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OCTONION WINDOWED LINEAR CANONICAL TRANSFORM

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Abstract. The Linear Canonical Transform (LCT) is a mathematical transform that generalizes several well-known transforms, including the Fourier transform, the fractional Fourier transform, and the Fresnel transform. It provides a unified framework for understanding and representing a wide range of linear and linear-like transforms, allowing for the analysis and manipulation of signals in various domains. Recently, Gao et al. extended the notion of LCT to octonion domains and showed its efficacy in precisely representing the non-transient octonion-valued signals. However, the octonion LCT exhibits limitations in effectively localizing the frequency characteristics of non-transient octonion-valued signals. As such, it is imperative to introduce the Octonion Windowed Linear Canonical Transform (OCWLCT) and explore its fundamental characteristics. We delve into the inversion formula and the orthogonality relation for the one-dimensional OCWLCT. Additionally, we derive the inversion formula for the three-dimensional Octonion Windowed Linear Canonical Transform (OCWLCT).

1. Introduction

The linear canonical transform has found wide applications in many fields of applied mathematics, optics, signal processing, filter design, radar system analysis, phase retrieval and pattern recognition [10, 14, 16]. It is a four parameter class of linear integral transform and was first introduced in 1970s [19]. The LCT is also called as ABCD transform, generalized Fresnel transform and the affine Fourier transform [1]. It is a generalization of many optical transforms like the Fourier transform (FT), the Fractional Fourier transform (FRFT) [20], the Lorenz transform [1], the Fresnel transform and scaling operations. With more degrees of freedom compared to the Fourier transform and Fractional Fourier transform, the LCT is more flexible but with similar computation cost as the conventional Fourier transform [13]. However, the LCT cannot reveal the local LCT frequency contents due to its global kernel. The windowed Fourier transform (WFT) with a local window function handles this kind of situation well, but unfortunately, the WFT often performs unsatisfactorily for its low resolution. Therefore in order to attain the local contents and high localization properties of signal, it is desirable to develop a new transform by replacing the Fourier transform kernel with the LCT kernel in the definition of WFT. This new transform is called windowed linear canonical transform (WLCT) and was first introduced by Bultheel and Martinez-Sulbaran [6]. The WLCT offers a flexible local frequency content, eliminates cross term and enjoys high resolution of a signal.

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In recent times, hyper-complex Fourier transforms have ignited a surge of interest, presenting a compelling approach for treating multi-channel signals as unified algebraic entities without sacrificing crucial spectral information. This newborn paradigm has yielded diverse applications in in signal and image compression, edge detection, and pattern recognition [9, 17, 21, 24, 25, 26]. The landscape of hyper-complex Fourier transforms is adorned with various formulations, each offering unique perspectives [8]. Among these, the two-dimensional quaternion Fourier transforms stand as fundamental and paramount. Quaternions, also known as the Cayley-Dickson algebra of order 4, have found substantial applications in filtering, image compression, reconstruction, and beyond [2, 22]. Yet, beyond the realm of quaternions lies a domain deserving of equal attention in hyper-complex signal processing: the octonions. Octonions, also known as Cayley-Dickson algebra of order 8, have been captivating modern signal and image processing, emerging as a demanding and vibrant research frontier since their discovery by Hahn and Snopek in 2011 [12]. For more about applications and some interesting properties of OFT, we refer to [3, 4, 5, 12, 18].

Recentl, W. B. Gao and B. Z. LI have expanded this fascination by introducing the octonion linear canonical transform, creatively blending it with the Octonion Fourier transform [11]. While octonions yield immense potential in diverse fields, their theoretical foundations are still in its infancy. Taking this opportunity, we extend the notion of windowed linear canonical transform to the octonion domains. Our exploration uncovers essential results, like the orthogonality relation and inversion formula, offering exciting opportunities for representing multi-dimensional data. From signal and image processing to remote sensing and data compression, this proposed transform has the potential to advance various disciplines.

The paper is organized as follows, Section 2 provides general definitions and fundamental properties of octonion algebra. Section 3 focuses on introducing the Octonion Windowed Linear Canonical Transform (OCWLCT) along with its basic properties. In Section 4, we discuss the potential applications of the proposed integral transform. Finally, our expedition concludes with an enlightening epilogue in Section 5.

2. Preliminaries

In the present section, we mainly review some basic facts and notations on the octonion algebra [7], which will be very useful in our study on octonion windowed linear canonical transform.

In accordance with Cayley-Dickson construction, the octonion algebra is denoted by $\mathbb{O}$ [7]. It is a non-associative and non-commutative algebra defined over $\mathbb{R}$ generated by the elements

$$\begin{aligned} &e_0=(1,0),\ e_1=(i,0),\ e_2=(j,0),\ e_3=(k,0),\\ &e_4=(0,1),\ e_5=(0,i),\ e_6=(0,j),\ e_7=(0,k). \end{aligned}$$

An arbitrary $o\in\mathbb{O}$ can be represented as

$$o=o_0+o_1e_1+o_2e_2+o_3e_3+o_4e_4+o_5e_5+o_6e_6+o_7e_7,$$

where $o_0, o_1, o_2, o_3, o_4, o_5, o_6, o_7 \in \mathbb{R}$. The conjugate of octonion is defined by

$$\overline{o} = o_0 - o_1e_1 - o_2e_2 - o_3e_3 - o_4e_4 - o_5e_5 - o_6e_6 - o_7e_7.$$

