Journal of Classical Analysis
Volume 22, Number 2 (2023), 149–163 doi:10.7153/jca-2023-22-10

CHARACTERIZATIONS OF CERTAIN SEQUENCES OF q–POLYNOMIALS

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Abstract. We provide a new characterization for those sequences of quasi-orthogonal polynomials which form also q-Appell sets.

1. Introduction

Throughout this paper, we use the following standard notations

$$\mathbb{N} := \{1,2,3,\ldots\}, \quad \mathbb{N}_0 = \{0,1,2,3,\ldots\} = \mathbb{N} \cup \{0\}.$$

Let $P_n(x)$, $n = 0,\,1,\,2,\,\ldots$ be a polynomial set, i.e. a sequence of polynomials with $P_n(x)$ of exact degree n. Assume further that

$$\frac{dP_n(x)}{dx} = P_n'(x) = nP_{n-1}(x) \quad \text{for} \quad n = 0,\,1,\,2,\,\ldots.$$

Such polynomial sets are called Appell sets and received considerable attention since P. Appell [2] introduced them in 1880.

Let q be an arbitrary real number (with $q \neq 0,1$) and define the q-derivative [6] of a function $f(x)$ by means of

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \qquad \text{if } x \neq 0 \tag{1}$$

and $D_q f(0) = f'(0)$ if f is differentiable at $x = 0$, which furnishes a generalization of the differential operator $\dfrac{d}{dx}$.

A basic (q-)analogue of Appell sequences was first introduced by Sharma and Chak [9] as those polynomial sets $\{P_n(x)\}_{n=0}^{\infty}$ which satisfy

$$D_q P_n(x) = [n]_q P_{n-1}(x), \quad n = 1,\,2,\,3,\,\cdots \tag{2}$$

where $[n]_q = (1-q^n)/(1-q)$. They called them q-harmonic. Later, Al-Salam [1] studied these families and referred to them as q-Appell sets in analogy with ordinary Appell sets. Note that when $q \to 1$, (2) reduces to

$$\frac{dP_n(x)}{dx} = nP_{n-1}(x), \quad n = 1,\,2,\,3,\,\cdots$$

Mathematics subject classification (2020): 33C65, 33C45, 33D05, 33D45, 11B68.

Keywords and phrases: Appell polynomial set, q-Bernoulli polynomials, q-Euler polynomials, orthogonal polynomials, quasi-orthogonal polynomials, q-difference equation.

so that we may think of q-Appell sets as a generalization of Appell sets. We will call these polynomial sets *q-Appell sets of type I*.

A sequence of polynomials $\{Q_n\}$, $n = 0, 1, 2, \cdots$ $\deg Q_n(x) = n$ is said to be quasi-orthogonal if there is an interval (a,b) and a non-decreasing function $\alpha(x)$ such that

$$\int_a^b x^m Q_n(x) d\alpha(x) \begin{cases} = 0 \text{ for } & 0 \leqslant m \leqslant n-2 \\ \neq 0 \text{ for } & 0 \leqslant m = n-1 \\ \neq 0 \text{ for } & 0 = m = n. \end{cases}$$

We say that two polynomial sets are related if one set is quasi-orthogonal with respect to the interval and the distribution of the orthogonality of the other set. Riesz [8] and Chihara [3] have shown that a necessary and sufficient condition for the quasi-orthogonality of the $\{Q_n(x)\}$ is that there exist non- zero constants, $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$, such that

$$\begin{aligned} Q_n(x) &= a_n P_n(x) + b_n P_{n-1}(x), \\ Q_0(x) &= a_0 P_0(x) \end{aligned} \qquad n \geqslant 1, \tag{3}$$

where the $\{P_n(x)\}_{n=0}^{\infty}$ are the related orthogonal polynomials.

In 1967, Al-Salam has given in a very short paper [1] a characterization of those sequences of orthogonal polynomials $\{P_n(x)\}$ which are also q-Appell sets. More precisely, He gave a characterization of those sequences of orthogonal polynomials for which $D_q P_n(x) = [n]_q P_{n-1}(x)$ for $n = 1, 2, 3, \cdots$.

The purpose of this paper is to study those classes of polynomial sets $\{P_n(x)\}$ that are at the same time quasi-orthogonal sets and q-Appell sets of type I. Extension will be done to those polynomials $\{P_n(x)\}$ that satisfy

$$D_q P_n(x) = [n]_q P_{n-1}(qx).$$

The later polynomials will be called q-Appell polynomials of type II and appear already in [5] where some of their properties are given.

