, \{\tau_j\}_{j=1}^{n})$ is said to be an *approximate Schauder frame (ASF)* for $\mathscr{X}$ if the map

$$S_{f,\tau} : \mathscr{X} \ni x \mapsto S_{f,\tau}x := \sum_{j=1}^{n} f_j(x)\tau_j \in \mathscr{X}.$$

is invertible. If $S_{f,\tau} = \lambda I_{\mathscr{X}}$, for some non zero scalar λ, then $(\{f_j\}_{j=1}^{n}, \{\tau_j\}_{j=1}^{n})$ is called a *tight ASF* for $\mathscr{X}$.

Following theorem says that we can recover the trace of frame operator using ASFs.

THEOREM 4. *Given a collection $\{\tau_j\}_{j=1}^{n}$ in $\mathscr{X}$ and a collection $\{f_j\}_{j=1}^{n}$ in $\mathscr{X}^*$, we have*

$$\operatorname{Tra}(S_{f,\tau}) = \sum_{j=1}^{n} f_j(\tau_j),$$

$$\operatorname{Tra}(S_{f,\tau}^2) = \sum_{j=1}^{n}\sum_{k=1}^{n} f_j(\tau_k)f_k(\tau_j).$$

Proof. From the expression of $S_{f,\tau}$ and from the definition of trace of operator in Banach space (see [34]) we get

$$\operatorname{Tra}(S_{f,\tau}) = \sum_{j=1}^{n} f_j(\tau_j).$$

Now

$$S_{f,\tau}^2 x = \sum_{j=1}^n f_j(x)(S_{f,\tau}\tau_j), \quad \forall x \in \mathscr{X}.$$

Again using the definition of trace we get

$$\operatorname{Tra}(S_{f,\tau}^2) = \sum_{j=1}^n f_j(S_{f,\tau}\tau_j) = \sum_{j=1}^n f_j\left(\sum_{k=1}^n f_k(\tau_j)\tau_k\right) = \sum_{j=1}^n\sum_{k=1}^n f_j(\tau_k)f_k(\tau_j). \quad \square$$

Now we can derive the first important result of the paper.

THEOREM 5. (First order Welch bound for Banach spaces) *Let* $\{\tau_j\}_{j=1}^n$ *be a collection in a finite dimensional Banach space* $\mathscr{X}$ *of dimension* d *and* $\{f_j\}_{j=1}^n$ *be a collection in* $\mathscr{X}^*$*. Let* $n \geqslant d$*. If the operator* $S_{f,\tau} : \mathscr{X} \ni x \mapsto S_{f,\tau}x := \sum_{j=1}^n f_j(x)\tau_j \in \mathscr{X}$ *is diagonalizable and its eigenvalues are all non negative, then*

$$\sum_{1\leqslant j,k\leqslant n} f_j(\tau_k)f_k(\tau_j) \geqslant \frac{1}{d}\left(\sum_{j=1}^n f_j(\tau_j)\right)^2 \tag{4}$$

and

$$\max_{1\leqslant j,k\leqslant n, j\neq k} |f_j(\tau_k)| \geqslant \sqrt{\frac{\frac{1}{d}\left(\sum_{j=1}^n f_j(\tau_j)\right)^2 - \sum_{j=1}^n |f_j(\tau_j)|^2}{n^2-n}}.$$

Furthermore, equality holds in (4) *if and only if* $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ *is tight ASF for* $\mathscr{X}$*. In particular, if* $f_j(\tau_j) = 1$ *for all* $1 \leqslant j \leqslant n$*, then*

$$\max_{1\leqslant j,k\leqslant n, j\neq k} |f_j(\tau_k)f_k(\tau_j)| \geqslant \frac{n-d}{d(n-1)}$$

and we have first order (discrete) Welch bound for Banach spaces

$$\max_{1\leqslant j,k\leqslant n, j\neq k} |f_j(\tau_k)| \geqslant \sqrt{\frac{n-d}{d(n-1)}}.$$

Proof. Let $\lambda_1,\dots,\lambda_d$ be eigenvalues of $S_{f,\tau}$. Then $\lambda_1,\dots,\lambda_d \geqslant 0$ and using the diagonalizability of $S_{f,\tau}$ we get

$$\begin{aligned}\left(\sum_{j=1}^n f_j(\tau_j)\right)^2 &= (\operatorname{Tra}(S_{f,\tau}))^2 = \left(\sum_{k=1}^d \lambda_k\right)^2 \leqslant d\sum_{k=1}^d \lambda_k^2\\ &= d\operatorname{Tra}(S_{f,\tau}^2) = d\sum_{j=1}^n\sum_{k=1}^n f_j(\tau_k)f_k(\tau_j).\end{aligned} \tag{5}$$

For the second inequality,

$$\begin{aligned}\frac{1}{d}\left(\sum_{j=1}^{n} f_j(\tau_j)\right)^2 &\leqslant \sum_{j=1}^{n}\sum_{k=1}^{n} f_j(\tau_k)f_k(\tau_j) = \sum_{j=1}^{n} f_j(\tau_j)^2 + \sum_{j,k=1,j\neq k}^{n} f_j(\tau_k)f_k(\tau_j)\\ &\leqslant \sum_{j=1}^{n} |f_j(\tau_j)|^2 + \sum_{j,k=1,j\neq k}^{n} |f_j(\tau_k)f_k(\tau_j)|\\ &\leqslant \sum_{j=1}^{n} |f_j(\tau_j)|^2 + (n^2-n)\max_{1\leqslant j,k\leqslant n, j\neq k} |f_j(\tau_k)f_k(\tau_j)|\end{aligned}$$

which gives

$$\max_{1\leqslant j,k\leqslant n, j\neq k} |f_j(\tau_k)f_k(\tau_j)| \geqslant \frac{\frac{1}{d}\left(\sum_{j=1}^{n} f_j(\tau_j)\right)^2 - \sum_{j=1}^{n} |f_j(\tau_j)|^2}{n^2-n}. \tag{6}$$

Now let $1 \leqslant j,k \leqslant n, j \neq k$ be fixed. Then

$$\begin{aligned}|f_j(\tau_k)f_k(\tau_j)| &\leqslant \max_{1\leqslant j,k\leqslant n, j\neq k} |f_j(\tau_k)| \max_{1\leqslant j,k\leqslant n, j\neq k} |f_k(\tau_j)|\\ &= \max_{1\leqslant j,k\leqslant n, j\neq k} |f_j(\tau_k)| \max_{1\leqslant j,k\leqslant n, j\neq k} |f_j(\tau_k)|\\ &= \left(\max_{1\leqslant j,k\leqslant n, j\neq k} |f_j(\tau_k)|\right)^2.\end{aligned}$$

Therefore

$$\max_{1\leqslant j,k\leqslant n, j\neq k} |f_j(\tau_k)f_k(\tau_j)| \leqslant \left(\max_{1\leqslant j,k\leqslant n, j\neq k} |f_j(\tau_k)|\right)^2.$$

Using (6) we now get

$$\max_{1\leqslant j,k\leqslant n, j\neq k} |f_j(\tau_k)| \geqslant \sqrt{\frac{\frac{1}{d}\left(\sum_{j=1}^{n} f_j(\tau_j)\right)^2 - \sum_{j=1}^{n} |f_j(\tau_j)|^2}{n^2-n}}. \tag{7}$$

Whenever $f_j(\tau_j) = 1$ for all $1 \leqslant j \leqslant n$, Inequality (6) gives

$$\max_{1\leqslant j,k\leqslant n, j\neq k} |f_j(\tau_k)f_k(\tau_j)| \geqslant \frac{\frac{n^2}{d}-n}{n^2-n} = \frac{n-d}{d(n-1)}$$

and (7) gives

$$\max_{1\leqslant j,k\leqslant n, j\neq k} |f_j(\tau_k)| \geqslant \sqrt{\frac{n-d}{d(n-1)}}.$$

Equality holds in Inequality (5) if and only if

$$\left(\sum_{k=1}^{d}\lambda_k\right)^2=\left(\sum_{k=1}^{d}1\right)\left(\sum_{k=1}^{d}\lambda_k^2\right)$$

if and only if

$$\lambda_k=a,\ \text{for some } a>0, \forall 1\leqslant k\leqslant n$$

if and only if $S_{f,\tau}$ is a tight ASF for $\mathscr{X}$. $\square$

Given a finite collection $\{\tau_j\}_{j=1}^n$ in a finite dimensional Hilbert space $\mathscr{H}$, the operator

$$S_\tau:\mathscr{H}\ni h\mapsto S_\tau h:=\sum_{j=1}^{n}\langle h,\tau_j\rangle\tau_j\in\mathscr{H}$$

is positive definite and hence diagonalizable and all eigenvalues are non negative. But we cannot say the same for $S_{f,\tau}$. Even in finite dimensions, frame operator S_τ, being a sum of rank one positive operators, is positive so that diagonalization is ensured. In Banach spaces (even finite dimensional) the frame operator $S_{f,\tau}$ need not be a sum of positive operators. Hence $S_{f,\tau}$ may not be diagonalizable. Thus even finite dimensional Banach space frame theory differs from (finite dimensional) Hilbert space frame theory. This is also the reason for additional assumptions in the statement of Theorem 5. We next derive higher order Welch bounds for Banach spaces. For this we need the notion of symmetric tensors. For this we need the concept of vector space of symmetric tensors. Given a vector space $\mathscr{V}$ of dimension d, let $\mathscr{V}^{\otimes m}$ be the vector space of m-tensors. A vector

$$\sum_{j=1}^{n}x_{j,1}\otimes\cdots\otimes x_{j,m}\in\mathscr{V}^{\otimes m}$$

is said to be symmetric if for every bijection $\sigma:\{1,\dots,m\}\to\{1,\dots,m\}$, we have

$$\sum_{j=1}^{n}x_{j,\sigma(1)}\otimes\cdots\otimes x_{j,\sigma(m)}=\sum_{j=1}^{n}x_{j,1}\otimes\cdots\otimes x_{j,m}.$$

Set of all symmetric m-tensors will form a vector space, denoted by $\operatorname{Sym}^m(\mathscr{V})$. It is known that (see [5, 14])

$$\dim(\operatorname{Sym}^m(\mathscr{V}))=\binom{d+m-1}{m},\quad \forall m\in\mathbb{N}.$$

THEOREM 6. (Welch bounds for Banach spaces) *Let $\{\tau_j\}_{j=1}^n$ be a collection in a finite dimensional Banach space $\mathscr{X}$ of dimension d and $\{f_j\}_{j=1}^n$ be a collection in $\mathscr{X}^*$. Let $n\geqslant d$ and $m\in\mathbb{N}$. If the operator $S_{f,\tau}:\operatorname{Sym}^m(\mathscr{X})\ni x\mapsto S_{f,\tau}x:=$*

$\sum_{j=1}^n f_j^{\otimes m}(x)\tau_j^{\otimes m} \in Sym^m(\mathscr{X})$ is diagonalizable and its eigenvalues are all non negative, then

$$\sum_{1\leqslant j,k\leqslant n} f_j(\tau_k)^m f_k(\tau_j)^m \geqslant \frac{1}{\binom{d+m-1}{m}}\left(\sum_{j=1}^n f_j(\tau_j)^m\right)^2 \tag{8}$$

and

$$\max_{1\leqslant j,k\leqslant n, j\neq k} |f_j(\tau_k)|^m \geqslant \sqrt{\frac{\frac{1}{\binom{d+m-1}{m}}\left(\sum_{j=1}^n f_j(\tau_j)^m\right)^2 - \sum_{j=1}^n |f_j(\tau_j)|^{2m}}{n^2-n}}.$$