Also the norm of octonions can be given as

$$|o| = \sqrt{o\overline{o}} = \sqrt{\overline{o}o}$$

$$\text{or} \qquad |o|^2 = \sum_{r=0}^{7} o_r^2$$

$$\text{and} \qquad |o_1o_2| = |o_1||o_2|.$$

Each $o \in \mathbb{O}$ can be represented in the quaternionic form as

$$o = a + be_4,$$

$$\text{where} \qquad a = o_0 + o_1e_1 + o_2e_2 + o_3e_3 \in \mathbb{H}$$

$$\text{and} \qquad b = o_4 + o_5e_1 + o_6e_2 + o_7e_3 \in \mathbb{H}.$$

Thus an octonion interms of quaternions is equal to the space $\mathbb{H}\bigoplus\mathbb{H}$, and the multiplication for any two pairs $\in \mathbb{H}\bigoplus\mathbb{H}$ is given as

$$(a,b)(c,d) = (ac - \overline{d}b, da + b\overline{c}).$$

The multiplication rules for octonion algebra are presented in Table 1.

*	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

Table 1: *Multiplication rules in octonion algebra*

The following useful properties are given in [3, 15].

PROPERTY 1. Let $a, b \in \mathbb{H}$. Then
(a) $e_4a = \overline{a}e_4$;
(b) $e_4(ae_4) = -\overline{a}$;
(c) $(ae_4)e_4 = -a$;
(d) $a(be_4) = (ba)e_4$;

(e) $\quad (ae_4)b = (a\overline{b})e_4;$
(f) $\quad (ae_4)(be_4) = -\overline{b}a.$

PROPERTY 2. The quaternionic form of any octonion $a+be_4\colon\ a,b\in\mathbb{H}$ satisfies

$$\overline{a+be_4} = \overline{a} - be_4,$$
$$|a+be_4|^2 = |a|^2 + |b|^2.$$

Now we will present the definition of octonion Fourier transform (OFT) and octonion linear canonical transform (OCLCT).

DEFINITION. [18] Let $f:\mathbb{R}\to\mathbb{O}$ be an octonion valued function, then one dimensional OFT is given by

$$F_1(f)(\omega) = \int_{\mathbb{R}} f(x)\mathbf{e}^{-e_4 2\pi x\omega}dx \tag{1}$$

and the inverse is given by

$$f(x) = F_1^{-1}\left(F_1(f)\right)(x) = \int_{\mathbb{R}} F_1(f)(\omega)\mathbf{e}^{e_4 2\pi x\omega}d\omega. \tag{2}$$

Also the three dimensional OFT [18, 23] of an octonion valued function $f:\mathbb{R}^3\to\mathbb{O}$ is given by

$$F_o(f)(\boldsymbol{\omega}) = \int_{\mathbb{R}^3} f(\mathbf{x})\mathbf{e}^{-e_1 2\pi x_1\omega_1}\mathbf{e}^{-e_2 2\pi x_2\omega_2}\mathbf{e}^{-e_4 2\pi x_3\omega_3}d\mathbf{x}, \tag{3}$$

where $\mathbf{x} = (x_1,x_2,x_3)\in\mathbb{R}^3$, $\boldsymbol{\omega} = (\omega_1,\omega_2,\omega_3)\in\mathbb{R}^3$.

Since the octonions are non-associative and non-commutative, the order of imaginary units in formula (3) remains unaltered, as the changes in the position of these imaginary units give other definition of hypercomplex Fourier transforms. Also the multiplication in the OFT is done from left to right because of non-associativity of octonions. Some properties of OFT presented in [3, 4, 12, 18] are:

- The inverse of 3D-OFT is

$$\begin{aligned} f(\mathbf{x}) &= F_o^{-1}\left(F_o(f)\right)(\mathbf{x}) \\ &= \int_{\mathbb{R}^3} F_o(f)(\boldsymbol{\omega})\mathbf{e}^{e_4 2\pi x_3\omega_3}\mathbf{e}^{e_2 2\pi x_2\omega_2}\mathbf{e}^{e_1 2\pi x_1\omega_1}d\boldsymbol{\omega}, \end{aligned} \tag{4}$$

  where $\mathbf{x} = (x_1,x_2,x_3)\in\mathbb{R}^3, \boldsymbol{\omega} = (\omega_1,\omega_2,\omega_3)\in\mathbb{R}^3$.

- The Parsevals theorem for OFT is

$$\|f\|_2 = \|F_o(f)\|_2.$$

- The Hausdorff-Young inequality for OFT is

$$\|F_o(f)\|_q = \|f\|_p\ ; \quad 1\leqslant p\leqslant 2 \ \text{ and } \ \frac{1}{p}+\frac{1}{q} = 1.$$

DEFINITION. [11] Let $f:\mathbb{R}\to\mathbb{O}$ be an octonion valued function, then the one dimensional octonion linear canonical transform (OCLCT) is given by

$$\mathscr{L}^{o}_{1,M}(f)(\omega)=\int_{\mathbb{R}} f(x)\mathscr{K}^{e_4}_{M}(x,\omega)dx, \tag{5}$$

where $M=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in\mathbb{R}^{2\times 2}$ be a matrix parameter satisfying $detM=1$, and the kernel signal is given by

$$\mathscr{K}^{e_4}_{M}(x,\omega)=\begin{cases} \frac{1}{\sqrt{2\pi|b|}}\mathbf{e}^{e_4\left(\frac{a}{2b}x^2-\frac{x\omega}{b}+\frac{d}{2b}\omega^2-\frac{\pi}{2}\right)}, & b\neq 0 \\ \sqrt{d}\mathbf{e}^{e_4\frac{cd}{2}\omega^2}\delta(x-d\omega), & b=0 \end{cases}.$$

Here we only consider the case $b\neq 0$, as for the case $b=0$ the OCLCT of a signal is essentially a chirp multiplication.