2. Preliminaries results and definitions

Let us introduce the so-called q-Pochhammer symbol

$$(x;q)_n = \begin{cases} (1-x)(1-xq)\dots(1-xq^{n-1}) & n = 1,2,\dots \\ 1 & n = 0. \end{cases}$$

For a non-negative integer n, the q-factorial is defined by

$$[n]_q! = \prod_{k=0}^{n} [k]_q \quad \text{for} \quad n \geqslant 1, \quad \text{and} \quad [0]_q! = 1.$$

The q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \quad (0 \leqslant k \leqslant n).$$

We will use the following two q-analogues of the exponential function e^x (see for example [6, 7] and the references therein)

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!}, \tag{4}$$

and

$$E_q(x) = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}}{[k]_q!} x^k. \tag{5}$$

These two functions are related by the equation (see [6])

$$e_q(x)E_q(-x) = 1. \tag{6}$$

The basic hypergeometric or q-hypergeometric function ${}_r\phi_s$ is defined by the series

$${}_r\phi_s \left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix} \middle| q; z \right) := \sum_{k=0}^{\infty} \frac{(a_1, \cdots, a_r; q)_k}{(b_1, \cdots, b_s; q)_k} \left((-1)^k q^{\binom{k}{2}} \right)^{1+s-r} \frac{z^k}{(q;q)_k},$$

where

$$(a_1, \cdots, a_r)_k := (a_1; q)_k \cdots (a_r; q)_k.$$

The Al Salam-Carlitz I polynomials [7, p. 534] have the q-hypergeometric representation

$$U_n^{(a)}(x;q) = (-a)^n q^{\binom{n}{2}} {}_2\phi_1 \left(\begin{matrix} q^{-n}, x^{-1} \\ 0 \end{matrix} \middle| q; \frac{qx}{a} \right).$$

The Al-Salam Carlitz I polynomials fulfil the three-term recurrence relation

$$xU_n^{(a)}(x;q) = U_{n+1}^{(a)}(x;q) + (a+1)q^n U_n^{(a)}(x;q) - aq^{n-1}(1-q^n)U_{n-1}^{(a)}(x;q),$$

and the q-derivative rule

$$D_q U_n^{(a)}(x;q) = [n]_q U_{n-1}^{(a)}(x;q).$$

It is therefore clear that the Al-Salam Carlitz I polynomials form a q-Appell set.

The Al-Salam-Carlitz II polynomials [7, p. 537] have the q-hypergeometric representation

$$V_n^{(a)}(x;q) = (-a)^n q^{\binom{-n}{2}} {}_2\phi_0 \left(\begin{matrix} q^{-n}, x \\ - \end{matrix} \middle| q; \frac{q^n}{a} \right).$$

Note that the Al Salam-Carlitz I polynomials and the Al Salam-Carlitz II polynomials are related in the following way:

$$U_n^{(a)}(x, q^{-1}) = V_n^{(a)}(x;q).$$

The Al-Salam Carlitz II polynomials fulfil the three-term recurrence relation

$$xV_n^{(a)}(x;q) = V_{n+1}^{(a)}(x;q) + (a+1)q^{-n}V_n^{(a)}(x;q) + aq^{-2n+1}(1-q^n)V_{n-1}^{(a)}(x;q),$$

and the q-derivative rule

$$D_qV_n^{(a)}(x;q) = q^{-n+1}[n]_qV_{n-1}^{(a)}(qx;q).$$

Let us introduce the modified Al-Salam Carlitz II polynomials $\mathscr{V}_n^{(a)}(x;q)$ by the relation

$$\mathscr{V}_n^{(a)}(x;q) = q^{\binom{n}{2}}V_n^{(a)}(x;q). \tag{7}$$

Then we have the following proposition.

PROPOSITION 1. *The polynomial sequence* $\{\mathscr{V}_n^{(a)}(x;q)\}_{n=0}^{\infty}$ *is a* q*-Appell polynomial set of type II.*

PROPOSITION 2. (See [4, Theorem 1]) *For* $\{Q_n(x)\}$ *to be a set of polynomials quasi-orthogonal with respect to an interval* (a,b) *and a distribution* $d\alpha(x)$*, it is necessary and sufficient that there exist a set of nonzero constants* $\{T_k\}_{k=0}^{\infty}$ *and a set of polynomials* $\{P_n(x)\}$ *orthogonal with respect to* (a,b) *and* $d\alpha(x)$ *such that*

$$P_n(x) = \sum_{k=0}^{n} T_kQ_k(x), \quad n \geqslant 0. \tag{8}$$

PROPOSITION 3. (See [4, Theorem 2]) *A necessary and suffucient condition that the set* $\{Q_n(x)\}_{n=0}^{\infty}$ *where each* $Q_n(x)$ *is a polynomial of degree precisely* n*, be quasi-orthogonal is that it satisfies*

$$Q_{n+1}(x) = (x+b_n)Q_n(x) - c_nQ_{n-1}(x) + d_n\sum_{k=0}^{n-2} T_kQ_k(x),$$

for all n*, with* $d_0 = d_1 = 0$.

PROPOSITION 4. (See [1, Theorem 4.1]) *If* $\{Q_n(x)\}_{n=0}^{\infty}$ *is a* q*-Appell set which are also orthogonal, then there exists a non zero constant* b *such that*

$$Q_n(x) = b^nU_n^{(a/b)}\left(\frac{x}{b}\right),$$

for all $n \geqslant 0$.