Furthermore, equality holds in (8) if and only if $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ is a tight ASF for $Sym^m(\mathscr{X})$. In particular, if $f_j(\tau_j)=1$ for all $1\leqslant j\leqslant n$, then

$$\max_{1\leqslant j,k\leqslant n, j\neq k} |f_j(\tau_k)f_k(\tau_j)|^m \geqslant \frac{n-\binom{d+m-1}{m}}{\binom{d+m-1}{m}(n-1)} == \frac{1}{n-1}\left[\frac{n}{\binom{d+m-1}{m}}-1\right]$$

and we have (discrete) Welch bounds for Banach spaces

$$\max_{1\leqslant j,k\leqslant n, j\neq k} |f_j(\tau_k)|^m \geqslant \sqrt{\frac{n-\binom{d+m-1}{m}}{\binom{d+m-1}{m}(n-1)}} = \sqrt{\frac{1}{n-1}\left[\frac{n}{\binom{d+m-1}{m}}-1\right]}. \tag{9}$$

Proof. We will do the proof of Theorem 5 for the space $\text{Sym}^m(\mathscr{X})$ (we refer to [50] for the tensor product of Banach spaces). Let $\lambda_1,\dots,\lambda_{\dim(\text{Sym}^m(\mathscr{X}))}$ be eigenvalues of $S_{f,\tau}$. Then

$$\begin{aligned}\left(\sum_{j=1}^n f_j(\tau_j)^m\right)^2 &= \left(\sum_{j=1}^n f_j^{\otimes m}(\tau_j^{\otimes m})\right)^2 = (\text{Tra}(S_{f,\tau}))^2 = \left(\sum_{l=1}^{\dim(\text{Sym}^m(\mathscr{X}))}\lambda_l\right)^2\\ &\leqslant \dim(\text{Sym}^m(\mathscr{X}))\sum_{l=1}^{\dim(\text{Sym}^m(\mathscr{X}))}\lambda_l^2 = \binom{d+m-1}{m}\text{Tra}(S_{f,\tau}^2)\\ &= \binom{d+m-1}{m}\sum_{j=1}^n\sum_{l=1}^n f_j^{\otimes m}(\tau_l^{\otimes m})f_l^{\otimes m}(\tau_j^{\otimes m})\\ &= \binom{d+m-1}{m}\sum_{j=1}^n\sum_{k=1}^n f_j(\tau_k)^m f_k(\tau_j)^m\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\binom{d+m-1}{m}}\left(\sum_{j=1}^{n} f_j(\tau_j)^m\right)^2 &= \sum_{1\leqslant j,k\leqslant n} f_j(\tau_k)^m f_k(\tau_j)^m \\ &= \sum_{1\leqslant j,k\leqslant n, j\neq k} f_j(\tau_k)^m f_k(\tau_j)^m + \sum_{j=1}^{n} f_j(\tau_j)^{2m} \\ &\leqslant \sum_{1\leqslant j,k\leqslant n, j\neq k} |f_j(\tau_k) f_k(\tau_j)|^m + \sum_{j=1}^{n} |f_j(\tau_j)|^{2m} \\ &\leqslant (n^2-n) \max_{1\leqslant j,k\leqslant n, j\neq k} |f_j(\tau_k) f_k(\tau_j)|^m + \sum_{j=1}^{n} |f_j(\tau_j)|^{2m}.\end{aligned}$$

Other parts are similar to the corresponding part in the proof of Theorem 5. □

REMARK 2. For $m \geqslant 2$, we call family of inequalities in (9) as *(discrete) higher order Welch bounds*.

REMARK 3. Note that Theorem 1 is a corollary of Theorem 6. In fact, Let $\{\tau_j\}_{j=1}^n$ be a finite collection in a finite dimensional Hilbert space $\mathscr{H}$ of dimension d. Define $f_j : \mathscr{H} \ni h \mapsto \langle h, \tau_j\rangle \in \mathbb{K}$, $\forall 1 \leqslant j \leqslant n$. Let $m \in \mathbb{N}$. Then the operator

$$S_\tau : \mathrm{Sym}^m(\mathscr{H}) \ni h \mapsto \sum_{j=1}^{n} f_j^{\otimes m}(h)\tau_j^{\otimes m} = \sum_{j=1}^{n} \langle h, \tau_j^{\otimes m}\rangle \tau_j^{\otimes m} \in \mathrm{Sym}^m(\mathscr{H})$$

is positive definite and we can apply Theorem 6.

Theorem 6 gives Welch bound for all natural numbers. One can now ask whether we can replace naturals by positive reals. Following results show that we can do this. For normalized tight frames for Hilbert spaces, these results are derived in [22] and [28].

THEOREM 7. *Let $\{\tau_j\}_{j=1}^n$ be a collection in a finite dimensional Banach space $\mathscr{X}$ of dimension d and $\{f_j\}_{j=1}^n$ be a collection in $\mathscr{X}^*$. Let $n \geqslant d$. If the operator $S_{f,\tau} : \mathscr{X} \ni x \mapsto S_{f,\tau}x := \sum_{j=1}^n f_j(x)\tau_j \in \mathscr{X}$ is diagonalizable and its eigenvalues are all non negative, then*

$$\mathrm{Tra}(S_{f,\tau}^r) \geqslant \frac{\left(\sum_{j=1}^n f_j(\tau_j)\right)^r}{d^{r-1}}, \quad \forall r \in [1,\infty)$$

and

$$\mathrm{Tra}(S_{f,\tau}^r) \leqslant \frac{\left(\sum_{j=1}^n f_j(\tau_j)\right)^r}{d^{r-1}}, \quad \forall r \in (0,1).$$

Proof. Let $\lambda_1,\dots,\lambda_d$ be eigenvalues of $S_{f,\tau}$. Let $r\in[1,\infty)$. Using Jensen's inequality, we have

$$\left(\frac{\sum_{j=1}^n f_j(\tau_j)}{d}\right)^r=\left(\frac{\operatorname{Tra}(S_{f,\tau})}{d}\right)^r=\left(\frac{\sum_{k=1}^d \lambda_k}{d}\right)^r\leqslant\frac{\sum_{k=1}^d \lambda_k^r}{d}=\frac{1}{d}\operatorname{Tra}(S_{f,\tau}^r)$$

which gives the first part. Second part again follows from Jensen's inequality. □

REMARK 4. If $f_j(\tau_j)=1$ for all $1\leqslant j\leqslant n$, then Theorem 7 says that

$$\operatorname{Tra}(S_{f,\tau}^r)\geqslant\frac{n^r}{d^{r-1}},\quad \forall r\in[1,\infty)$$

and

$$\operatorname{Tra}(S_{f,\tau}^r)\leqslant\frac{n^r}{d^{r-1}},\quad \forall r\in(0,1).$$

THEOREM 8. *Let $\{\tau_j\}_{j=1}^n$ be a collection in a finite dimensional Banach space $\mathscr{X}$ of dimension d and $\{f_j\}_{j=1}^n$ be a collection in $\mathscr{X}^*$. Let $2<p<\infty$. If the operator $S_{f,\tau}:\mathscr{X}\ni x\mapsto S_{f,\tau}x:=\sum_{j=1}^n f_j(x)\tau_j\in\mathscr{X}$ is diagonalizable and its eigenvalues are all non negative, then*

$$\sum_{1\leqslant j,k\leqslant n}|f_j(\tau_k)f_k(\tau_j)|^{\frac{p}{2}}\geqslant n(n-1)\left(\frac{n-d}{d(n-1)}\right)^{\frac{p}{2}}+n.$$

Proof. Define $r:=2p/(p-2)$ and q be the conjugate index of $p/2$. Then $q=r/2$. Using Theorem 5 and Holder's inequality, we have

$$\begin{aligned}\frac{n^2}{d}-n&\leqslant\sum_{1\leqslant j,k\leqslant n,j\neq k}|f_j(\tau_k)f_k(\tau_j)|\\&\leqslant\left(\sum_{1\leqslant j,k\leqslant n,j\neq k}|f_j(\tau_k)f_k(\tau_j)|^{\frac{p}{2}}\right)^{\frac{2}{p}}\left(\sum_{1\leqslant j,k\leqslant n,j\neq k}1\right)^{\frac{1}{q}}\\&=\left(\sum_{1\leqslant j,k\leqslant n,j\neq k}|f_j(\tau_k)f_k(\tau_j)|^{\frac{p}{2}}\right)^{\frac{2}{p}}(n^2-n)^{\frac{1}{q}}\\&=\left(\sum_{1\leqslant j,k\leqslant n,j\neq k}|f_j(\tau_k)f_k(\tau_j)|^{\frac{p}{2}}\right)^{\frac{2}{p}}(n^2-n)^{\frac{2}{r}}\\&=\left(\sum_{1\leqslant j,k\leqslant n,j\neq k}|f_j(\tau_k)f_k(\tau_j)|^{\frac{p}{2}}\right)^{\frac{2}{p}}(n^2-n)^{\frac{p-2}{p}}\end{aligned}$$

which gives

$$\left(\frac{n^2}{d}-n\right)^{\frac{p}{2}} \leqslant \left(\sum_{1\leqslant j,k\leqslant n, j\neq k} |f_j(\tau_k)f_k(\tau_j)|^{\frac{p}{2}}\right)(n^2-n)^{\frac{p}{2}-1}.$$

Therefore

$$\begin{aligned} n(n-1)\left(\frac{n-d}{d(n-1)}\right)^{\frac{p}{2}}+n &= \frac{1}{(n^2-n)^{\frac{p}{2}-1}}\left(\frac{n^2}{d}-n\right)^{\frac{p}{2}}+n \\ &\leqslant \sum_{1\leqslant j,k\leqslant n, j\neq k} |f_j(\tau_k)f_k(\tau_j)|^{\frac{p}{2}} + \sum_{j=1}^{n} |f_j(\tau_j)f_j(\tau_j)|^{\frac{p}{2}} \\ &= \sum_{1\leqslant j,k\leqslant n} |f_j(\tau_k)f_k(\tau_j)|^{\frac{p}{2}}. \quad \square \end{aligned}$$

Some of the proofs of Theorem 6 (for instance see [49]) use the Gram matrix and Frobenius norm/Hilbert-Schmidt norm. We now give Welch bound for Banach spaces using matrices. First we need a definition.

DEFINITION 3. Let $\{\tau_j\}_{j=1}^n$ be a collection in a Banach space $\mathscr{X}$ and $\{f_j\}_{j=1}^n$ be a collection in $\mathscr{X}^*$. We define the *Gram matrix* $G_{f,\tau}$ of $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ as

$$G_{f,\tau} := [f_j(\tau_k)]_{1\leqslant j,k\leqslant n} = \begin{pmatrix} f_1(\tau_1) & f_1(\tau_2) & \cdots & f_1(\tau_n) \\ f_2(\tau_1) & f_2(\tau_2) & \cdots & f_2(\tau_n) \\ \vdots & \vdots & & \vdots \\ f_n(\tau_1) & f_n(\tau_2) & \cdots & f_n(\tau_n) \end{pmatrix}_{n\times n} \in \mathbb{M}_n(\mathbb{K}).$$

In terms of analysis and synthesis operators, $G_{f,\tau} = \theta_f\theta_\tau$. We observe that Definition 3 reduces to the definition of Gram matrix in Hilbert spaces. Indeed, let $\{\tau_j\}_{j=1}^n$ be a collection in a Hilbert space $\mathscr{H}$. Now define $f_j(h) := \langle h, \tau_j\rangle$ for all $h \in \mathscr{H}$, for all $1 \leqslant j \leqslant n$.