The inverse for the one dimensional OCLCT is

$$f(x)=\int_{\mathbb{R}}\mathscr{L}^{o}_{1,M}(f)(\omega)\mathscr{K}^{-e_4}_{M}(x,\omega)d\omega,$$

where $\mathscr{K}^{-e_4}_{M}(x,\omega)=\mathscr{K}^{e_4}_{M^{-1}}(\omega,x)=\frac{1}{\sqrt{2\pi|b|}}\mathbf{e}^{-e_4\left(\frac{a}{2b}x^2-\frac{x\omega}{b}+\frac{d}{2b}\omega^2-\frac{\pi}{2}\right)}$, $M^{-1}=\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\in\mathbb{R}^{2\times 2}$ and $b\neq 0$.

DEFINITION. [11] Let $f:\mathbb{R}^3\to\mathbb{O}$ be an octonion valued signal, then the three dimensional OCLCT is defined as

$$\mathscr{L}^{o}_{M_1,M_2,M_3}(f)(\boldsymbol{\omega})=\int_{\mathbb{R}^3} f(\mathbf{x})\mathscr{K}^{e_1}_{M_1}(x_1,\omega_1)\mathscr{K}^{e_2}_{M_2}(x_2,\omega_2)\mathscr{K}^{e_4}_{M_3}(x_3,\omega_3)d\mathbf{x}, \tag{6}$$

where $\mathbf{x}=(x_1,x_2,x_3)\in\mathbb{R}^3$, $\boldsymbol{\omega}=(\omega_1,\omega_2,\omega_3)\in\mathbb{R}^3$, $M_k=\begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix}\in\mathbb{R}^{2\times 2}$ be a matrix parameter satisfying $det(M_K)=1$, for $k=1,2,3$ and the kernel signals are given by

$$\mathscr{K}^{e_1}_{M_1}(x_1,\omega_1)=\begin{cases} \frac{1}{\sqrt{2\pi|b_1|}}\mathbf{e}^{e_1\left(\frac{a_1}{2b_1}x_1^2-\frac{x_1\omega_1}{b_1}+\frac{d_1}{2b_1}\omega_1^2-\frac{\pi}{2}\right)}, & b_1\neq 0 \\ \sqrt{d_1}\mathbf{e}^{e_1\frac{c_1d_1}{2}\omega_1^2}\delta(x_1-d_1\omega_1), & b_1=0 \end{cases} \tag{7}$$

$$\mathscr{K}^{e_2}_{M_2}(x_2,\omega_2)=\begin{cases} \frac{1}{\sqrt{2\pi|b_2|}}\mathbf{e}^{e_2\left(\frac{a_2}{2b_2}x_2^2-\frac{x_2\omega_2}{b_2}+\frac{d_2}{2b_2}\omega_2^2-\frac{\pi}{2}\right)}, & b_2\neq 0 \\ \sqrt{d_2}\mathbf{e}^{e_2\frac{c_2d_2}{2}\omega_2^2}\delta(x_2-d_2\omega_2), & b_2=0 \end{cases} \tag{8}$$

$$\mathscr{K}_{M_3}^{e_4}(x_3,\omega_3)=\begin{cases}\frac{1}{\sqrt{2\pi|b_3|}}\mathbf{e}^{e_4\left(\frac{a_3}{2b_3}x_3^2-\frac{x_3\omega_3}{b_3}+\frac{d_3}{2b_3}\omega_3^2-\frac{\pi}{2}\right)}, & b_3\neq 0\\ \sqrt{d_3}\mathbf{e}^{e_4\frac{c_3d_3}{2}\omega_3^2}\delta(x_3-d_3\omega_3), & b_3=0,\end{cases}\tag{9}$$

where $\delta(x)$ represents the Dirac function.

The OCLCT of a signal is essentially a chirp multiplication for $b_k=0$, $(k=1,2,3)$. Thus we will set $b_k\neq 0$, $(k=1,2,3)$ from onwards unless stated otherwise.

Using the definition of OCLCT, we can formulate the definition of octonion windowed linear canonical transform (OCWLCT).

3. Octonion windowed linear canonical transform

DEFINITION. Let $M=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\in\mathbb{R}^{2\times 2}$ be a matrix parameter satisfying $det(M)=1$ and $\phi\in\mathbb{L}^2(\mathbb{R},\mathbb{O})\setminus\{0\}$ be a window function . Then the octonion windowed linear canonical transform (OCWLCT) is given by

$$\mathscr{G}_{\phi}^{M}f(\omega,u)=\int_{\mathbb{R}}f(x)\overline{\phi(x-u)}\mathscr{K}_M^{e_4}(x,\omega)dx,\tag{10}$$

where $\mathscr{K}_M^{e_4}(x,\omega)$ is the octonion LCT kernel and is given by

$$\mathscr{K}_M^{e_4}(x,\omega)=\begin{cases}\frac{1}{\sqrt{2\pi|b|}}\mathbf{e}^{e_4\left(\frac{a}{2b}x^2-\frac{x\omega}{b}+\frac{d}{2b}\omega^2-\frac{\pi}{4}\right)}, & b\neq 0\\ \sqrt{d}\mathbf{e}^{e_4\frac{cd}{2}\omega^2}\delta(x-d\omega), & b=0\end{cases}.$$

Here we only consider the case $b\neq 0$, as the OCWLCT is just a chirp multiplication for the case $b=0$.