3. Some notes on q-Appell polynomials of type II

As mentioned earlier in the manuscript, q-Appell polynomials of type II are those polynomial sets $\{P_n\}$ satisfying the relation

$$D_q P_n(x) = [n]_q P_{n-1}(qx).$$

Let us recall that the following Cauchy product for infinite series applies

$$\left(\sum_{n=0}^{\infty} A_n\right)\left(\sum_{n=0}^{\infty} B_n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} A_k B_{n-k}\right). \tag{9}$$

In particular, if $A_n = \dfrac{a_n x^n}{[n]_q!}$ and $B_n = \dfrac{b_n x^n}{[n]_q!}$, then we have

$$\left(\sum_{n=0}^{\infty} \frac{a_n x^n}{[n]_q!}\right)\left(\sum_{n=0}^{\infty} \frac{b_n x^n}{[n]_q!}\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a_k b_{n-k}\right)\frac{x^n}{[n]_q!}. \tag{10}$$

3.1. Four equivalent statements

In this section, we give several characterizations of q-Appell sets of type II.

THEOREM 1. *Let $\{f_n(x)\}_{n=0}^{\infty}$ be a sequence of polynomials. Then the following are all equivalent:*

1. *$\{f_n(x)\}_{n=0}^{\infty}$ is a q-Appell set of type II.*
2. *There exists a sequence $(a_k)_{k\geqslant 0}$; independent of n; $a_0 \neq 0$; such that*

$$f_n(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} a_k x^{n-k}.$$

3. *$\{f_n(x)\}_{n=0}^{\infty}$ is generated by*

$$A(t)E_q(xt) = \sum_{n=0}^{\infty} f_n(x)\frac{t^n}{[n]_q!},$$

where

$$A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{[n]_q!}, \tag{11}$$

is called the determining function *for $\{f_n(x)\}_{n=0}^{\infty}$.*

4. *There exists a sequence $(a_k)_{k\geqslant 0}$; independent of n; $a_0 \neq 0$; such that*

$$f_n(x) = \left(\sum_{k=0}^{\infty} \frac{a_k q^{\binom{n-k}{2}}}{[k]_q!} D_q{}^k\right) x^n.$$

Proof. First, we prove that $(1) \implies (2) \implies (3) \implies (1)$.

$(1) \implies (2)$. Since $\{f_n(x)\}_{n=0}^{\infty}$ is a polynomial set, it is possible to write

$$f_n(x) = \sum_{k=0}^{n} a_{n,k} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} x^{n-k}, \quad n = 1,2,\ldots, \tag{12}$$

where the coefficients $a_{n,k}$ depend on n and k and $a_{n,0} \neq 0$. We need to prove that these coefficients are independent of n. By applying the operator D_q to each member of (12) and taking into account that $\{f_n(x)\}_{n=0}^{\infty}$ is a q-Appell polynomial set of type II, we obtain

$$f_{n-1}(qx) = \sum_{k=0}^{n-1} a_{n,k} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q q^{\binom{n-1-k}{2}} (qx)^{n-1-k}, \quad n = 1,2,\ldots, \tag{13}$$

since $D_q x^0 = 0$. Shifting index $n \to n+1$ in (13) and making the substitution $x \to xq^{-1}$, we get

$$f_n(x) = \sum_{k=0}^{n} a_{n+1,k} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} x^{n-k}, \quad n = 0,1,\ldots, \tag{14}$$

Comparing (12) and (14), we have $a_{n+1,k} = a_{n,k}$ for all k and n, which means that $a_{n,k} = a_k$ is independent of n.

$(2) \implies (3)$. From (2), and the identity (10), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} f_n(x) \frac{t^n}{[n]_q!} &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} a_k x^{n-k} \right) \frac{t^n}{[n]_q!} \\
&= \left(\sum_{n=0}^{\infty} a_n \frac{t^n}{[n]_q!} \right) \left(\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{[n]_q!} (xt)^n \right) \\
&= A(t) E_q(xt).
\end{aligned}$$

$(3) \implies (1)$. Assume that $\{f_n(x)\}_{n=0}^{\infty}$ is generated by

$$A(t) E_q(xt) = \sum_{n=0}^{\infty} f_n(x) \frac{t^n}{[n]_q!}.$$

Then, applying the operator D_q to each side of this equation,

$$t A(t) E_q(qxt) = \sum_{n=0}^{\infty} D_q f_n(x) \frac{t^n}{[n]_q!}.$$

Moreover, we have

$$t A(t) E_q(qxt) = \sum_{n=0}^{\infty} f_n(qx) \frac{t^{n+1}}{[n]_q!} = \sum_{n=0}^{\infty} [n]_q f_{n-1}(qx) \frac{t^n}{[n]_q!}.$$

By comparing the coefficients of t^n, we obtain (1).