THEOREM 9. *Let $\{\tau_j\}_{j=1}^n$ be a collection in a Banach space $\mathscr{X}$ and $\{f_j\}_{j=1}^n$ be a collection in $\mathscr{X}^*$. If the Gram matrix $G_{f,\tau} = [f_j(\tau_k)]_{1\leqslant j,k\leqslant n}$ is diagonalizable and its eigenvalues are all non negative, then*

$$\sum_{1\leqslant j,k\leqslant n} f_j(\tau_k)f_k(\tau_j) \geqslant \frac{1}{rank(G_{f,\tau})}\left(\sum_{j=1}^{n} f_j(\tau_j)\right)^2$$

and

$$\max_{1\leqslant j,k\leqslant n, j\neq k} |f_j(\tau_k)| \geqslant \sqrt{\frac{\frac{1}{rank(G_{f,\tau})}\left(\sum_{j=1}^{n} f_j(\tau_j)\right)^2 - \sum_{j=1}^{n} |f_j(\tau_j)|^2}{n^2-n}}.$$

In particular, if $f_j(\tau_j)=1$ *for all* $1\leqslant j\leqslant n$, *then*

$$\max_{1\leqslant j,k\leqslant n, j\neq k}|f_j(\tau_k)f_k(\tau_j)|\geqslant \frac{n-rank(G_{f,\tau})}{rank(G_{f,\tau})(n-1)}$$

and

$$\max_{1\leqslant j,k\leqslant n, j\neq k}|f_j(\tau_k)|\geqslant \sqrt{\frac{n-rank(G_{f,\tau})}{rank(G_{f,\tau})(n-1)}}.$$

Proof. Let $\lambda_1,\dots,\lambda_{(\text{rank}G_{f,\tau})}$ be the eigenvalues of $G_{f,\tau}$. Then

$$\begin{aligned}\left(\sum_{j=1}^{n} f_j(\tau_j)\right)^2 &= (\text{Trace}(G_{f,\tau}))^2=\left(\sum_{k=1}^{\text{rank}(G_{f,\tau})}\lambda_k\right)^2\leqslant \text{rank}(G_{f,\tau})\sum_{k=1}^{\text{rank}(G_{f,\tau})}\lambda_k^2\\ &=\text{rank}(G_{f,\tau})\,\text{Trace}(G_{f,\tau}^2)=\text{rank}(G_{f,\tau})\sum_{j=1}^{n}\sum_{k=1}^{n}f_j(\tau_k)f_k(\tau_j)\\ &\leqslant \text{rank}(G_{f,\tau})\left(\sum_{1\leqslant j,k\leqslant n, j\neq k}|f_j(\tau_k)f_k(\tau_j)|+\sum_{j=1}^{n}|f_j(\tau_j)|^2\right)\\ &\leqslant \text{rank}(G_{f,\tau})\left((n^2-n)\max_{1\leqslant j,k\leqslant n, j\neq k}|f_j(\tau_k)f_k(\tau_j)|+\sum_{j=1}^{n}|f_j(\tau_j)|^2\right).\quad\square\end{aligned}$$

Given a finite collection $\{\tau_j\}_{j=1}^n$ in a Hilbert space $\mathscr{H}$, the Gram matrix

$$[\langle\tau_k,\tau_j\rangle]_{1\leqslant j,k\leqslant n}\in\mathbb{M}_n(\mathbb{K})$$

is always positive definite and hence diagonalizable and all eigenvalues are non negative. For Banach spaces, the Gram matrix $G_{f,\tau}$ need not be positive definite. Therefore we imposed conditions on $G_{f,\tau}$ in the statement of Theorem 9. In the following result, given a matrix G, $G^{\circ m}$ denotes the Hadamard/Schur/pointwise product of G with itself, m times [31].

THEOREM 10. *Let* $\{\tau_j\}_{j=1}^n$ *be a collection in a Banach space* $\mathscr{X}$ *and* $\{f_j\}_{j=1}^n$ *be a collection in* $\mathscr{X}^*$. *Let* $m\in\mathbb{N}$. *If the Hadamard product* $G_{f,\tau}^{\circ m}$ *is diagonalizable and its eigenvalues are all non negative, then*

$$\sum_{1\leqslant j,k\leqslant n}f_j(\tau_k)^m f_k(\tau_j)^m=\sum_{j=1}^{n}\sum_{k=1}^{n}f_j(\tau_k)^m f_k(\tau_j)^m\geqslant\frac{1}{rank(G_{f,\tau}^{\circ m})}\left(\sum_{j=1}^{n}f_j(\tau_j)^m\right)^2.$$

and

$$\max_{1\leqslant j,k\leqslant n, j\neq k}|f_j(\tau_k)f_k(\tau_j)|^m\geqslant\frac{\frac{1}{rank(G_{f,\tau}^{\circ m})}\left(\sum_{j=1}^{n}f_j(\tau_j)^m\right)^2-\sum_{j=1}^{n}|f_j(\tau_j)|^{2m}}{n^2-n},$$

$$\max_{1\leqslant j,k\leqslant n, j\neq k}|f_j(\tau_k)|^m \geqslant \sqrt{\frac{\frac{1}{rank(G^{\circ m}_{f,\tau})}\left(\sum_{j=1}^n f_j(\tau_j)^m\right)^2-\sum_{j=1}^n |f_j(\tau_j)|^{2m}}{n^2-n}}.$$

In particular, if $f_j(\tau_j)=1$ *for all* $1\leqslant j\leqslant n$*, then*

$$\max_{1\leqslant j,k\leqslant n, j\neq k}|f_j(\tau_k)f_k(\tau_j)|^m \geqslant \frac{n-rank(G^{\circ m}_{f,\tau})}{rank(G^{\circ m}_{f,\tau})(n-1)}$$

and

$$\max_{1\leqslant j,k\leqslant n, j\neq k}|f_j(\tau_k)|^m \geqslant \sqrt{\frac{n-rank(G^{\circ m}_{f,\tau})}{rank(G^{\circ m}_{f,\tau})(n-1)}}.$$

Proof. We note that

$$G^{\circ m}_{f,\tau}=[(f_j(\tau_k))^m]_{1\leqslant j,k\leqslant n}=[(f_j^{\otimes m}(\tau_k^{\otimes m}))]_{1\leqslant j,k\leqslant n}.$$

Now the proof is similar to the proof of Theorem 9. □

3. Applications of discrete Welch bounds for Banach spaces

We begin by defining the RMS of vectors and functionals in Banach spaces.

DEFINITION 4. Let $\{\tau_j\}_{j=1}^n$ be a collection in a finite dimensional Banach space $\mathscr{X}$ of dimension d and $\{f_j\}_{j=1}^n$ be a collection in $\mathscr{X}^*$ satisfying $f_j(\tau_j)=1$ for all $1\leqslant j\leqslant n$. Assume that the frame operator $S_{f,\tau}$ is diagonalizable and its eigenvalues are all non negative. We define the *root-mean-square (RMS) absolute cross relation* of $(\{f_j\}_{j=1}^n,\{\tau_j\}_{j=1}^n)$ as

$$I_{\mathrm{RMS}}(\{f_j\}_{j=1}^n,\{\tau_j\}_{j=1}^n):=\left(\frac{1}{n(n-1)}\sum_{1\leqslant j,k\leqslant n, j\neq k} f_j(\tau_k)f_k(\tau_j)\right)^{\frac{1}{2}}.$$

Theorem 5 gives the following result.

THEOREM 11. *Let* $\{\tau_j\}_{j=1}^n$ *and* $\{f_j\}_{j=1}^n$ *be as in Definition* 4*. Then*

$$\max_{1\leqslant j,k\leqslant n, j\neq k}|f_j(\tau_k)|\geqslant I_{RMS}(\{f_j\}_{j=1}^n,\{\tau_j\}_{j=1}^n)\geqslant\left(\frac{n-d}{d(n-1)}\right)^{\frac{1}{2}}.$$

Here is another notion similar to that of frame potential in Hilbert spaces.

DEFINITION 5. Let $\{\tau_j\}_{j=1}^n$ be a collection in a finite dimensional Banach space $\mathscr{X}$ of dimension d and $\{f_j\}_{j=1}^n$ be a collection in $\mathscr{X}^*$ satisfying $f_j(\tau_j)=1$ for all $1\leqslant j\leqslant n$. Assume that the frame operator $S_{f,\tau}$ is diagonalizable and its eigenvalues are all non negative. We define the *pseudo frame potential* of $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ as

$$PFP(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n) := \sum_{j=1}^{n}\sum_{k=1}^{n} f_j(\tau_k)f_k(\tau_j).$$

Note that we defined the notion pseudo frame potential and not frame potential. The reason is that frame potential for Banach spaces cannot be defined in the way in Definition 5. One has to go to the theory of p-summing operators (see [19,61]) to define frame potential in Banach spaces, see [12]. Theorem 5 again gives the following result.

THEOREM 12. *Let $\{\tau_j\}_{j=1}^n$ and $\{f_j\}_{j=1}^n$ be as in Definition 5. Then*

$$n^2 \max_{1\leqslant j,k\leqslant n} |f_j(\tau_k)|^2 \geqslant PFP(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n) \geqslant \frac{n^2}{d}. \tag{10}$$

We next introduce the notions of Grassmannian frames and equiangular frames for Banach spaces. First we need a definition.

DEFINITION 6. Let $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ be an ASF for $\mathscr{X}$ satisfying $f_j(\tau_j)=1$, $\forall 1\leqslant j\leqslant n$. Assume that the frame operator $S_{f,\tau}$ is diagonalizable and its eigenvalues are all non negative. We define the *frame correlation* of $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ as

$$\mathscr{M}(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n) := \max_{1\leqslant j,k\leqslant n, j\neq k} |f_j(\tau_k)|.$$

DEFINITION 7. Let $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ be an ASF for $\mathscr{X}$ satisfying $\|f_j\|=1$, $\|\tau_j\|=1$, $f_j(\tau_j)=1$, $\forall 1\leqslant j\leqslant n$. Assume that the frame operator $S_{f,\tau}$ is diagonalizable and its eigenvalues are all non negative. ASF $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ is said to be a *Grassmannian frame* for $\mathscr{X}$ if

$$\mathscr{M}(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n) = \inf\Big\{\mathscr{M}(\{g_j\}_{j=1}^n, \{\omega_j\}_{j=1}^n) : (\{g_j\}_{j=1}^n, \{\omega_j\}_{j=1}^n) \text{ is an ASF for } \mathscr{X} \text{ satisfying } \|g_j\|=1, \|\omega_j\|=1, g_j(\omega_j)=1, \forall 1\leqslant j\leqslant n \text{ and the frame operator } S_{g,\omega} \text{is diagonalizable and its eigenvalues are all non negative}\Big\}.$$

THEOREM 13. *Grassmannian frames exist in every dimension for every Banach space.*

Proof. Our arguments are motivated by the arguments given in [3] for Hilbert spaces. We give arguments only for real Banach spaces and complex case follows by considering real and imaginary parts (of the linear functionals). Define

$$\begin{aligned}
&S^n_{\mathscr{X}} := \{(x_1,\dots,x_n) : x_1,\dots,x_n \in \mathscr{X}, \|x_1\| = \cdots = \|x_n\| = 1\},\\
&S^n_{\mathscr{X}^*} := \{(\phi_1,\dots,\phi_n) : \phi_1,\dots,\phi_n \in \mathscr{X}^*, \|\phi_1\| = \cdots = \|\phi_n\| = 1\},\\
&\mathscr{W} := \{((x_1,\dots,x_n),(\phi_1,\dots,\phi_n)) : ((x_1,\dots,x_n),(\phi_1,\dots,\phi_n)) \in S^n_{\mathscr{X}} \times S^n_{\mathscr{X}^*},\\
&\qquad \phi_j(x_j) = 1, \forall 1 \leqslant j \leqslant n, (\{\phi_j\}_{j=1}^n, \{x_j\}_{j=1}^n) \text{ is an ASF for } \mathscr{X} \text{ and the frame}\\
&\qquad \text{operator } S_{\phi,x} \text{is diagonalizable and its eigenvalues are all non negative}\},\\
&\Phi : \mathscr{W} \to [0,1], \quad \Phi((x_1,\dots,x_n),(\phi_1,\dots,\phi_n)) := \mathscr{M}(\{\phi_j\}_{j=1}^n, \{x_j\}_{j=1}^n).
\end{aligned}$$