For a fixed u, we have

$$\mathscr{G}_{\phi}^{M}f(\omega,u)=\mathscr{L}_M^{\mathscr{O}}\left\{f(x)\overline{\phi(x-u)}\right\}(\omega).\tag{11}$$

This implies

$$f(x)\overline{\phi(x-u)}=(\mathscr{L}_M^{\mathscr{O}})^{-1}\left\{\mathscr{G}_{\phi}^{M}f(\omega,u)\right\}=\int_{\mathbb{R}}\mathscr{G}_{\phi}^{M}f(\omega,u)\overline{\mathscr{K}_M^{e_4}(x,\omega)}d\omega.\tag{12}$$

Now we derive some important properties of one dimensional octonion windowed linear canonical transform.

THEOREM 1. (Boundedness) *Let $\phi\in\mathbb{L}^2(\mathbb{R},\mathbb{O})\setminus\{0\}$ be a window function and $f\in\mathbb{L}^2(\mathbb{R},\mathbb{O})$, then*

$$\left|\mathscr{G}_{\phi}^{M}f(\omega,u)\right|\leqslant\frac{1}{\sqrt{2\pi|b|}}\|f\|_{\mathbb{L}^2(\mathbb{R},\mathbb{O})}\|\phi\|_{\mathbb{L}^2(\mathbb{R},\mathbb{O})}.$$

Proof. By Cauchy-Schwartz inequality, we have

$$\begin{aligned}
\left|\mathscr{G}_{\phi}^{M} f(\omega,u)\right|^2 &= \left|\int_{\mathbb{R}} f(x)\overline{\phi(x-u)}\mathscr{K}_M^{e_4}(x,\omega)dx\right|^2 \\
&\leqslant \left(\int_{\mathbb{R}} \left|f(x)\overline{\phi(x-u)}\mathscr{K}_M^{e_4}(x,\omega)\right| dx\right)^2 \\
&= \left(\frac{1}{\sqrt{2\pi|b|}}\int_{\mathbb{R}} \left|f(x)\overline{\phi(x-u)}\right| dx\right)^2 \\
&\leqslant \frac{1}{2\pi|b|}\left(\int_{\mathbb{R}} |f(x)|^2 dx\right)\left(\int_{\mathbb{R}} \left|\overline{\phi(x-u)}\right|^2 dx\right) \\
&= \frac{1}{2\pi|b|}\|f\|^2_{\mathbb{L}^2(\mathbb{R},\mathbb{O})}\|\phi\|^2_{\mathbb{L}^2(\mathbb{R},\mathbb{O})}.
\end{aligned}$$

We apply the change of variables $x-u=t$ in the last step. Then we have

$$\left|\mathscr{G}_{\phi}^{M} f(\omega,u)\right| \leqslant \frac{1}{\sqrt{2\pi|b|}}\|f\|_{\mathbb{L}^2(\mathbb{R},\mathbb{O})}\|\phi\|_{\mathbb{L}^2(\mathbb{R},\mathbb{O})}.$$

This completes the proof. $\square$

THEOREM 2. (Linearity) *Let $\phi \in \mathbb{L}^2(\mathbb{R},\mathbb{O}) \setminus \{0\}$ be a window function and $f,g \in \mathbb{L}^2(\mathbb{R},\mathbb{O})$, then OCWLCT is a linear operator*

$$\mathscr{G}_{\phi}^{M}\left[(\alpha f+\beta g)\right](\omega,u) = \alpha\mathscr{G}_{\phi}^{M} f(\omega,u)+\beta\mathscr{G}_{\phi}^{M} g(\omega,u)$$

for arbitrary constants α and β.

Proof. By definition of OCWLCT, we have

$$\begin{aligned}
\mathscr{G}_{\phi}^{M}\left[(\alpha f+\beta g)\right](\omega,u) &= \int_{\mathbb{R}} \left[\alpha f+\beta g\right](x)\overline{\phi(x-u)}\mathscr{K}_M^{e_4}(x,\omega)dx \\
&= \alpha\int_{\mathbb{R}} f(x)\overline{\phi(x-u)}\mathscr{K}_M^{e_4}(x,\omega)dx \\
&\quad +\beta\int_{\mathbb{R}} g(x)\overline{\phi(x-u)}\mathscr{K}_M^{e_4}(x,\omega)dx \\
&= \alpha\mathscr{G}_{\phi}^{M} f(\omega,u)+\beta\mathscr{G}_{\phi}^{M} g(\omega,u).
\end{aligned}$$

This completes the proof. $\square$

THEOREM 3. (Shift) *Let $\phi \in \mathbb{L}^2(\mathbb{R},\mathbb{O}) \setminus \{0\}$ be a window function and $f \in \mathbb{L}^2(\mathbb{R},\mathbb{O})$ then we have*

$$\mathscr{G}_{\phi}^{M}\left\{f(x-k)\right\}(\omega,u) = \mathbf{e}^{e_4 k\omega c}\mathbf{e}^{-e_4\frac{ak^2}{2}c}\mathscr{G}_{\phi}^{M} f(\omega-ka,u-k).$$