Next, $(2) \iff (4)$ is obvious. This ends the proof of the theorem. $\square$

3.2. Algebraic structure

We denote a given polynomial set $\{f_n(x)\}_{n=0}^{\infty}$ by a single symbol f and refer to $f_n(x)$ as the n-th component of f. We define (see [2, 10]) on the set $\mathscr{P}$ of all polynomial sets the following operation $+$. This operation is given by the rule that $f+g$ is the polynomial set whose n-th component is $f_n(x)+g_n(x)$ provided that the degree of $f_n(x)+g_n(x)$ is exactly n. We also define the operation $*$ (which appears here for the fist time) such that if f and g are two sets whose n-th components are, respectively,

$$f_n(x)=\sum_{k=0}^{n}\alpha(n,k)x^k, \quad g_n(x)=\sum_{k=0}^{n}\beta(n,k)x^k,$$

then $f*g$ is the polynomial set whose n-th component is

$$(f*g)_n(x)=\sum_{k=0}^{n}\alpha(n,k)q^{-\binom{k}{2}}g_k(x).$$

If λ is a real or complex number, then λf is defined as the polynomial set whose n-th component is $\lambda f_n(x)$. We obviously have

$$\begin{aligned} f+g=g+f \quad &\text{for all} \quad f,g\in\mathscr{P},\\ \lambda f*g=(f*\lambda g)&=\lambda(f*g). \end{aligned}$$

Clearly, the operation $*$ is not commutative on $\mathscr{P}$. One commutative subclass is the set $\mathscr{A}$ of all Appell polynomials (see [2]).

In what follows, $\mathscr{A}(q)$ denotes the class of all q-Appell sets of type II.

In $\mathscr{A}(q)$ the identity element (with respect to $*$) is the q-Appell set of type II $\mathscr{I}=\left\{q^{\binom{n}{2}}x^n\right\}$. Note that $\mathscr{I}$ has the determining function $A(t)=1$. This is due to the identity (5). Next we state the following Lemma.

LEMMA 1. *Let $f,g,h\in\mathscr{A}(q)$ with the determining functions $A(t)$, $B(t)$ and $C(t)$ respectively. Then*

1. *$f+g\in\mathscr{A}(q)$ if $A(0)+B(0)\neq 0$,*
2. *$f+g$ belongs to the determining function $A(t)+B(t)$,*
3. *$f+(g+h)=(f+g)+h$.*

Next we state and prove the following theorem.

THEOREM 2. *If $f,g,h\in\mathscr{A}(q)$ with the determining functions $A(t)$, $B(t)$ and $C(t)$ respectively, then*

1. *$f*g\in\mathscr{A}(q)$,*
2. *$f*g=g*f$,*

3. *$f*g$ belongs to the determining function $A(t)B(t)$,*
4. *$f*(g*h)=(f*g)*h$.*

Proof. It is enough to prove the first part of the theorem. The rest follows directly. According to Theorem 1, we may put

$$f_n(x)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{\binom{n-k}{2}}a_kx^{n-k}=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{\binom{k}{2}}a_{n-k}x^k$$

so that

$$A(t)=\sum_{n=0}^{\infty}a_n\frac{t^n}{[n]_q!}.$$

Hence

$$\begin{aligned}\sum_{n=0}^{\infty}(f*g)_n(x)\frac{t^n}{[n]_q!}&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q a_{n-k}g_k(x)\right)\frac{t^n}{[n]_q!}\\&=\left(\sum_{n=0}^{\infty}a_n\frac{t^n}{[n]_q!}\right)\left(\sum_{n=0}^{\infty}g_n(x)\frac{t^n}{[n]_q!}\right)\\&=A(t)B(t)E_q(xt).\end{aligned}$$

This ends the proof of the theorem. □

COROLLARY 1. *Let $f\in\mathscr{A}(q)$ then there is a set $g\in\mathscr{A}(q)$ such that*

$$f*g=g*f=\mathscr{I}.$$

Indeed g belongs to the determining function $(A(t))^{-1}$ where $A(t)$ is the determining function for f.

In view of Corollary 1 we shall denote this element g by f^{-1}. We are further motivated by Theorem 2 and its corollary to define $f^0=\mathscr{I}$, $f^n=f*(f^{n-1})$ where n is a non-negative integer, and $f^{-n}=f^{-1}*(f^{-n+1})$. We note that we have proved that the system $(\mathscr{A}(q),*)$ is a commutative group. In particular this leads to the fact that if

$$f*g=h$$

and if any two of the elements f,g,h are q-Appell of type II then the third is also q-Appell of type II.