We re-norm $\mathscr{X}^n \times \mathscr{X}^{*n}$ by

$$\|((x_1,\dots,x_n),(\phi_1,\dots,\phi_n))\| := \sum_{j=1}^n (\|x_j\| + \|\phi_j\|)$$

and consider $\mathscr{W}$ in this norm. Then $\mathscr{W}$ is compact. We show that Φ is continuous on $\mathscr{W}$ which proves the theorem. In fact, in this situation, Φ being continuous on a compact set attains its infimum. But then, from the definition of Grassmannian frame, every element in $\mathscr{W}$ attaining infimum is a Grassmannian frame. Let $(\{\tau_j\}_{j=1}^n, \{f_j\}_{j=1}^n) \in \mathscr{W}$. Let $\varepsilon > 0$ be given. Define

$$R := 1 + \max_{1\leqslant j,k\leqslant n} \{\|\tau_j\|, \|f_k\|\}$$

and let

$$0 < \delta < \frac{\sqrt{1+\varepsilon} - 1}{R}.$$

Now for any given $(\{\omega_j\}_{j=1}^n, \{g_j\}_{j=1}^n) \in \mathscr{W}$, with

$$\|((\omega_1,\dots,\omega_n),(g_1,\dots,g_n)) - ((\tau_1,\dots,\tau_n),(f_1,\dots,f_n))\| < \delta,$$

if we define $h_j := g_j - f_j, \rho_j := \omega_j - \tau_j, \forall 1 \leqslant j \leqslant n$, then $\|h_j\| < \delta, \|\rho_j\| < \delta$. Then

$$\begin{aligned}
&|\Phi((\omega_1,\dots,\omega_n),(g_1,\dots,g_n)) - \Phi((\tau_1,\dots,\tau_n),(f_1,\dots,f_n))|\\
&= \left| \max_{1\leqslant j,k\leqslant n, j\neq k} |g_j(\omega_k)| - \max_{1\leqslant j,k\leqslant n, j\neq k} |f_j(\tau_k)| \right| \leqslant \max_{1\leqslant j,k\leqslant n, j\neq k} \left||g_j(\omega_k)| - |f_j(\tau_k)|\right|\\
&\leqslant \max_{1\leqslant j,k\leqslant n, j\neq k} \left|g_j(\omega_k) - f_j(\tau_k)\right| = \max_{1\leqslant j,k\leqslant n, j\neq k} \left|(f_j+h_j)(\tau_k+\rho_k) - f_j(\tau_k)\right|\\
&= \max_{1\leqslant j,k\leqslant n, j\neq k} \left|f_j(\tau_k) + f_j(\rho_k) + h_j(\tau_k) + h_j(\rho_k) - f_j(\tau_k)\right|\\
&\leqslant \max_{1\leqslant j,k\leqslant n, j\neq k} |f_j(\rho_k)| + \max_{1\leqslant j,k\leqslant n, j\neq k} |h_j(\tau_k)| + \max_{1\leqslant j,k\leqslant n, j\neq k} |h_j(\rho_k)|\\
&\leqslant \max_{1\leqslant j,k\leqslant n, j\neq k} \|f_j\|\|\rho_k\| + \max_{1\leqslant j,k\leqslant n, j\neq k} \|h_j\|\|\tau_k\| + \max_{1\leqslant j,k\leqslant n, j\neq k} \|h_j\|\|\rho_k\|\\
&\leqslant R\cdot\delta + \delta\cdot R + \delta^2 \leqslant R\cdot\delta + \delta\cdot R + R\delta^2 < \varepsilon. \quad \square
\end{aligned}$$

DEFINITION 8. Let $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ be an ASF for $\mathscr{X}$ satisfying $f_j(\tau_j) = 1$, $\forall 1 \leqslant j \leqslant n$. Assume that the frame operator $S_{f,\tau}$ is diagonalizable and its eigenvalues are all non negative. ASF $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ is said to be γ-*equiangular* if there exists $\gamma \geqslant 0$ such that

$$|f_j(\tau_k)|^2 = \gamma, \quad \forall 1 \leqslant j,k \leqslant n, j \neq k.$$

Following theorem again follows from Theorem 5.

THEOREM 14. *Let $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ be as in Definition 7. Then*

$$\mathscr{M}(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n) \geqslant \sqrt{\frac{n-d}{d(n-1)}} =: \gamma. \tag{11}$$

If the ASF is γ-equiangular, then we have equality in (11).

4. Continuous Welch bounds for Banach spaces

Given a finite collection in a Banach space and in its dual, three maps defined in Definition 1 are well-defined. We cannot do this for collections indexed with a measure space and hence certain conditions have to be imposed on the collections to get well-defined bounded linear operators. First we recall the notion of weak integral also known as Pettis integrals [58]. Let (Ω, μ) be a measure space and $\mathscr{X}$ be a Banach space. A function $f : \Omega \to \mathscr{X}$ is said to be weak integrable or Pettis integrable if following conditions hold.

(i) For every $\phi \in \mathscr{X}^*$, the map $\phi f : \Omega \to \mathbb{K}$ is measurable and $\phi f \in \mathscr{L}^1(\Omega, \mu)$.

(ii) For every measurable subset $E \in \Omega$, there exists an (unique) element $x_E \in \mathscr{X}$ such that

$$\phi(x_E) = \int_E \phi(f(\alpha))\, d\mu(\alpha), \quad \forall \phi \in \mathscr{X}^*.$$

The element $x_E \in \mathscr{X}$ is denoted by $\int_E f(\alpha)\, d\mu(\alpha)$. With this notion, we have

$$\phi\left(\int_E f(\alpha)\, d\mu(\alpha)\right) = \int_E \phi(f(\alpha))\, d\mu(\alpha), \quad \forall \phi \in \mathscr{X}^*, \forall E \in \Omega.$$

Following definition is motivated by the notion of continuous frames for Hilbert spaces [1, 35], continuous framings for Banach spaces [42], continuous Schauder frames for Banach spaces [23] and approximate Schauder frames for Banach spaces [25, 60]. We wish to say that there is a long way which led to the notion of approximate Schauder frames. First, the Han-Larson-Naimark dilation theorem [16] for Hilbert space frames [30] motivated the notion of unconditional Schauder frames (also known as framings) due to Casazza, Han and Larson [11]. A decade later, Casazza, Dilworth, Odell, Schlumprecht and Zsak [9] introduced the notion of Schauder frames.

DEFINITION 9. Let (Ω,μ) be a measure space. Let $\{\tau_\alpha\}_{\alpha\in\Omega}$ be a collection in a Banach space $\mathscr{X}$ and $\{f_\alpha\}_{\alpha\in\Omega}$ be a collection in $\mathscr{X}^*$. The pair $(\{f_\alpha\}_{\alpha\in\Omega},\{\tau_\alpha\}_{\alpha\in\Omega})$ is said to be a *continuous approximate Schauder frame* for $\mathscr{X}$ if the following holds.

(i) For every $x\in\mathscr{X}$ and for every $\phi\in\mathscr{X}^*$, the map

$$\Omega\ni\alpha\mapsto f_\alpha(x)\phi(\tau_\alpha)\in\mathbb{K}$$

is measurable and integrable.

(ii) The *frame operator*

$$S_{f,\tau}:\mathscr{X}\ni x\mapsto S_{f,\tau}x:=\int_\Omega f_\alpha(x)\tau_\alpha\,d\mu(\alpha)\in\mathscr{X}$$

is a well-defined invertible bounded linear operator, where the integral is weak integral.

If $S_{f,\tau}=\lambda I_{\mathscr{X}}$, for some non zero scalar λ, then $(\{f_\alpha\}_{\alpha\in\Omega},\{\tau_\alpha\}_{\alpha\in\Omega})$ is called a *tight continuous ASF* for $\mathscr{X}$. If we do not demand the invertibility of $S_{f,\tau}$, then we say that $(\{f_\alpha\}_{\alpha\in\Omega},\{\tau_\alpha\}_{\alpha\in\Omega})$ is a *continuous approximate Bessel family* for $\mathscr{X}$.

Our first observation is that there is a large supply of continuous ASFs for finite dimensional Banach spaces. Here is a result of existence of them for Banach spaces. Our result is motivated by the work in [45].

THEOREM 15. *Let $\mathscr{X}$ be a finite dimensional Banach space of dimension d. Let (Ω,μ) be a finite measure space such that there are measurable subsets $\Omega_1,\dots,\Omega_n$ satisfying*

$$\Omega=\Omega_1\cup\cdots\cup\Omega_n,\quad \Omega_j\cap\Omega_k=\emptyset,\quad \forall 1\leqslant j,k\leqslant n,\, j\neq n.$$

If $n\geqslant d$, then there exists a continuous ASF $(\{f_\alpha\}_{\alpha\in\Omega},\{\tau_\alpha\}_{\alpha\in\Omega})$ for $\mathscr{X}$.

Proof. Let $(\{g_j\}_{j=1}^n,\{\omega_j\}_{j=1}^n)$ be an ASF for $\mathscr{X}$. Note that ASF always exists. Infact, it is easy to see that spanning collection can be turned to an ASF. It is also known that a pair of basis for the space and its dual is an ASF [41]. Define

$$f_\alpha:=\frac{g_j}{\sqrt{\mu(\Omega_j)}},\quad \forall\alpha\in\Omega_j,\forall 1\leqslant j\leqslant n,\quad \tau_\alpha:=\frac{\omega_j}{\sqrt{\mu(\Omega_j)}},\quad \forall\alpha\in\Omega_j,\forall 1\leqslant j\leqslant n.$$

Then

$$\begin{aligned}\int_\Omega f_\alpha(x)\tau_\alpha\,d\mu(\alpha)&=\sum_{j=1}^n\int_{\Omega_j} f_\alpha(x)\tau_\alpha\,d\mu(\alpha)=\sum_{j=1}^n\int_{\Omega_j}\frac{g_j(x)}{\sqrt{\mu(\Omega_j)}}\frac{\omega_j}{\sqrt{\mu(\Omega_j)}}\,d\mu(\alpha)\\&=\sum_{j=1}^n g_j(x)\omega_j,\quad \forall x\in\mathscr{X}.\end{aligned}$$

Hence $(\{f_\alpha\}_{\alpha\in\Omega},\{\tau_\alpha\}_{\alpha\in\Omega})$ is a continuous ASF for $\mathscr{X}$. $\square$

Like in discrete case, we can get trace of frame operator using continuous ASFs.

THEOREM 16. *Let* $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ *be a continuous approximate Bessel family for* $\mathscr{X}$. *Then*

$$Tra(S_{f,\tau}) = \int_\Omega f_\alpha(\tau_\alpha)\,d\mu(\alpha),$$
$$Tra(S^2_{f,\tau}) = \int_\Omega \int_\Omega f_\beta(\tau_\alpha) f_\alpha(\tau_\beta)\,d\mu(\alpha)\,d\mu(\beta).$$

Proof. Let $\{\omega_j\}_{j=1}^d$ be any basis for $\mathscr{X}$, where d is the dimension of $\mathscr{X}$. Let $\{\zeta_j\}_{j=1}^d$ be the dual basis associated with $\{\omega_j\}_{j=1}^d$. Then $S_{f,\tau}x = \sum_{j=1}^n \zeta_j(x)(S_{f,\tau}\omega_j)$ which gives

$$\begin{aligned}
\mathrm{Tra}(S_{f,\tau}) &= \sum_{j=1}^n \zeta_j(S_{f,\tau}\omega_j) = \sum_{j=1}^n \zeta_j\left(\int_\Omega f_\alpha(\omega_j)\tau_\alpha\,d\mu(\alpha)\right)\\
&= \sum_{j=1}^n \int_\Omega f_\alpha(\omega_j)\zeta_j(\tau_\alpha)\,d\mu(\alpha) = \int_\Omega f_\alpha\left(\sum_{j=1}^n \zeta_j(\tau_\alpha)\omega_j\right)d\mu(\alpha)\\
&= \int_\Omega f_\alpha(\tau_\alpha)\,d\mu(\alpha)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Tra}(S^2_{f,\tau}) &= \sum_{j=1}^n \zeta_j(S^2_{f,\tau}\omega_j) = \sum_{j=1}^n \zeta_j\left(\int_\Omega f_\alpha(S_{f,\tau}\omega_j)\tau_\alpha\,d\mu(\alpha)\right)\\
&= \sum_{j=1}^n \int_\Omega f_\alpha(S_{f,\tau}\omega_j)\zeta_j(\tau_\alpha)\,d\mu(\alpha) = \int_\Omega f_\alpha\left(\sum_{j=1}^n \zeta_j(\tau_\alpha)S_{f,\tau}\omega_j\right)d\mu(\alpha)\\
&= \int_\Omega f_\alpha(S_{f,\tau}\tau_\alpha)\,d\mu(\alpha) = \int_\Omega f_\alpha\left(\int_\Omega f_\beta(\tau_\alpha)\tau_\beta\,d\mu(\beta)\right)d\mu(\alpha)\\
&= \int_\Omega\int_\Omega f_\beta(\tau_\alpha)f_\alpha(\tau_\beta)\,d\mu(\alpha)\,d\mu(\beta). \qquad \square
\end{aligned}$$

Note that we did not assume any condition on set of vectors and functionals to derive Theorem 4. The reason is that frame operator always exists in discrete case. To get the existence of frame operator we assumed Besselness in Theorem 16. We now derive continuous versions of Theorem 5 and Theorem 6.