Proof. By definition of OCWLCT, we have for any real no k

$$\begin{aligned}
\mathscr{G}_{\phi}^{M}\{f(x-k)\}(\omega,u) &= \int_{\mathbb{R}} f(x-k)\overline{\phi(x-u)}\mathscr{K}_{M}^{e_4}(x,\omega)dx \\
&= \int_{\mathbb{R}} f(t)\overline{\phi(t-(u-k))}\mathscr{K}_{M}^{e_4}(t+k,\omega)dt \\
&= \int_{\mathbb{R}} f(t)\overline{\phi(t-(u-k))} \\
&\quad \times \frac{1}{\sqrt{2\pi|b|}}\mathbf{e}^{e_4\left(\frac{a}{2b}(t+k)^2-\frac{(t+k)\omega}{b}+\frac{d}{2b}\omega^2-\frac{\pi}{2}\right)}dt \\
&= \frac{1}{\sqrt{2\pi|b|}}\int_{\mathbb{R}} f(t)\overline{\phi(t-(u-k))} \\
&\quad \times \mathbf{e}^{e_4\left(\frac{a}{2b}(t^2+k^2+2tk)-\frac{t\omega}{b}-\frac{k\omega}{b}+\frac{d}{2b}\omega^2-\frac{\pi}{2}\right)}dt \\
&= \frac{1}{\sqrt{2\pi|b|}}\int_{\mathbb{R}} f(t)\overline{\phi(t-(u-k))} \\
&\quad \times \mathbf{e}^{e_4\left(\frac{a}{2b}t^2-\frac{t}{b}(\omega-ka)+\frac{d}{2b}\omega^2-\frac{\pi}{2}\right)} \\
&\quad \times \mathbf{e}^{e_4\left(\frac{1}{2}\frac{a}{b}k^2\right)}\mathbf{e}^{e_4\left(\frac{-k\omega}{b}\right)}dt.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\mathscr{G}_{\phi}^{M}\{f(x-k)\}(\omega,u) &= \frac{1}{\sqrt{2\pi|b|}}\int_{\mathbb{R}} f(t)\overline{\phi(t-(u-k))} \\
&\quad \times \mathbf{e}^{e_4\left(\frac{a}{2b}t^2-\frac{t}{b}(\omega-ka)+\frac{d}{2b}(\omega-ka)^2-\frac{\pi}{2}\right)} \\
&\quad \times \mathbf{e}^{e_4\frac{d}{2b}\left(2(\omega-ka)ka+(ka)^2\right)}\mathbf{e}^{e_4\left(\frac{a}{2b}k^2\right)}\mathbf{e}^{e_4\left(\frac{-k\omega}{b}\right)}dt.
\end{aligned}$$

By using the definition of OCWLCT, the above equation can be written as

$$\begin{aligned}
\mathscr{G}_{\phi}^{M}\{f(x-k)\}(\omega,u) &= \mathbf{e}^{e_4\frac{d}{2b}\left(2(\omega-ka)ka+(ka)^2\right)}\mathbf{e}^{e_4\left(\frac{a}{2b}k^2\right)} \\
&\quad \times \mathbf{e}^{e_4\left(\frac{-k\omega}{b}\right)}\mathscr{G}_{\phi}^{M}f(\omega-ka,u-k) \\
&= \mathbf{e}^{e_4\frac{d}{2b}\left(2\omega ka-2(ka)^2+(ka)^2\right)}\mathbf{e}^{e_4\left(\frac{a}{2b}k^2\right)} \\
&\quad \times \mathbf{e}^{e_4\left(\frac{-k\omega}{b}\right)}\mathscr{G}_{\phi}^{M}f(\omega-ka,u-k) \\
&= \mathbf{e}^{e_4\frac{d}{b}(\omega ka)}\mathbf{e}^{-e_4\frac{d}{2b}(ka)^2}\mathbf{e}^{e_4\left(\frac{a}{2b}k^2\right)} \\
&\quad \times \mathbf{e}^{e_4\left(\frac{-k\omega}{b}\right)}\mathscr{G}_{\phi}^{M}f(\omega-ka,u-k) \\
&= \mathbf{e}^{e_4k\omega\left(\frac{ad}{b}-\frac{1}{b}\right)}\mathbf{e}^{-e_4\frac{ak^2}{2}\left(\frac{ad}{b}-\frac{1}{b}\right)}\mathscr{G}_{\phi}^{M}f(\omega-ka,u-k) \\
&= \mathbf{e}^{e_4k\omega c}\mathbf{e}^{-e_4\frac{ak^2}{2}c}\mathscr{G}_{\phi}^{M}f(\omega-ka,u-k).
\end{aligned}$$

This completes the proof. $\square$

THEOREM 4. (parity) *Let* $\phi \in \mathbb{L}^2(\mathbb{R},\mathbb{O})\setminus\{0\}$ *be a window function and* $f \in \mathbb{L}^2(\mathbb{R},\mathbb{O})$ *then we have*

$$\mathcal{G}^M_{P\phi}\{Pf\}(\omega,u) = \mathcal{G}^M_{\phi} f(-\omega,-u),$$

where $P\phi(x) = \phi(-x)$ *for every* $\phi \in \mathbb{L}^2(\mathbb{R})$.