PROPOSITION 5. *If f is a q-Appell set of type II with the determining function $A(t)$, if we put*

$$A^{-1}(t)=\sum_{n=0}^{\infty}b_n\frac{t^n}{[n]_q!},$$

therefore

$$x^n=q^{-\binom{n}{2}}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q b_kf_{n-k}(x).$$

Proof. Since f is a q-Appell set of type II, we have

$$\begin{aligned}\sum_{n=0}^{\infty} q^{\binom{n}{2}} x^n \frac{t^n}{[n]_q!} &= (A(t))^{-1} A(t) E_q(xt) \\ &= \left(\sum_{n=0}^{\infty} b_n \frac{t^n}{[n]_q!}\right)\left(\sum_{n=0}^{\infty} f_n(x) \frac{t^n}{[n]_q!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q b_k f_{n-k}(x)\right) \frac{t^n}{[n]_q!}.\end{aligned}$$

The result follows by comparing the coefficients of t^n. □

4. Characterization results

4.1. Quasi-orthogonal q-Appell polynomials of type I

In this section, we characterize quasi-orthogonal polynomial sets that are also q-Appell set of type I.

THEOREM 3. *If $\{Q_n(x)\}_{n=0}^{\infty}$ is a q-Appell set which are quasi-orthogonal. Then, there exist three real numbers b, c and λ, such that*

$$Q_{n+1}(x) = (x + bq^n) Q_n(x) - cq^{n-1} [n]_q Q_{n-1}(x) + d_n \sum_{k=0}^{n-2} \frac{\lambda^k}{[k]_q!} Q_k(x). \tag{15}$$

Proof. Assume that $\{Q_n(x)\}_{n=0}^{\infty}$ is a q-Appell set which are quasi-orthogonal and $\{P_n(x)\}_{n=0}^{\infty}$ the related orthogonal family. From Proposition 3, there exist three sequences $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ and $\{d_n\}_{n=0}^{\infty}$ with $d_0 = d_1 = 0$ such that

$$Q_{n+1}(x) = (x + b_n) Q_n(x) - c_n Q_{n-1}(x) + d_n \sum_{k=0}^{n-2} T_k Q_k(x). \tag{16}$$

If we q-differentiate (16) and using the fact that $\{Q_n(x)\}_{n=0}^{\infty}$ is a q-Appell, we get after some simplifications

$$Q_n(x) = \left(x + \frac{b_n}{q}\right) Q_{n-1}(x) - \frac{c_n}{q} \frac{[n-1]_q}{[n]_q} Q_{n-2}(x) + \frac{d_n}{q[n]_q} \sum_{k=0}^{n-3} [k+1]_q T_{k+1} Q_k(x). \tag{17}$$

Next, if we replace n by $n-1$ in (16), we obtain

$$Q_n(x) = (x + b_{n-1}) Q_n(x) - c_{n-1} Q_{n-2}(x) + d_{n-1} \sum_{k=0}^{n-3} T_k Q_k(x). \tag{18}$$

If we compare (17) and (18), we see that we should have

$$b_n = q b_{n-1}, \quad c_n = q \frac{[n]_q}{[n-1]_q} c_{n-1}, \tag{19}$$

and

$$d_n[k+1]_qT_{k+1} = q[n]_qd_{n-1}T_k, \qquad k = 0,\ 1,\ \cdots n-3. \tag{20}$$

Equation (19) gives

$$b_n = q^n b_0, \quad \text{and} \quad c_n = q^{n-1}[n]_q c_1.$$

Next, (20) gives for $k = 0$ and $k = n-3$ the relations

$$d_n = \frac{q[n]_q}{T_1} d_{n-1} \quad \text{and} \quad d_n = \frac{q[n]_qT_{n-1}}{[n-2]_qT_{n-2}} d_{n-1}. \tag{21}$$

If, for a given $k \geqslant 2$, $d_k = 0$, it follows from (21) that $d_k = 0$ for all k. In this case (16) becomes a three-term recurrence relation

$$Q_{n+1}(x) = (x+b_n)Q_n(x) - c_nQ_{n-1}(x). \tag{22}$$

In this case, from Proposition 4, it is seen that $\{Q_n(x)\}_{n=0}^{\infty}$ is essentially the sequence of Al-Salam Carlitz I polynomials. Thus, in this case, $\{Q_n(x)\}_{n=0}^{\infty}$ is not a sequence of quasi-orthogonal polynomials. Thus, we must have $d_k \neq 0$ for $k \geqslant 2$.

Again, using (21), we have for all $n \geqslant 0$ $\dfrac{T_{n-1}}{[n]_qT_n} = \dfrac{1}{T_1}$. This last relation gives $T_n = \dfrac{T_1^n}{[n]_q!}$. Seting $b_0 = b$, $c_1 = c$ and $T_1 = \lambda$, this ends the proof or the theorem. □

THEOREM 4. *Let $\{Q_n(x)\}_{n=0}^{\infty}$ be a monic polynomial set with $Q_0(x) = 1$. The following assertions are equivalent:*

1. *$\{Q_n(x)\}_{n=0}^{\infty}$ is quasi-orthogonal and is a q-Appell set, $n \geqslant 1$.*
2. *There exists three constants α, β and λ ($\beta, \lambda \neq 0$) such that*

$$Q_n(x) = \beta^n U_n^{(\alpha/\beta)}\left(\frac{x}{\beta};q\right) - \frac{\beta^n[n]_q}{\lambda} U_{n-1}^{(\alpha/\beta)}\left(\frac{x}{\beta};q\right), \quad n \geqslant 1,$$

where $U_n^{(a)}(x;q)$ are the Al-Salam Carlitz I polynomials.