THEOREM 17. (First order continuous Welch bound for Banach spaces) *Let* (Ω,μ) *be a* σ*-finite measure space and* $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ *be a continuous Bessel family for finite dimensional Banach space* $\mathscr{X}$ *of dimension* d. *If the diagonal* $\Delta := \{(\alpha,\alpha) : \alpha\in\Omega\}$ *is measurable in the measure space* $\Omega\times\Omega$,

$$\int_{\Omega\times\Omega} |f_\alpha(\tau_\beta)f_\beta(\tau_\alpha)|\,d(\mu\times\mu)(\alpha,\beta) < \infty,$$

and the operator $S_{f,\tau} : \mathscr{X} \ni x \mapsto S_{f,\tau}x := \int_\Omega f_\alpha(x)\tau_\alpha\, d\mu(\alpha) \in \mathscr{X}$ *is diagonalizable and its eigenvalues are all non negative, then*

$$\int_{\Omega\times\Omega} f_\alpha(\tau_\beta) f_\beta(\tau_\alpha)\, d(\mu\times\mu)(\alpha,\beta) \geqslant \frac{1}{d}\left(\int_\Omega f_\alpha(\tau_\alpha)\, d\mu(\alpha)\right)^2 \tag{12}$$

and

$$\sup_{\alpha,\beta\in\Omega,\alpha\neq\beta} |f_\alpha(\tau_\beta)| \geqslant \sqrt{\frac{\frac{1}{d}\left(\int_\Omega f_\alpha(\tau_\alpha)\, d\mu(\alpha)\right)^2 - \int_\Delta |f_\alpha(\tau_\alpha)|^2\, d(\mu\times\mu)(\alpha,\alpha)}{(\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)}}.$$

Furthermore, equality holds in (12) *if and only if* $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ *is a tight continuous ASF for* $\mathscr{X}$. *In particular, if* $f_\alpha(\tau_\alpha) = 1$ *for all* $\alpha \in \Omega$, *then*

$$\sup_{\alpha,\beta\in\Omega,\alpha\neq\beta} |f_\alpha(\tau_\beta) f_\beta(\tau_\alpha)| \geqslant \frac{1}{(\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)}\left[\frac{\mu(\Omega)^2}{d} - (\mu\times\mu)(\Delta)\right]$$

and we have first order continuous Welch bound for Banach spaces

$$\sup_{\alpha,\beta\in\Omega,\alpha\neq\beta} |f_\alpha(\tau_\beta)| \geqslant \sqrt{\frac{1}{(\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)}\left[\frac{\mu(\Omega)^2}{d} - (\mu\times\mu)(\Delta)\right]}.$$

Proof. Let $\lambda_1,\dots,\lambda_d$ be eigenvalues of the frame operator $S_{f,\tau}$. Then $\lambda_1,\dots,\lambda_d \geqslant 0$. Now using Theorem 16 we get

$$\begin{aligned}\left(\int_\Omega f_\alpha(\tau_\alpha)\, d\mu(\alpha)\right)^2 &= (\mathrm{Tra}(S_{f,\tau}))^2 = \left(\sum_{k=1}^d \lambda_k\right)^2 \leqslant d\sum_{k=1}^d \lambda_k^2 \\ &= d\,\mathrm{Tra}(S_{f,\tau}^2) = d\int_\Omega\int_\Omega f_\beta(\tau_\alpha) f_\alpha(\tau_\beta)\, d\mu(\alpha)\, d\mu(\beta).\end{aligned}$$

which gives the first inequality. Using Fubini's theorem, now we get

$$\begin{aligned}&\frac{1}{d}\left(\int_\Omega f_\alpha(\tau_\alpha)\, d\mu(\alpha)\right)^2 \leqslant \int_\Omega\int_\Omega f_\alpha(\tau_\beta) f_\beta(\tau_\alpha)\, d\mu(\alpha)\, d\mu(\beta) \\ &= \int_{\Omega\times\Omega} f_\alpha(\tau_\beta) f_\beta(\tau_\alpha)\, d(\mu\times\mu)(\alpha,\beta) \\ &= \int_\Delta f_\alpha(\tau_\beta) f_\beta(\tau_\alpha)\, d(\mu\times\mu)(\alpha,\beta) + \int_{(\Omega\times\Omega)\setminus\Delta} f_\alpha(\tau_\beta) f_\beta(\tau_\alpha)\, d(\mu\times\mu)(\alpha,\beta) \\ &= \int_\Delta |f_\alpha(\tau_\alpha)|^2\, d(\mu\times\mu)(\alpha,\alpha) + \int_{(\Omega\times\Omega)\setminus\Delta} f_\alpha(\tau_\beta) f_\beta(\tau_\alpha)\, d(\mu\times\mu)(\alpha,\beta) \\ &\leqslant \int_\Delta |f_\alpha(\tau_\alpha)|^2\, d(\mu\times\mu)(\alpha,\alpha) + (\mu\times\mu)((\Omega\times\Omega)\setminus\Delta) \sup_{\alpha,\beta\in\Omega,\alpha\neq\beta} |f_\alpha(\tau_\beta) f_\beta(\tau_\alpha)|. \quad \square\end{aligned}$$

THEOREM 18. (Continuous Welch bounds for Banach spaces) *Let $m\in\mathbb{N}$, (Ω,μ) be a σ-finite measure space and $(\{f_\alpha\}_{\alpha\in\Omega},\{\tau_\alpha\}_{\alpha\in\Omega})$ be a continuous Bessel family for finite dimensional Banach space $\mathscr{X}$ of dimension d. If the diagonal $\Delta:=\{(\alpha,\alpha):\alpha\in\Omega\}$ is measurable in the measure space $\Omega\times\Omega$,*

$$\int_{\Omega\times\Omega}|f_\alpha(\tau_\beta)f_\beta(\tau_\alpha)|^m\,d(\mu\times\mu)(\alpha,\beta)<\infty,$$

and the operator $S_{f,\tau}:\text{Sym}^m(\mathscr{X})\ni x\mapsto\int_\Omega f_\alpha^{\otimes m}(x)\tau_\alpha^{\otimes m}\,d\mu(\alpha)\in\text{Sym}^m(\mathscr{X})$ is diagonalizable and its eigenvalues are all non negative, then

$$\int_{\Omega\times\Omega}f_\alpha(\tau_\beta)^m f_\beta(\tau_\alpha)^m\,d(\mu\times\mu)(\alpha,\beta)\geqslant\frac{1}{\binom{d+m-1}{m}}\left(\int_\Omega f_\alpha(\tau_\alpha)^m\,d\mu(\alpha)\right)^2 \tag{13}$$

and

$$\sup_{\alpha,\beta\in\Omega,\alpha\neq\beta}|f_\alpha(\tau_\beta)|^m\geqslant\sqrt{\frac{\frac{1}{\binom{d+m-1}{m}}\left(\int_\Omega f_\alpha(\tau_\alpha)^m\,d\mu(\alpha)\right)^2-\int_\Delta|f_\alpha(\tau_\alpha)|^{2m}\,d(\mu\times\mu)(\alpha,\alpha)}{(\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)}}. \tag{14}$$

Furthermore, equality holds in (13) if and only if $(\{f_\alpha\}_{\alpha\in\Omega},\{\tau_\alpha\}_{\alpha\in\Omega})$ is a tight continuous ASF for $\text{Sym}^m(\mathscr{X})$. In particular, if $f_\alpha(\tau_\alpha)=1$ for all $\alpha\in\Omega$, then

$$\sup_{\alpha,\beta\in\Omega,\alpha\neq\beta}|f_\alpha(\tau_\beta)f_\beta(\tau_\alpha)|^m\geqslant\frac{1}{(\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)}\left[\frac{\mu(\Omega)^2}{\binom{d+m-1}{m}}-(\mu\times\mu)(\Delta)\right]$$

and we have continuous Welch bound for Banach spaces

$$\sup_{\alpha,\beta\in\Omega,\alpha\neq\beta}|f_\alpha(\tau_\beta)|^m\geqslant\sqrt{\frac{1}{(\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)}\left[\frac{\mu(\Omega)^2}{\binom{d+m-1}{m}}-(\mu\times\mu)(\Delta)\right]}. \tag{15}$$

Proof. Let $\lambda_1,\dots,\lambda_{\dim(\text{Sym}^m(\mathscr{X}))}$ be eigenvalues of $S_{f,\tau}$. Then

$$\begin{aligned}
&\left(\int_\Omega f_\alpha(\tau_\alpha)^m\,d\mu(\alpha)\right)^2=\left(\int_\Omega f_\alpha^{\otimes m}(\tau_\alpha^{\otimes m})\,d\mu(\alpha)\right)^2\\
&=(\text{Tra}(S_{f,\tau}))^2=\left(\sum_{l=1}^{\dim(\text{Sym}^m(\mathscr{X}))}\lambda_l\right)^2\\
&\leqslant\dim(\text{Sym}^m(\mathscr{X}))\sum_{l=1}^{\dim(\text{Sym}^m(\mathscr{X}))}\lambda_l^2=\binom{d+m-1}{m}\text{Tra}(S_{f,\tau}^2)\\
&=\binom{d+m-1}{m}\int_\Omega\int_\Omega f_\alpha^{\otimes m}(\tau_\beta^{\otimes m})f_\beta^{\otimes m}(\tau_\alpha^{\otimes m})\,d\mu(\alpha)\,d\mu(\beta)\\
&=\binom{d+m-1}{m}\int_\Omega\int_\Omega f_\alpha(\tau_\beta)^m f_\beta(\tau_\alpha)^m\,d\mu(\alpha)\,d\mu(\beta)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\binom{d+m-1}{m}}\left(\int_{\Omega} f_\alpha(\tau_\alpha)^m\,d\mu(\alpha)\right)^2\\
&=\int_{\Omega}\int_{\Omega} f_\alpha(\tau_\beta)^m f_\beta(\tau_\alpha)^m\,d\mu(\alpha)\,d\mu(\beta)\\
&=\int_{\Omega\times\Omega} f_\alpha(\tau_\beta)^m f_\beta(\tau_\alpha)^m\,d(\mu\times\mu)(\alpha,\beta)\\
&=\int_{(\Omega\times\Omega)\setminus\Delta} f_\alpha(\tau_\beta)^m f_\beta(\tau_\alpha)^m\,d(\mu\times\mu)(\alpha,\beta)+\int_{\Delta} f_\alpha(\tau_\alpha)^{2m}\,d(\mu\times\mu)(\alpha,\alpha)\\
&\leqslant \int_{(\Omega\times\Omega)\setminus\Delta} |f_\alpha(\tau_\beta)f_\beta(\tau_\alpha)|^m\,d(\mu\times\mu)(\alpha,\beta)+\int_{\Delta} |f_\alpha(\tau_\alpha)|^{2m}\,d(\mu\times\mu)(\alpha,\alpha)\\
&\leqslant (\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)\sup_{\alpha,\beta\in\Omega,\alpha\neq\beta}|f_\alpha(\tau_\beta)f_\beta(\tau_\alpha)|^m+\int_{\Delta}|f_\alpha(\tau_\alpha)|^{2m}\,d(\mu\times\mu)(\alpha,\alpha).
\end{aligned}$$

□

REMARK 5. For $m \geqslant 2$, we call family of inequalities in (15) as *continuous higher order Welch bounds*.