Proof. For every $f \in \mathbb{L}^2(\mathbb{R})$, we have

$$\begin{aligned}
\mathcal{G}^M_{P\phi}\{Pf\}(\omega,u) &= \int_{\mathbb{R}} f(-x)\overline{\phi(-(x-u))}\mathcal{K}^{e_4}_M(x,\omega)dx \\
&= \int_{\mathbb{R}} f(-x)\overline{\phi(-(x-u))}\frac{1}{\sqrt{2\pi|b|}}\mathbf{e}^{e_4\left(\frac{a}{2b}x^2-\frac{x\omega}{b}+\frac{d}{2b}\omega^2-\frac{\pi}{2}\right)}dx \\
&= \int_{\mathbb{R}} f(-x)\overline{\phi(-(x-u))} \\
&\qquad \times \frac{1}{\sqrt{2\pi|b|}}\mathbf{e}^{e_4\left(\frac{a}{2b}(-x)^2-\frac{(-x)(-\omega)}{b}+\frac{d}{2b}(-\omega)^2-\frac{\pi}{2}\right)}dx.
\end{aligned}$$

This completes the proof according to definition of OCWLCT. □

Now we will derive the relationship between one dimensional OFT and the one dimensional OCWLCT as follows:

$$\begin{aligned}
\mathcal{G}^M_{\phi} f(\omega,u) &= \frac{1}{\sqrt{2\pi|b|}}\int_{\mathbb{R}} f(x)\overline{\phi(x-u)}\mathbf{e}^{e_4\left(\frac{a}{2b}x^2-\frac{x\omega}{b}+\frac{d}{2b}\omega^2-\frac{\pi}{2}\right)}dx, \quad b\neq 0 \\
&= \frac{1}{\sqrt{2\pi|b|}}\int_{\mathbb{R}} f(x)\overline{\phi(x-u)}\mathbf{e}^{e_4\left(\frac{a}{2b}x^2\right)}\mathbf{e}^{e_4\left(\frac{d}{2b}\omega^2-\frac{\pi}{2}\right)}\mathbf{e}^{-e_4\left(\frac{x\omega}{b}\right)}dx, \; b\neq 0 \\
&= \frac{1}{\sqrt{2\pi|b|}}\int_{\mathbb{R}} f(x)\overline{\phi(x-u)}\mathbf{e}^{e_4\left(\frac{a}{2b}x^2\right)}\mathbf{e}^{-e_4\left(\frac{2\pi x\omega}{2\pi b}\right)}dx\,\mathbf{e}^{e_4\left(\frac{d}{2b}\omega^2-\frac{\pi}{2}\right)}, \; b\neq 0 \\
&= \frac{1}{\sqrt{2\pi|b|}}F_1(g)\left(\frac{\omega}{2\pi|b|}\right)\mathbf{e}^{e_4\left(\frac{d}{2b}\omega^2-\frac{\pi}{2}\right)}.
\end{aligned}$$

Hence,

$$\mathcal{G}^M_{\phi} f(\omega,u) = \frac{1}{\sqrt{2\pi|b|}}F_1(g)\left(\frac{\omega}{2\pi|b|}\right)\mathbf{e}^{e_4\left(\frac{d}{2b}\omega^2-\frac{\pi}{2}\right)}, \tag{13}$$

where $b\neq 0$ and $g(x) = f(x)\overline{\phi(x-u)}\mathbf{e}^{e_4\left(\frac{a}{2b}x^2\right)}$.

THEOREM 5. (Inversion formula for one dimensional Octonion windowed linear canonical transform) *Let* $\phi \in \mathbb{L}^2(\mathbb{R},\mathbb{O})\setminus\{0\}$ *be a window function,* $0 < \|\phi\|^2 < \infty$ *and* $f \in \mathbb{L}^2(\mathbb{R},\mathbb{O})$. *Then we have*

$$f(x) = \frac{1}{\overline{\phi(x-u)}}\int_{\mathbb{R}} \mathcal{G}^M_{\phi} f(\omega,u)\overline{\mathcal{K}^{e_4}_M(x,\omega)}d\omega. \tag{14}$$

Proof. From (13), we have

$$\mathscr{G}_\phi^M f(\omega,u) = \mathbf{e}^{e_4\left(\frac{d}{2b}\omega^2-\frac{\pi}{2}\right)} \frac{1}{\sqrt{2\pi|b|}} F_1(g)\left(\frac{\omega}{2\pi|b|}\right),$$

where $b \neq 0$ and $g(x) = f(x)\overline{\phi(x-u)}\mathbf{e}^{e_4\left(\frac{a}{2b}x^2\right)}$. This implies,

$$\sqrt{2\pi|b|}\mathscr{G}_\phi^M f(\omega,u)\mathbf{e}^{-e_4\left(\frac{d}{2b}\omega^2-\frac{\pi}{2}\right)} = F_1(g)\left(\frac{\omega}{2\pi|b|}\right).$$

Using (2), we obtain

$$f(x)\overline{\phi(x-u)}\mathbf{e}^{e_4\left(\frac{a}{2b}x^2\right)} = \int_{\mathbb{R}} \sqrt{2\pi|b|}\mathscr{G}_\phi^M f(\omega,u)\mathbf{e}^{-e_4\left(\frac{d}{2b}\omega^2-\frac{\pi}{2}\right)} \times \mathbf{e}^{e_4\left(\frac{2\pi x\omega}{2\pi b}\right)} d\left(\frac{\omega}{2\pi|b|}\right).$$

Hence,

$$f(x) = \frac{1}{\overline{\overline{\phi(x-u)}}} \int_{\mathbb{R}} \mathscr{G}_\phi^M f(\omega,u) \frac{1}{\sqrt{2\pi|b|}} \mathbf{e}^{-e_4\left(\frac{d}{2b}\omega^2-\frac{\pi}{2}\right)} \times \mathbf{e}^{e_4\left(\frac{2\pi x\omega}{2\pi b}\right)}\mathbf{e}^{-e_4\left(\frac{a}{2b}x^2\right)} d\omega.$$

Or

$$f(x) = \frac{1}{\overline{\overline{\phi(x-u)}}} \int_{\mathbb{R}} \mathscr{G}_\phi^M f(\omega,u)\overline{\mathscr{K}_M^{e_4}(x,\omega)} d\omega.$$