Proof. Suppose first that $\{Q_n(x)\}_{n=0}^{\infty}$ is quasi-orthogonal and is a q-Appell set, $n \geqslant 1$. Then, by Theorem 3, the Q_n's satisfy a recurrence relation of the form (15). Let us define the polynomial set $\{P_n(x)\}_{n=0}^{\infty}$ by

$$P_n(x) = \frac{[n]_q!}{\lambda^n} \sum_{k=0}^{n} \frac{\lambda^k}{[k]_q!} Q_k(x). \tag{23}$$

It is not difficult to see that

$$\begin{aligned} D_qP_n(x) &= \frac{[n]_q!}{\lambda^n} \sum_{k=1}^{n} \frac{\lambda^k}{[k]_q!}[k]_qQ_{k-1}(x) \\ &= [n]_q \frac{[n-1]_q!}{\lambda^{n-1}} \sum_{k=0}^{n-1} \frac{\lambda^k}{[k]_q!} Q_k(x) \\ &= [n]_qP_{n-1}(x). \end{aligned}$$

Hence, $\{P_n(x)\}_{n=0}^{\infty}$ is a q-Appell set. Moreover, $\{P_n(x)\}_{n=0}^{\infty}$ are the orthogonal set (see Proposition 2) related to $\{Q_n(x)\}_{n=0}^{\infty}$. By Proposition 4, there exist α and β such that

$$P_n(x) = \beta^n U_n^{(\alpha/\beta)}\left(\frac{x}{\beta};q\right).$$

Next, from (23), it follows that

$$Q_n(x) = \frac{[n]_q!}{\lambda^n}\left(P_n(x) - P_{n-1}(x)\right).$$

The first implication of the theorem follows.

Conversely, assume that there exists three constants α, β and λ $(\beta,\lambda \neq 0)$ such that

$$Q_n(x) = \beta^n U_n^{(\alpha/\beta)}\left(\frac{x}{\beta};q\right) - \frac{\beta^n[n]_q}{\lambda}U_{n-1}^{(\alpha/\beta)}\left(\frac{x}{\beta};q\right), \quad n \geqslant 1.$$

It can be seen that $\{Q_n(x)\}_{n=0}^{\infty}$ is quasi orthogonal set. It remains to prove that $\{Q_n(x)\}_{n=0}^{\infty}$ is q-Appell. Using the fact that $D_q[f(ax)] = a[D_qf](ax)$. We have $D_qU_n^{(\alpha/\beta)}\left(\frac{x}{\beta};q\right) = \frac{1}{\beta}U_{n-1}^{(\alpha/\beta)}\left(\frac{x}{\beta};q\right)$. It follows that $D_qQ_n(x) = [n]_qQ_{n-1}(x)$. This ends the proof of the theorem. $\square$

4.2. Orthogonal q-Appell polynomials of type II

In this section we determine those real sets in $\mathscr{A}(q)$ which are also orthogonal. It is well known [11] that a set of real orthogonal polynomials satisfies a recurrence relation of the form

$$P_{n+1}(x) = (A_nx + B_n)P_n(x) + C_nP_{n-1}(x), \quad n \geqslant 1, \tag{24}$$

with

$$P_0(x) = 1, \quad P_1(x) = A_0x + B_0.$$

Here A_n, B_n and C_n are real constants which do not depend on n.

If we q-differentiate (24) and assume that the polynomial set $\{P_n(x)\}$ is q-Appell of type II, we get:

$$[n+1]_qP_n(qx) = [n]_q\left(A_nx + B_n\right)P_{n-1}(qx) + A_nP_n(qx) + [n-1]_qC_nP_{n-2}(qx). \tag{25}$$

Substituting n by $n+1$ and x by xq^{-1} in (25), it follows that

$$P_{n+1}(x) = \left(\frac{[n+1]_qq^{-1}A_{n+1}}{[n+2]_q - A_{n+1}}x + \frac{[n+1]_qB_{n+1}}{[n+2]_q - A_{n+1}}\right)P_n(x) + \frac{[n]_qC_{n+1}}{[n+2]_q - A_{n+1}}P_{n-1}(x). \tag{26}$$