COROLLARY 1. *Theorem 6 is a corollary of Theorem 18.*

Proof. Take $\Omega = \{1,\dots,n\}$ and μ as the counting measure. □

THEOREM 19. *Theorem 3 is a corollary of Theorem 18.*

Proof. Let $\{\tau_\alpha\}_{\alpha\in\Omega}$ be a normalized continuous Bessel family for $\mathscr{H}$ of dimension d. We define

$$f_\alpha : \mathscr{H} \ni h \mapsto \langle h, \tau_\alpha\rangle \in \mathbb{K}, \quad \forall \alpha \in \Omega$$

and apply Theorem 18. □

As observed in [40], we again observe that given a measure space Ω, the diagonal Δ need not be measurable (see [21]). Now we derive continuous versions of Theorem 7 and Theorem 8.

THEOREM 20. *Let (Ω,μ) be a σ-finite measure space and $(\{f_\alpha\}_{\alpha\in\Omega},\{\tau_\alpha\}_{\alpha\in\Omega})$ be a continuous Bessel family for finite dimensional Banach space $\mathscr{X}$ of dimension d. If the diagonal $\Delta := \{(\alpha,\alpha) : \alpha\in\Omega\}$ is measurable in the measure space $\Omega\times\Omega$ and the operator $S_{f,\tau} : \mathscr{X} \ni x \mapsto S_{f,\tau}x := \int_\Omega f_\alpha(x)\tau_\alpha\,d\mu(\alpha) \in \mathscr{X}$ is diagonalizable and its eigenvalues are all non negative, then*

$$\frac{1}{d}\operatorname{Tra}(S_{f,\tau})^r \geqslant \left(\frac{1}{d}\int_\Omega f_\alpha(\tau_\alpha)\,d\mu(\alpha)\right)^r, \quad \forall r\in[1,\infty)$$

and

$$\frac{1}{d}\operatorname{Tra}(S_{f,\tau})^r \leqslant \left(\frac{1}{d}\int_{\Omega} f_\alpha(\tau_\alpha)\,d\mu(\alpha)\right)^r, \quad \forall r \in (0,1).$$

In particular, if $f_\alpha(\tau_\alpha) = 1$ *for all* $\alpha \in \Omega$*, then*

$$\frac{1}{\mu(\Omega)}\operatorname{Tra}(S_{f,\tau})^r \geqslant \left(\frac{\mu(\Omega)}{d}\right)^{r-1}, \quad \forall r \in [1,\infty)$$

and

$$\frac{1}{\mu(\Omega)}\operatorname{Tra}(S_{f,\tau})^r \leqslant \left(\frac{\mu(\Omega)}{d}\right)^{r-1}, \quad \forall r \in (0,1).$$

Proof. Let $\lambda_1,\dots,\lambda_d$ be eigenvalues of $S_{f,\tau}$. Let $r \in [1,\infty)$. Jensen's inequality gives

$$\left(\frac{1}{d}\sum_{k=1}^{d}\lambda_k\right)^r \leqslant \frac{1}{d}\sum_{k=1}^{d}\lambda_k^r.$$

Therefore

$$\left(\frac{1}{d}\int_{\Omega} f_\alpha(\tau_\alpha)\,d\mu(\alpha)\right)^r = \left(\frac{1}{d}\operatorname{Tra}(S_{f,\tau})\right)^r \leqslant \frac{1}{d}\operatorname{Tra}(S_{f,\tau}^r).$$

Similarly the case $r \in (0,1)$ follows by using Jensen's inequality. $\square$

THEOREM 21. *Let* $2 < p < \infty$, (Ω,μ) *be a* σ*-finite measure space and* $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ *be a continuous Bessel family for finite dimensional Banach space* $\mathscr{X}$ *of dimension* d *such that* $f_\alpha(\tau_\alpha) = 1$ *for all* $\alpha \in \Omega$*. If the diagonal* $\Delta := \{(\alpha,\alpha) : \alpha \in \Omega\}$ *is measurable in the measure space* $\Omega \times \Omega$*,*

$$\int_{\Omega\times\Omega} |f_\alpha(\tau_\beta) f_\beta(\tau_\alpha)|\,d(\mu\times\mu)(\alpha,\beta) < \infty,$$

and the operator $S_{f,\tau} : \mathscr{X} \ni x \mapsto S_{f,\tau}x := \int_\Omega f_\alpha(x)\tau_\alpha\,d\mu(\alpha) \in \mathscr{X}$ *is diagonalizable and its eigenvalues are all non negative, then*

$$\begin{aligned}&\int_{\Omega\times\Omega} |f_\alpha(\tau_\beta) f_\beta(\tau_\alpha)|^{\frac{p}{2}}\,d(\mu\times\mu)(\alpha,\beta)\\ &\geqslant \frac{1}{(\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)^{\frac{p}{2}-1}}\left(\frac{\mu(\Omega)^2}{d} - (\mu\times\mu)(\Delta)\right)^{\frac{p}{2}} + (\mu\times\mu)(\Delta).\end{aligned}$$

Proof. Define $r := 2p/(p-2)$ and q be the conjugate index of $p/2$. Then $q = r/2$. Using Theorem 17 and Holder's inequality, we have

$$\begin{aligned}
&\frac{\mu(\Omega)^2}{d} - (\mu\times\mu)(\Delta) \leqslant \int_{(\Omega\times\Omega)\setminus\Delta} |f_\alpha(\tau_\beta)f_\beta(\tau_\alpha)|\, d(\mu\times\mu)(\alpha,\beta)\\
&\leqslant \left(\int_{(\Omega\times\Omega)\setminus\Delta} |f_\alpha(\tau_\beta)f_\beta(\tau_\alpha)|^{\frac{p}{2}}\, d(\mu\times\mu)(\alpha,\beta)\right)^{\frac{2}{p}} \left(\int_{(\Omega\times\Omega)\setminus\Delta} d(\mu\times\mu)(\alpha,\beta)\right)^{\frac{1}{q}}\\
&= \left(\int_{(\Omega\times\Omega)\setminus\Delta} |f_\alpha(\tau_\beta)f_\beta(\tau_\alpha)|^{\frac{p}{2}}\, d(\mu\times\mu)(\alpha,\beta)\right)^{\frac{2}{p}} (\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)^{\frac{1}{q}}\\
&= \left(\int_{(\Omega\times\Omega)\setminus\Delta} |f_\alpha(\tau_\beta)f_\beta(\tau_\alpha)|^{\frac{p}{2}}\, d(\mu\times\mu)(\alpha,\beta)\right)^{\frac{2}{p}} (\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)^{\frac{2}{r}}\\
&= \left(\int_{(\Omega\times\Omega)\setminus\Delta} |f_\alpha(\tau_\beta)f_\beta(\tau_\alpha)|^{\frac{p}{2}}\, d(\mu\times\mu)(\alpha,\beta)\right)^{\frac{2}{p}} (\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)^{\frac{p-2}{p}}
\end{aligned}$$

which gives

$$\begin{aligned}
&\left(\frac{\mu(\Omega)^2}{d} - (\mu\times\mu)(\Delta)\right)^{\frac{p}{2}}\\
&\leqslant \left(\int_{(\Omega\times\Omega)\setminus\Delta} |f_\alpha(\tau_\beta)f_\beta(\tau_\alpha)|^{\frac{p}{2}}\, d(\mu\times\mu)(\alpha,\beta)\right) (\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)^{\frac{p}{2}-1}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{1}{(\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)^{\frac{p}{2}-1}}\left(\frac{\mu(\Omega)^2}{d} - (\mu\times\mu)(\Delta)\right)^{\frac{p}{2}} + (\mu\times\mu)(\Delta)\\
&= \frac{1}{(\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)^{\frac{p}{2}-1}}\left(\frac{\mu(\Omega)^2}{d} - (\mu\times\mu)(\Delta)\right)^{\frac{p}{2}}\\
&\quad + \int_{\Delta} |f_\alpha(\tau_\beta)f_\beta(\tau_\alpha)|^{\frac{p}{2}}\, d(\mu\times\mu)(\alpha,\beta)\\
&\leqslant \int_{(\Omega\times\Omega)\setminus\Delta} |f_\alpha(\tau_\beta)f_\beta(\tau_\alpha)|^{\frac{p}{2}}\, d(\mu\times\mu)(\alpha,\beta) + \int_{\Delta} |f_\alpha(\tau_\alpha)f_\alpha(\tau_\alpha)|^{\frac{p}{2}}\, d(\mu\times\mu)(\alpha,\alpha)\\
&= \int_{\Omega\times\Omega} |f_\alpha(\tau_\beta)f_\beta(\tau_\alpha)|^{\frac{p}{2}}\, d(\mu\times\mu)(\alpha,\beta). \quad \square
\end{aligned}$$

5. Applications of continuous Welch bounds for Banach spaces

Here we list continuous versions of corresponding concepts, results and open problems stated in Section 3. Throughout this section, (Ω,μ) is a σ-finite measure space. We furthermore assume that the diagonal Δ is measurable.

DEFINITION 10. Let $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ be a continuous Bessel family for a finite dimensional Banach space $\mathscr{X}$ of dimension d satisfying $f_\alpha(\tau_\alpha) = 1$ for all

$\alpha \in \Omega$. Assume that the frame operator $S_{f,\tau}$ is diagonalizable and its eigenvalues are all non negative. Also assume that

$$\int_{\Omega\times\Omega} |f_\alpha(\tau_\beta) f_\beta(\tau_\alpha)|\, d(\mu\times\mu)(\alpha,\beta) < \infty.$$

We define the *continuous root-mean-square (RMS) absolute cross relation* of $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ as

$$\begin{aligned}&I_{\mathrm{CRMS}}(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})\\ &:= \left(\frac{1}{(\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)} \int_{(\Omega\times\Omega)\setminus\Delta} f_\alpha(\tau_\beta) f_\beta(\tau_\alpha)\, d(\mu\times\mu)(\alpha,\beta)\right)^{\frac{1}{2}}.\end{aligned}$$

THEOREM 22. *Let* $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ *be as in Definition* 10. *Then*

$$\begin{aligned}\sup_{\alpha,\beta\in\Omega,\alpha\neq\beta} |f_\alpha(\tau_\beta)| &\geqslant I_{CRMS}(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})\\ &\geqslant \left(\frac{1}{(\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)}\left[\frac{\mu(\Omega)^2}{d} - (\mu\times\mu)(\Delta)\right]\right)^{\frac{1}{2}}.\end{aligned}$$

DEFINITION 11. Let $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ be a continuous Bessel family for a finite dimensional Banach space $\mathscr{X}$ of dimension d satisfying $f_\alpha(\tau_\alpha) = 1$ for all $\alpha \in \Omega$. Assume that the frame operator $S_{f,\tau}$ is diagonalizable and its eigenvalues are all non negative. Also assume that

$$\int_{\Omega\times\Omega} |f_\alpha(\tau_\beta) f_\beta(\tau_\alpha)|\, d(\mu\times\mu)(\alpha,\beta) < \infty.$$

We define the *continuous pseudo frame potential* of $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ as

$$CPFP(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega}) := \int_{\Omega\times\Omega} f_\alpha(\tau_\beta) f_\beta(\tau_\alpha)\, d(\mu\times\mu)(\alpha,\beta).$$

THEOREM 23. *Let* $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ *be as in Definition* 11. *Then*

$$\mu(\Omega)^2 \sup_{\alpha,\beta\in\Omega} |f_\alpha(\tau_\beta)|^2 \geqslant CPFP(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega}) \geqslant \frac{\mu(\Omega)^2}{d}. \tag{16}$$

DEFINITION 12. Let $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ be a continuous ASF for $\mathscr{X}$ satisfying $f_\alpha(\tau_\alpha) = 1$ for all $\alpha \in \Omega$. Assume that the frame operator $S_{f,\tau}$ is diagonalizable and its eigenvalues are all non negative. Also assume that

$$\int_{\Omega\times\Omega} |f_\alpha(\tau_\beta) f_\beta(\tau_\alpha)|\, d(\mu\times\mu)(\alpha,\beta) < \infty.$$

We define the *continuous frame correlation* of $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ as

$$\mathscr{M}(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega}) := \sup_{\alpha,\beta\in\Omega,\alpha\neq\beta} |f_\alpha(\tau_\beta)|.$$