This completes the proof. □

THEOREM 6. (Orthogonality relation) *Let* $\phi,\psi \in \mathbb{L}^2(\mathbb{R},\mathbb{O}) \setminus \{0\}$ *be window functions and* $f,g \in \mathbb{L}^2(\mathbb{R},\mathbb{O})$ *then we have*

$$\left\langle \mathscr{G}_\phi^M f(\omega,u), \mathscr{G}_\psi^M g(\omega,u)\right\rangle = \langle f,g\rangle \langle \phi,\psi\rangle. \tag{15}$$

Proof. Using the definition of OCWLCT and the inner product relation, we have

$$\begin{aligned}
&\left\langle \mathscr{G}_\phi^M f(\omega,u), \mathscr{G}_\psi^M g(\omega,u)\right\rangle \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathscr{G}_\phi^M\{f\}(\omega,u), \overline{\mathscr{G}_\psi^M\{g\}(\omega,u)} d\omega du \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \left[\mathscr{G}_\phi^M\{f\}(\omega,u) \int_{\mathbb{R}} \overline{g(x)(\overline{\psi(x-u)})\mathscr{K}_M^{e_4}(x,\omega) dx}\right] d\omega du \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \left[[\mathscr{G}_\phi^M\{f\}(\omega,u)\mathscr{K}_M^{-e_4}(x,\omega)\psi(x-u)\overline{g(x)}\right] d\omega du dx \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \left[\int_{\mathbb{R}} \mathscr{G}_\phi^M\{f\}(\omega,u)\mathscr{K}_M^{-e_4}(x,\omega) d\omega\right] \psi(x-u)\overline{g(x)} du dx.
\end{aligned}$$

By using (14), we have

$$\left\langle \mathcal{G}_\phi^M f(\omega,u), \mathcal{G}_\psi^M g(\omega,u) \right\rangle = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x)\overline{\phi(x-u)})\psi(x-u)\overline{g(x)}dudx.$$

Using the change of variables $x-u=y$, we have

$$\begin{aligned}\left\langle \mathcal{G}_\phi^M f(\omega,u), \mathcal{G}_\psi^M g(\omega,u) \right\rangle &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x)\overline{\phi(y)})\psi(y)\overline{g(x)}dydx \\ &= \int_{\mathbb{R}} f(x)\overline{g(x)}dx \int_{\mathbb{R}} \psi(y)\overline{\phi(y)}dy \\ &= \langle f,g\rangle \langle \phi,\psi\rangle .\end{aligned}$$

This completes the proof. □

Now we will give the definition of three dimensional octonion windowed linear canonical transform (OCWLCT).

DEFINITION. Let $f:\mathbb{R}^3 \to \mathbb{O}$ be an octonion valued signal, then three dimensional octonion windowed linear canonical transform (OCWLCT) is given by

$$\begin{aligned}\mathcal{G}_\phi^{M_1,M_2,M_3}\{f\}(\boldsymbol{\omega},\mathbf{u}) = \int_{\mathbb{R}^3} & f(\mathbf{x})\overline{\phi(\mathbf{x}-\mathbf{u})}\mathcal{K}_{M_1}^{e_1}(x_1,\omega_1)\mathcal{K}_{M_2}^{e_2}(x_2,\omega_2) \\ & \times \mathcal{K}_{M_3}^{e_4}(x_3,\omega_3)d\mathbf{x},\end{aligned} \tag{16}$$

where $\mathbf{x}=(x_1,x_2,x_3)\in\mathbb{R}^3$, $\mathbf{u}=(u_1,u_2,u_3)\in\mathbb{R}^3$, $\boldsymbol{\omega}=(\omega_1,\omega_2,\omega_3)\in\mathbb{R}^3$, $M_k = \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix} \in \mathbb{R}^{2\times 2}$ be a matrix parameter satisfying $det(M_k)=1$ for $k=1,2,3$ and $\mathcal{K}_{M_1}^{e_1}(x_1,\omega_1)$, $\mathcal{K}_{M_2}^{e_2}(x_2,\omega_2)$ and $\mathcal{K}_{M_3}^{e_4}(x_3,\omega_3)$ are given by (7), (8) and (9) respectively.

Next, we will prove the inversion formula for the three dimensional OCWLCT, but in its proof we need definition of QWLCT.

The Quaternion windowed linear canonical transform of a function $f:\mathbb{R}^2\to\mathbb{H}$ with a window function ϕ is given by

$$\mathcal{G}_\phi^{M_1,M_2}\{f\}(\boldsymbol{\omega},\mathbf{u}) = \int_{\mathbb{R}^2} f(\mathbf{x})\overline{\phi(\mathbf{x}-\mathbf{u})}\mathcal{K}_{M_1}^{e_1}(x_1,\omega_1)\mathcal{K}_{M_2}^{e_2}(x_2,\omega_2)d\mathbf{x}, \tag{17}$$

and its inverse is given by

$$f(\mathbf{x}) = \int_{\mathbb{R}^2} \mathcal{G}_\phi^{M_1,M_2}\{f\}(\boldsymbol{\omega},\mathbf{u})\overline{\phi(\mathbf{x}-\mathbf{u})}\mathcal{K}_{M_2}^{-e_2}(\omega_2,x_2)\mathcal{K}_{M_1}^{-e_1}(\omega_1,x_1)d\boldsymbol{\omega}, \tag{18}$$

where $\mathbf{x}=(x_1,x_2), \boldsymbol{\omega}=(\omega_1,\omega_2)$ and $\mathcal{K}_{M_1}^{e_1}(x_1,\omega_1)$ and $\mathcal{K}_{M_2}^{e_2}(x_2,\omega_2)$ are given by (7) and (8), respectively.