By comparing (24) and (26) we get

$$\frac{[n+1]_qA_{n+1}}{[n+2]_q - A_{n+1}} = qA_n, \qquad \frac{[n+1]_qB_{n+1}}{[n+2]_q - A_{n+1}} = B_n \quad \text{and} \quad \frac{[n]_qC_{n+1}}{[n+2]_q - A_{n+1}} = C_n,$$

so that

$$A_n = q^n, \quad B_n = B_0 \quad \text{and} \quad C_n = C_1(1-q^n).$$

Hence, $\{P_n(x)\}$ is given by

$$P_{n+1}(x) = (q^n x + B_0)P_n(x) + C_1(1-q^n)P_{n-1}(x), \tag{27}$$

$$P_0(x) = 1, \quad P_1(x) = x + B_0.$$

From the recurrence relation of the Al-Salam Carlitz II polynomials (see [7, p. 538]), one can see that the polynomial sequence $\{R_n(x)\}$ with

$$R_n(x) = \beta^n q^{\binom{n}{2}} V_n^{(\frac{\alpha}{\beta})}\left(\frac{x}{\beta}; q\right),$$

satisfies the recurrence relation

$$xR_n(x) = R_{n+1}(x) + (q^n x - (\alpha+\beta))R_n(x) - \alpha\beta(1-q^n)R_{n-1}(x), \tag{28}$$

with $R_0(x) = 1$ and $R_1(x) = x - (\alpha+\beta)$. It is therefore clear that

$$P_n(x) = \beta^n q^{\binom{n}{2}} V_n^{(\frac{\alpha}{\beta})}\left(\frac{x}{\beta}; q\right). \tag{29}$$

where $\alpha+\beta = -B_0$ and $\alpha\beta = -C_1$.

We thus have the following theorem.

THEOREM 5. *The set of q-Appell polynomials of type II which are also orthogonal is given* (27) *or* (29).

4.3. Quasi-orthogonal q-Appell polynomials of type II

THEOREM 6. *If $\{Q_n(x)\}_{n=0}^{\infty}$ is a q-Appell set of type II of quasi-orthogonal polynomials, then there exist three reel numbers B_0, C_1 and λ, such that*

$$Q_{n+1}(x) = (q^n x + B_0)Q_n(x) + C_1(1-q^n)Q_{n-1}(x) + \frac{[n]_q!}{\lambda^n}\sum_{k=0}^{n-2}\frac{\lambda^k}{[k]_q!}Q_k(x). \tag{30}$$

Proof. Assume that $\{Q_n(x)\}_{n=0}^{\infty}$ is a q-Appell set which is quasi-orthogonal and $\{P_n(x)\}_{n=0}^{\infty}$ the related orthogonal family. From Proposition 3, there exist four sequences $\{A_n\}_{n=0}^{\infty}$, $\{B_n\}_{n=0}^{\infty}$, $\{C_n\}_{n=0}^{\infty}$ and $\{E_n\}_{n=0}^{\infty}$ with $E_0 = E_1 = 0$ such that

$$Q_{n+1}(x) = (A_n x + B_n)Q_n(x) + C_n Q_{n-1}(x) + E_n \sum_{k=0}^{n-2} T_k Q_k(x). \tag{31}$$

If we q-differentiate (31) and use the fact that $\{Q_n(x)\}_{n=0}^{\infty}$ is a q-Appell set of type II, we get after some simplifications

$$\begin{aligned}Q_{n+1}(x) &= \left(\frac{[n+1]_q q^{-1}A_{n+1}}{[n+2]_q - A_{n+1}}x + \frac{[n+1]_q B_{n+1}}{[n+2]_q - A_{n+1}}\right)Q_n(x)\\ &\quad + \frac{[n]_q C_{n+1}}{[n+2]_q - A_{n+1}}Q_{n-1}(x)\\ &\quad + \frac{E_{n+1}}{[n+2]_q - A_{n+1}}\sum_{k=0}^{n-2}[k+1]_q T_{k+1}Q_k(x). \end{aligned} \tag{32}$$

By comparing (31) and (32) we get

$$A_n = q^n, \quad B_n = B_0 \quad \text{and} \quad C_n = C_1(1-q^n),$$

and

$$E_n T_k = \frac{E_{n+1}[k+1]_q T_{k+1}}{[n+2]_q - A_{n+1}} = \frac{[k+1]_q T_{k+1}}{[n+1]_q}E_{n+1},$$

For $k=0$ and $k=n-2$, we obtain the following

$$E_{n+1} = \frac{[n+1]_q}{T_1}E_n, \qquad T_n = \frac{E_{n+1}}{E_{n+2}}\frac{[n+2]_q}{[n]_q}T_{n-1}. \tag{33}$$

If, for a given $k \geqslant 2$, $E_k = 0$, it follows from (33) that $E_k = 0$ for all k. In this case (31) becomes a three-term recurrence relation

$$Q_{n+1}(x) = (A_n x + B_n)Q_n(x) + C_n Q_{n-1}(x). \tag{34}$$

In this case, from Theorem 5, it is seen that $\{Q_n(x)\}_{n=0}^{\infty}$ is essentially the sequence of Al-Salam Carlitz II polynomials. Thus, in this case, $\{Q_n(x)\}_{n=0}^{\infty}$ is not a sequence of quasi-orthogonal polynomials. Thus, we must have $E_k \neq 0$ for $k \geqslant 2$.