DEFINITION 13. Let $(\{f_\alpha\}_{\alpha\in\Omega},\{\tau_\alpha\}_{\alpha\in\Omega})$ be a continuous ASF for $\mathscr{X}$ satisfying $\|f_\alpha\|=1, \|\tau_\alpha\|=1, f_\alpha(\tau_\alpha)=1$ for all $\alpha\in\Omega$. Assume that the frame operator $S_{f,\tau}$ is diagonalizable and its eigenvalues are all non negative. Also assume that

$$\int_{\Omega\times\Omega}|f_\alpha(\tau_\beta)f_\beta(\tau_\alpha)|\,d(\mu\times\mu)(\alpha,\beta)<\infty.$$

Continuous ASF $(\{f_\alpha\}_{\alpha\in\Omega},\{\tau_\alpha\}_{\alpha\in\Omega})$ is said to be a *continuous Grassmannian frame* for $\mathscr{X}$ if

$$\mathscr{M}(\{f_\alpha\}_{\alpha\in\Omega},\{\tau_\alpha\}_{\alpha\in\Omega})=\inf\Bigg\{\mathscr{M}(\{g_\alpha\}_{\alpha\in\Omega},\{\omega_\alpha\}_{\alpha\in\Omega}):(\{g_\alpha\}_{\alpha\in\Omega},\{\omega_\alpha\}_{\alpha\in\Omega})\text{ is a continuous ASF for }\mathscr{X}\text{ satisfying }\|g_\alpha\|=1,\|\omega_\alpha\|=1, g_\alpha(\omega_\alpha)=1,\forall\alpha\in\Omega,\text{ the frame operator }S_{g,\omega}\text{ is diagonalizable and its eigenvalues are all non negative and }\int_{\Omega\times\Omega}|g_\alpha(\omega_\beta)g_\beta(\omega_\alpha)|\,d(\mu\times\mu)(\alpha,\beta)<\infty\Bigg\}.$$

DEFINITION 14. Let $(\{f_\alpha\}_{\alpha\in\Omega},\{\tau_\alpha\}_{\alpha\in\Omega})$ be a continuous ASF for $\mathscr{X}$ satisfying $f_\alpha(\tau_\alpha)=1$ for all $\alpha\in\Omega$. Assume that the frame operator $S_{f,\tau}$ is diagonalizable and its eigenvalues are all non negative. Continuous ASF $(\{f_\alpha\}_{\alpha\in\Omega},\{\tau_\alpha\}_{\alpha\in\Omega})$ is said to be γ*-equiangular* if there exists $\gamma\geqslant 0$ such that

$$|f_\alpha(\tau_\beta)|=\gamma,\quad \forall\alpha,\beta\in\Omega,\alpha\neq\beta.$$

THEOREM 24. *Let* $(\{f_\alpha\}_{\alpha\in\Omega},\{\tau_\alpha\}_{\alpha\in\Omega})$ *be as in Definition* 13. *Then*

$$\mathscr{M}(\{f_\alpha\}_{\alpha\in\Omega},\{\tau_\alpha\}_{\alpha\in\Omega})\geqslant\left(\frac{1}{(\mu\times\mu)((\Omega\times\Omega)\setminus\Delta)}\left[\frac{\mu(\Omega)^2}{d}-(\mu\times\mu)(\Delta)\right]\right)^{\frac{1}{2}}=:\gamma. \tag{17}$$

If the continuous ASF is γ*-equiangular, then we have equality in* (17).

6. Problems for future research

Based on the results in this paper, results and conjectures in Hilbert space frame theory, we formulate following problems for future research.

(I) Recall the definition of Gerzon's bound which allows us to recall the bounds which are in the same way to discrete Welch bounds in Hilbert spaces.

DEFINITION 15. [32] Given $d\in\mathbb{N}$, define *Gerzon's bound*

$$\mathscr{Z}(d,\mathbb{K}):=\begin{cases} d^2 & \text{if}\quad \mathbb{K}=\mathbb{C}\\ \frac{d(d+1)}{2} & \text{if}\quad \mathbb{K}=\mathbb{R}.\end{cases}$$

THEOREM 25. *Define* $m := \dim_{\mathbb{R}}(\mathbb{K})/2$. *If* $\{\tau_j\}_{j=1}^n$ *is any collection of unit vectors in* $\mathbb{K}^d$, *then*

(i) [8, 44] (Bukh-Cox bound)

$$\max_{1\leqslant j,k\leqslant n, j\neq k}|\langle \tau_j, \tau_k\rangle| \geqslant \frac{\mathscr{Z}(n-d,\mathbb{K})}{n(1+m(n-d-1)\sqrt{m^{-1}+n-d})-\mathscr{Z}(n-d,\mathbb{K})}$$
$$\text{if} \quad n>d.$$

(ii) [15, 46] (Orthoplex/Rankin bound)

$$\max_{1\leqslant j,k\leqslant n, j\neq k}|\langle \tau_j, \tau_k\rangle| \geqslant \frac{1}{\sqrt{d}} \quad \text{if} \quad n>\mathscr{Z}(d,\mathbb{K}).$$

(iii) [27, 32] (Levenstein bound)

$$\max_{1\leqslant j,k\leqslant n, j\neq k}|\langle \tau_j, \tau_k\rangle| \geqslant \sqrt{\frac{n(m+1)-d(md+1)}{(n-d)(md+1)}} \quad \text{if} \quad n>\mathscr{Z}(d,\mathbb{K}).$$

(iv) [43, 67] (Exponential bound)

$$\max_{1\leqslant j,k\leqslant n, j\neq k}|\langle \tau_j, \tau_k\rangle| \geqslant 1-2n^{\frac{-1}{d-1}}.$$

Theorem 25, Theorem 5 and Theorem 17 give the following question.

PROBLEM 1. What is the discrete and continuous versions of

(i) Bukh-Cox bound for Banach spaces?

(ii) Orthoplex/Rankin bound for Banach spaces?

(iii) Levenstein bound for Banach spaces?

(iv) Exponential bound for Banach spaces?

(II) Benedetto and Fickus characterized unit norm frames for finite dimensional Hilbert spaces using frame potential (see [4]). Theorems 12 and 23 give the following problem.

PROBLEM 2.

(i) What is a characterization of $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ by using equality in (10)?

(ii) What is a characterization of $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ by using equality in (16)?

(III) Celebrated Zauner's conjecture for Hilbert spaces (see [2, 26, 36, 47, 68] for Zauner's conjecture and its connections with Hilbert's 12-problem and Stark conjecture) asserts the existence of d^2 unit vectors $\{\tau_j\}_{j=1}^{d^2}$ in $\mathbb{C}^d$ for every d such that

$$|\langle\tau_j,\tau_k\rangle|^2=\frac{1}{d+1},\quad \forall 1\leqslant j,k\leqslant d^2, j\neq k.$$

Motivated by this conjecture and Definitions 8 and 14 we formulate following problems.

PROBLEM 3. Given a Banach space $\mathscr{X}$ of dimension d and a $\gamma>0$, for which $n\in\mathbb{N}$, there exists a collection $\{\tau_j\}_{j=1}^n$ in $\mathscr{X}$ and a collection $\{\tau_j\}_{j=1}^n$ in $\mathscr{X}^*$ satisfying the following.

(a) $\|f_j\|=\|\tau_j\|=f_j(\tau_j)=1, \forall 1\leqslant j\leqslant n.$

(b) For every $x\in\mathscr{X}$,

$$x=\frac{d}{n}\sum_{j=1}^{n}f_j(x)\tau_j.$$

(c) For every $1\leqslant j,k\leqslant n, j\neq k$,

$$|f_j(\tau_k)|=\gamma.$$

PROBLEM 4. Given a Banach space $\mathscr{X}$ of dimension d and a $\gamma>0$, for which measure spaces (Ω,μ), there exists a continuous Bessel family $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ for $\mathscr{X}$ satisfying the following.

(a) $\|f_\alpha\|=\|\tau_\alpha\|=f_\alpha(\tau_\alpha)=1, \forall\alpha\in\Omega.$

(b) For every $x\in\mathscr{X}$,

$$x=\frac{d}{\mu(\Omega)}\int_\Omega f_\alpha(x)\tau_\alpha\,d\mu(\alpha).$$

(c) For every $\alpha,\beta\in\Omega, \alpha\neq\beta$,

$$|f_\alpha(\tau_\beta)|=\gamma.$$

(IV) In [55] Strohmer and Heath showed that there is a relation between the number of elements in the ASF and dimension of the space. Inequalities (11) and (17) then give following problems.

PROBLEM 5.

(i) Whether the equality in (11) implies that the ASF $(\{f_j\}_{j=1}^n,\{\tau_j\}_{j=1}^n)$ is γ-equiangular?

(ii) Whether the validity of (11) implies there is a relation between the number of elements in the ASF and dimension of the space?

PROBLEM 6.

(i) Whether the equality in (17) implies that the continuous ASF $(\{f_\alpha\}_{\alpha\in\Omega}, \{\tau_\alpha\}_{\alpha\in\Omega})$ is γ-equiangular?

(ii) Whether the validity of (17) implies there is a relation between the measure of Ω and dimension of the space?

(V) In Theorem 13 we showed that (discrete) Grassmannian frames always exist. However, the arguments used in the proof of that theorem cannot be carried to measure spaces. Hence we have following problem.

PROBLEM 7. Classify measure spaces and (finite dimensional) Banach spaces so that continuous Grassmannian ASFs exist.

(VI) Theorem 18 gives the following problem.

PROBLEM 8. Classify measure spaces (Ω, μ) such that Theorem 18 holds? In other words, given a measure space (Ω, μ), does the validity of (13) or (14) implies conditions on measure space (Ω, μ), say σ-finite?

Acknowledgements. Author thank the reviewer for his/her extensive study of the manuscript, finding mistakes and suggestions for better presentation. The author is supported by Indian Statistical Institute, Bangalore, through the J. C. Bose Fellowship of Prof. B. V. Rajarama Bhat. He thanks Prof. B. V. Rajarama Bhat for the Post Doc position.

REFERENCES

[1] S. TWAREQUE ALI, J.-P. ANTOINE, AND J.-P. GAZEAU, *Continuous frames in Hilbert space*, Ann. Physics, 222 (1): 1–37, 1993.

[2] MARCUS APPLEBY, INGEMAR BENGTSSON, STEVEN FLAMMIA, AND DARDO GOYENECHE, *Tight frames, Hadamard matrices and Zauner's conjecture*, J. Phys. A, 52 (29): 295301, 26, 2019.

[3] J. J. BENEDETTO AND J. D KOLESAR, *Geometric properties of grassmannian frames for $\mathbb{R}^2$ and $\mathbb{R}^3$*, EURASIP J. Adv. Signal Process. 2006, (049850), 2006.

[4] JOHN J. BENEDETTO AND MATTHEW FICKUS, *Finite normalized tight frames*, Adv. Comput. Math., 18 (2–4): 357–385, 2003.

[5] CRISTIANO BOCCI AND LUCA CHIANTINI, *An introduction to algebraic statistics with tensors*, vol. 118 of Unitext, Springer, Cham, 2019.

[6] BERNHARD G. BODMANN AND JOHN HAAS, *Frame potentials and the geometry of frames*, J. Fourier Anal. Appl., 21 (6): 1344–1383, 2015.

[7] ANDRIY BONDARENKO, DANYLO RADCHENKO, AND MARYNA VIAZOVSKA, *Optimal asymptotic bounds for spherical designs*, Ann. of Math. (2), 178 (2): 443–452, 2013.

[8] BORIS BUKH AND CHRISTOPHER COX, *Nearly orthogonal vectors and small antipodal spherical codes*, Israel J. Math., 238 (1): 359–388, 2020.

[9] P. G. CASAZZA, S. J. DILWORTH, E. ODELL, TH. SCHLUMPRECHT, AND A. ZSÁK, *Coefficient quantization for frames in Banach spaces*, J. Math. Anal. Appl., 348 (1): 66–86, 2008.