THEOREM 7. (Inversion formula for three dimensional OCWLCT) *Let $f:\mathbb{R}^3\to\mathbb{O}$ be an octonion valued function, then we have,*

$$f(\mathbf{x}) = \int_{\mathbb{R}^3} \mathcal{G}_\phi^{M_1,M_2,M_3}\{f\}(\boldsymbol{\omega},\mathbf{u})\mathcal{K}_{M_3^{-1}}^{e_4}(\omega_3,x_3)\mathcal{K}_{M_2^{-1}}^{e_2}(\omega_2,x_2)\mathcal{K}_{M_1^{-1}}^{e_1}(\omega_1,x_1)d\boldsymbol{\omega}. \tag{19}$$

Proof. Using the definition of QWLCT, the one dimensional OCWLCT and three dimensional OCWLCT, we have

$$\begin{aligned}\mathscr{G}_{\phi}^{M_1,M_2,M_3}\{f\}(\boldsymbol{\omega},\mathbf{u}) &= \int_{\mathbb{R}} \mathscr{G}_{\phi}^{M_1,M_2}\{f\}(\omega_1,u_1)(\omega_2,u_2)\overline{\phi(x-u)}\mathscr{K}_{M_3}^{e_4}(x_3,\omega_3)dx_3 \\ &= \mathscr{G}_{\phi}^{M_3}\left[\mathscr{G}_{\phi}^{M_1,M_2}\{f\}\right](\omega_1,u_1)(\omega_2,u_2).\end{aligned}$$

By assumption $f:\mathbb{R}^3\to\mathbb{O}$, then $\mathscr{G}_{\phi}^{M_1,M_2}:\mathbb{R}^3\to\mathbb{O}$. Therefore,

$$\begin{aligned}&\int_{\mathbb{R}^3} \mathscr{G}_{\phi}^{M_1,M_2,M_3}\{f\}(\boldsymbol{\omega},\mathbf{u})\mathscr{K}_{M_3^{-1}}^{e_4}(\omega_3,x_3)\mathscr{K}_{M_2^{-1}}^{e_2}(\omega_2,x_2)\mathscr{K}_{M_1^{-1}}^{e_1}(\omega_1,x_1)d\boldsymbol{\omega} \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}} \mathscr{G}_{\phi}^{M_3}\left[\mathscr{G}_{\phi}^{M_1,M_2}\{f\}\right](\boldsymbol{\omega},\mathbf{u})\mathscr{K}_{M_3^{-1}}^{e_4}(\omega_3,x_3)d\omega_3 \\ &\times \mathscr{K}_{M_2^{-1}}^{e_2}(\omega_2,x_2)\mathscr{K}_{M_1^{-1}}^{e_1}(\omega_1,x_1)d\omega_1 d\omega_2.\end{aligned}$$

By using Theorem 5 and relation (18), we have

$$\begin{aligned}&\int_{\mathbb{R}^3} \mathscr{G}_{\phi}^{M_1,M_2,M_3}\{f\}(\boldsymbol{\omega},\mathbf{u})\mathscr{K}_{M_3^{-1}}^{e_4}(\omega_3,x_3)\mathscr{K}_{M_2^{-1}}^{e_2}(\omega_2,x_2)\mathscr{K}_{M_1^{-1}}^{e_1}(\omega_1,x_1)d\boldsymbol{\omega} \\ &\quad= \int_{\mathbb{R}^2} \mathscr{G}_{\phi}^{M_1,M_2}\{f\}(\boldsymbol{\omega},\mathbf{u})\overline{\phi(x-u)}\mathscr{K}_{M_2^{-1}}^{e_2}(\omega_2,x_2)\mathscr{K}_{M_1^{-1}}^{e_1}(\omega_1,x_1)d\omega_1 d\omega_2 \\ &\quad= f(\mathbf{x}).\end{aligned}$$

This completes the proof. □

4. Potential applications

As highlighted in the introduction, hyper-complex algebra-based transforms have emerged as indispensable tools within the realms of contemporary science and engineering. Their application spans across diverse domains, signifying their significance at the forefront of modern advances. Among these transformative algebraic concepts, octonions have attracted substantial scientific interest, finding utility in an array of crucial areas. These encompass structural design, seismic signal analysis for earthquake prediction, computer graphics, aerospace engineering, quantum mechanics, time-frequency analysis, optics, signal processing, image refinement, pattern recognition, artificial intelligence, and beyond.

In existing literature, the Octonion-based linear canonical transform has been a prevalent choice, yet it is tailored solely for multi-channel stationary signals, thereby excluding non-stationary signals. In contrast, the Three-Dimensional Windowed Octonion-based LCT emerges as a pivotal tool that transcends this limitation. Designed to accommodate both multi-channel stationary and non-stationary signals, its real-world application presents a significant advantage. This advantage lies in its capability to manipulate signals through a window function, an approach that concurrently localizes hyper-complex signals within the time and frequency domains. This dynamic ability enhances its relevance and utility in addressing complex signal processing challenges.

Conclusion

In this article, we have introduced a novel integral transform within the fascinating domain of octonions. Through a comprehensive exploration, we have delved into its fundamental mathematical properties and carefully examined the impact of translation on this innovative transform. These valuable findings hold significant importance and will undoubtedly prove to be a guiding light for both the mathematical and signal processing communities, offering valuable insights and heuristic knowledge.

Conflict of interest. The authors declare that there are no competing interests.

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