Again, using (33), we have for all $n \geqslant 0$ the identities $E_n = \dfrac{[n]_q!}{T_1^n}$ and $\dfrac{T_{n-1}}{[n]_q T_n} = \dfrac{1}{T_1}$. This last relation gives $T_n = \dfrac{T_1^n}{[n]_q!}$. Seting $T_1 = \lambda$, this ends the proof of the theorem. □

THEOREM 7. *Let $\{Q_n(x)\}_{n=0}^{\infty}$ be a polynomial set. The following assertions are equivalent:*

1. *$\{Q_n(x)\}_{n=0}^{\infty}$ is quasi-orthogonal and is a q-Appell set of type II.*
2. *There exists three constants α, β and γ ($\beta,\gamma \neq 0$) such that*

$$Q_n(x) = \beta^n q^{\binom{n}{2}} V_n^{(\frac{\alpha}{\beta})}\left(\frac{x}{\beta};q\right) - \frac{\beta^{n-1}q^{\binom{n-1}{2}}[n]_q!}{\lambda^n}V_{n-1}^{(\frac{\alpha}{\beta})}\left(\frac{x}{\beta};q\right), \quad (n \geqslant 1),$$

where $V_n^{(a)}(x;q)$ are the Al-Salam Carlitz II polynomials.

Proof. Suppose first that $\{Q_n(x)\}_{n=0}^{\infty}$ is quasi-orthogonal and is a q-Appell set of type II. Then, by Theorem 6, the Q_n's satisfy a recurrence relation of the form (30). Let us define the polynomial set $\{P_n(x)\}_{n=0}^{\infty}$ by

$$P_n(x) = \frac{[n]_q!}{\lambda^n}\sum_{k=0}^{n}\frac{\lambda^k}{[k]_q!}Q_k(x). \tag{35}$$

It is not difficult to see that

$$\begin{aligned} D_qP_n(x) &= \frac{[n]_q!}{\lambda^n}\sum_{k=1}^{n}\frac{\lambda^k}{[k]_q!}[k]_qQ_{k-1}(qx) \\ &= [n]_q\frac{[n-1]_q!}{\lambda^{n-1}}\sum_{k=0}^{n-1}\frac{\lambda^k}{[k]_q!}Q_k(qx) \\ &= [n]_qP_{n-1}(qx). \end{aligned}$$

Hence, $\{P_n(x)\}_{n=0}^{\infty}$ is a q-Appell set of type II. Moreover, $\{P_n(x)\}_{n=0}^{\infty}$ is the orthogonal set (see Proposition 2) related to $\{Q_n(x)\}_{n=0}^{\infty}$. By Theorem 5, there exist α and β such that

$$P_n(x) = \beta^n q^{\binom{n}{2}} V_n^{\left(\frac{\alpha}{\beta}\right)}\left(\frac{x}{\beta};q\right).$$

Next, from (35), it follows easily that

$$\begin{aligned} Q_n(x) &= P_n(x) - \frac{[n]_q!}{\lambda^n}P_{n-1}(x) \\ &= \beta^n q^{\binom{n}{2}} V_n^{\left(\frac{\alpha}{\beta}\right)}\left(\frac{x}{\beta};q\right) - \frac{\beta^{n-1}q^{\binom{n-1}{2}}[n]_q!}{\lambda^n}V_{n-1}^{\left(\frac{\alpha}{\beta}\right)}\left(\frac{x}{\beta};q\right) \end{aligned}$$

The first implication of the theorem follows.

Conversely, assume that there exist three constants α, β and γ $(\beta,\gamma \neq 0)$ such that

$$Q_n(x) = \beta^n q^{\binom{n}{2}} V_n^{\left(\frac{\alpha}{\beta}\right)}\left(\frac{x}{\beta};q\right) - \frac{\beta^{n-1}q^{\binom{n-1}{2}}[n]_q!}{\lambda^n}V_{n-1}^{\left(\frac{\alpha}{\beta}\right)}\left(\frac{x}{\beta};q\right), \quad (n \geqslant 1).$$

It can be seen that $\{Q_n(x)\}_{n=0}^{\infty}$ is a quasi-orthogonal set. It remains to prove that $\{Q_n(x)\}_{n=0}^{\infty}$ is a q-Appell set. Using the fact that $D_q[f(ax)] = a[D_qf](ax)$, we get

$$D_qV_n^{(\alpha/\beta)}\left(\frac{x}{\beta};q\right) = \frac{[n]_qq^{-n+1}}{\beta}V_{n-1}^{(\alpha/\beta)}\left(\frac{qx}{\beta};q\right).$$

It follows that $D_qQ_n(x) = [n]_qQ_{n-1}(qx)$. This ends the proof of the theorem. $\square$

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(Received August 11, 2023)

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