[10] PETER G. CASAZZA, MATTHEW FICKUS, JELENA KOVAČEVIĆ, MANUEL T. LEON, AND JANET C. TREMAIN, *A physical interpretation of tight frames*, In *Harmonic analysis and applications*, Appl. Numer. Harmon. Anal., p. 51–76, Birkhäuser Boston, Boston, MA, 2006.

[11] PETER G. CASAZZA, DEGUANG HAN, AND DAVID R. LARSON, *Frames for Banach spaces*, In *The functional and harmonic analysis of wavelets and frames (San Antonio, TX, 1999)*, vol. 247 of Contemp. Math., p. 149–182, Amer. Math. Soc., Providence, RI, 1999.

[12] J. A. CHÁVEZ-DOMÍNGUEZ, D. FREEMAN, AND K. KORNELSON, *Frame potential for finite-dimensional Banach spaces*, Linear Algebra Appl., 578: 1–26, 2019.

[13] OLE CHRISTENSEN, SOMANTIKA DATTA, AND RAE YOUNG KIM, *Equiangular frames and generalizations of the Welch bound to dual pairs of frames*, Linear Multilinear Algebra, 68 (12): 2495–2505, 2020.

[14] PIERRE COMON, GENE GOLUB, LEK-HENG LIM, AND BERNARD MOURRAIN, *Symmetric tensors and symmetric tensor rank*, SIAM J. Matrix Anal. Appl., 30 (3): 1254–1279, 2008.

[15] JOHN H. CONWAY, RONALD H. HARDIN, AND NEIL J. A. SLOANE, *Packing lines, planes, etc.: packings in Grassmannian spaces*, Experiment. Math., 5 (2): 139–159, 1996.

[16] WOJCIECH CZAJA, *Remarks on Naimark's duality*, Proc. Amer. Math. Soc., 136 (3): 867–871, 2008.

[17] S. DATTA, S. HOWARD, AND D. COCHRAN, *Geometry of the Welch bounds*, Linear Algebra Appl., 437 (10): 2455–2470, 2012.

[18] SOMANTIKA DATTA, *Welch bounds for cross correlation of subspaces and generalizations*, Linear Multilinear Algebra, 64 (8): 1484–1497, 2016.

[19] JOE DIESTEL, HANS JARCHOW, AND ANDREW TONGE, *Absolutely summing operators*, vol. 43 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1995.

[20] CUNSHENG DING AND TAO FENG, *Codebooks from almost difference sets*, Des. Codes Cryptogr., 46 (1): 113–126, 2008.

[21] JOZEF DRAVECKÝ, *Spaces with measurable diagonal*, Mat. Časopis Sloven. Akad. Vied, 25 (1): 3–9, 1975.

[22] M. EHLER AND K. A. OKOUDJOU, *Minimization of the probabilistic p-frame potential*, J. Statist. Plann. Inference, 142 (3): 645–659, 2012.

[23] JOSEPH EISNER AND DANIEL FREEMAN, *Continuous Schauder frames for Banach spaces*, J. Fourier Anal. Appl., 26 (4): Paper No. 66, 30, 2020.

[24] SIMON FOUCART AND HOLGER RAUHUT, *A mathematical introduction to compressive sensing*, Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, New York, 2013.

[25] D. FREEMAN, E. ODELL, TH. SCHLUMPRECHT, AND A. ZSÁK, *Unconditional structures of translates for $L_p(\mathbb{R}^d)$*, Israel J. Math., 203 (1): 189–209, 2014.

[26] M. C. HOANG, C. A. FUCHS AND B. C STACEY, *The SIC question: History and state of play*, Axioms, 6 (3), 2017.

[27] JOHN I. HAAS, NATHANIEL HAMMEN, AND DUSTIN G. MIXON, *The Levenstein bound for packings in projective spaces*, Proceedings, volume 10394, Wavelets and Sparsity XVII, SPIE Optical Engineering+Applications, San Diego, California, United States of America, 2017.

[28] MARINA HAIKIN, RAM ZAMIR, AND MATAN GAVISH, *Frame moments and Welch bound with erasures*, 2018 IEEE International Symposium on Information Theory (ISIT), p. 2057–2061, 2018.

[29] PETR HÁJEK, VICENTE MONTESINOS SANTALUCÍA, JON VANDERWERFF, AND VÁCLAV ZIZLER, *Biorthogonal systems in Banach spaces*, vol. 26 of CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, New York, 2008.

[30] DEGUANG HAN AND DAVID R. LARSON, *Frames, bases and group representations*, Mem. Amer. Math. Soc., 147 (697): x+94, 2000.

[31] ROGER A. HORN, *The Hadamard product*, In *Matrix theory and applications (Phoenix, AZ, 1989)*, vol. 40 of Proc. Sympos. Appl. Math., p. 87–169. Amer. Math. Soc., Providence, RI, 1990.

[32] JOHN JASPER, EMILY J. KING, AND DUSTIN G. MIXON, *Game of Sloanes: best known packings in complex projective space*, Proc. SPIE 11138, Wavelets and Sparsity XVIII, 2019.

[33] ZILIN JIANG, JONATHAN TIDOR, YUAN YAO, SHENGTONG ZHANG, AND YUFEI ZHAO, *Equiangular lines with a fixed angle*, Ann. of Math. (2), 194 (3): 729–743, 2021.

[34] W. B. JOHNSON AND A. SZANKOWSKI, *The trace formula in Banach spaces*, Israel J. Math., 203 (1): 389–404, 2014.

[35] GERALD KAISER, *A friendly guide to wavelets*, Modern Birkhäuser Classics, Birkhäuser/Springer, New York, 2011.

[36] GENE S. KOPP, *SIC-POVMs and the Stark Conjectures*, Int. Math. Res. Not. IMRN, (18): 3812–13838, 2021.

[37] J. KOVACEVIC AND A. CHEBIRA, *Life beyond bases: The advent of frames (part I)*, IEEE Signal Processing Magazine, 24 (4): 86–104, 2007.

[38] J. KOVACEVIC AND A. CHEBIRA, *Life beyond bases: The advent of frames (part II)*, IEEE Signal Processing Magazine, 24 (5): 115–125, 2007.

[39] K. MAHESH KRISHNA, *Modular Welch bounds with applications*, arXiv:2201.00319v1 [OA] 2 January, 2022.

[40] K. MAHESH KRISHNA, *Continuous Welch bounds with applications*, Commun. Korean Math. Soc, https://doi.org/10.4134/CKMS.c220200.

[41] K. MAHESH KRISHNA AND P. SAM JOHNSON, *Towards characterizations of approximate Schauder frame and its duals for Banach spaces*, J. Pseudo-Differ. Oper. Appl., 12 (1): Paper No. 9, 13, 2021.

[42] FENGJIE LI, PENGTONG LI, AND DEGUANG HAN, *Continuous framings for Banach spaces*, J. Funct. Anal., 271 (4): 992–1021, 2016.

[43] DAVID J. LOVE AND ROBERT W. HEATH, JR, *Grassmannian beamforming for multiple-input multiple-output wireless systems*, IEEE Trans. Inform. Theory, 49 (10): 2735–2747, 2003.

[44] MIXON DUSTIN G. MAGSINO, MARK AND HANS PARSHALL, *A Delsarte-style proof of the Bukh-Cox bound*, arXiv:1902.00926v3 [MG] 1 May, 2019.

[45] A. RAHIMI, B. DARABY, AND Z. DARVISHI, *Construction of continuous frames in Hilbert spaces*, Azerb. J. Math., 7 (1): 49–58, 2017.

[46] R. A. RANKIN, *The closest packing of spherical caps in n dimensions*, Proc. Glasgow Math. Assoc., 2: 139–144, 1955.

[47] JOSEPH M. RENES, ROBIN BLUME-KOHOUT, A. J. SCOTT, AND CARLTON M. CAVES, *Symmetric informationally complete quantum measurements*, J. Math. Phys., 45 (6): 2171–2180, 2004.

[48] C. ROSE, S. ULUKUS, AND R. D. YATES, *Wireless systems and interference avoidance*, EEE Transactions on Wireless Communications, 1 (3): 415–428, 2002.

[49] MOSHE ROSENFELD, *In praise of the Gram matrix*, In *The mathematics of Paul Erdős, II*, vol. 14 of Algorithms Combin., p. 318–323, Springer, Berlin, 1997.

[50] RAYMOND A. RYAN, *Introduction to tensor products of Banach spaces*, Springer Monographs in Mathematics, Springer-Verlag London, Ltd., London, 2002.

[51] DILIP V. SARWATE, *Bounds on crosscorrelation and autocorrelation of sequences*, IEEE Trans. Inform. Theory, 25 (6): 720–724, 1979.

[52] DILIP V. SARWATE, *Meeting the Welch bound with equality*, In *Sequences and their applications (Singapore, 1998)*, Springer Ser. Discrete Math. Theor. Comput. Sci., p. 79–102, Springer, London, 1999.

[53] A. J. SCOTT, *Tight informationally complete quantum measurements*, J. Phys. A, 39 (43): 13507–13530, 2006.

[54] P. D. SEYMOUR AND THOMAS ZASLAVSKY, *Averaging sets: a generalization of mean values and spherical designs*, Adv. in Math., 52 (3): 213–240, 1984.

[55] THOMAS STROHMER AND ROBERT W. HEATH, JR, *Grassmannian frames with applications to coding and communication*, Appl. Comput. Harmon. Anal., 14 (3): 257–275, 2003.

[56] MÁTYÁS A. SUSTIK, JOEL A. TROPP, INDERJIT S. DHILLON, AND ROBERT W. HEATH, JR, *On the existence of equiangular tight frames*, Linear Algebra Appl., 426 (2–3): 619–635, 2007.

[57] MÁTYÁS A. SUSTIK, JOEL A. TROPP, INDERJIT S. DHILLON, AND ROBERT W. HEATH, JR, *On the existence of equiangular tight frames*, Linear Algebra Appl., 426 (2–3): 619–635, 2007.

[58] MICHEL TALAGRAND, *Pettis integral and measure theory*, Mem. Amer. Math. Soc., 51 (307): ix+224, 1984.

[59] YAN SHUO TAN, *Energy optimization for distributions on the sphere and improvement to the Welch bounds*, Electron. Commun. Probab., 22: Paper No. 43, 12, 2017.

[60] S. M. THOMAS, *Approximate Schauder frames for $\mathbb{R}^n$*, Masters Thesis, St. Louis University, St. Louis, MO, 2012.

[61] NICOLE TOMCZAK-JAEGERMANN, *Banach-Mazur distances and finite-dimensional operator ideals*, vol. 38 of Pitman Monographs and Surveys in Pure and Applied Mathematics, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1989.

[62] JOEL A. TROPP, INDERJIT S. DHILLON, ROBERT W. HEATH, JR., AND THOMAS STROHMER, *Designing structured tight frames via an alternating projection method*, IEEE Trans. Inform. Theory, 51 (1): 188–209, 2005.

[63] SHAYNE WALDRON, *Generalized Welch bound equality sequences are tight frames*, IEEE Trans. Inform. Theory, 49 (9): 2307–2309, 2003.
[64] SHAYNE WALDRON, *A sharpening of the Welch bounds and the existence of real and complex spherical t-designs*, IEEE Trans. Inform. Theory, 63 (11): 6849–6857, 2017.
[65] SHAYNE F. D. WALDRON, *An introduction to finite tight frames*, Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, New York, 2018.
[66] L. WELCH, *Lower bounds on the maximum cross correlation of signals*, IEEE Transactions on Information Theory, 20 (3): 397–399, 1974.
[67] PENGFEI XIA, SHENGLI ZHOU, AND GEORGIOS B. GIANNAKIS, *Achieving the Welch bound with difference sets*, IEEE Trans. Inform. Theory, 51 (5): 1900–1907, 2005.
[68] GERHARD ZAUNER, *Quantum designs: foundations of a noncommutative design theory*, Int. J. Quantum Inf., 9 (1): 445–507, 2011.

(Received December 22, 2022)

K. Mahesh Krishna
Post Doctoral Fellow
Statistics and Mathematics Unit
Indian Statistical Institute, Bangalore Centre
Karnataka 560 059, India
e-mail: kmaheshak@gmail.com