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GENERALIZATION OF GRACE'S THEOREM, SCHUR–SZEGÖ COMPOSITION AND COHN–EGERVÁRY THEOREM FOR BICOMPLEX POLYNOMIALS

ASHISH KUMAR* AND B. A. ZARGAR

Abstract. The aim of this paper is to extend the domain of the Grace's theorem, Schur-Szegö composition theorem and Cohn-Egerváry theorem from the set of complex numbers to the set of bicomplex numbers.

1. Introduction

Corrado Segre published a paper [5] in 1892, in which he studied an infinite set of algebra whose elements he called bicomplex numbers. The work of Segre remained unnoticed for almost a century, but recently mathematicians have started taking interest in the subject and a new theory of special functions has started coming up [1, 3]. In this paper, we introduce the mathematical tools necessary to investigate Grace's theorem, Schur-Szegö composition theorem and Cohn-Egerváry theorem for bicomplex polynomials. This paper has four sections viz, first section covers some basic knowledge about bicomplex numbers, second section contains main results, third section contains required lemmas and the final section contains proofs of the results.

1.1. Preliminary definitions and notations

The set $\mathbb{BC}$ of bicomplex numbers is defined as $\mathbb{BC} = \{Z : Z = z_1 + jz_2; z_1, z_2 \in \mathbb{C}\}$, where $\mathbb{C}$ is the set of complex numbers with the imaginary unit i such that $ij = ji = k$ and $i^2 = j^2 = -k^2 = -1$. Here k is known as a hyperbolic imaginary unit. The bicomplex numbers are the complex numbers with complex coefficients whereas, quaternions is a hypercomplex number that can be presented as linear combination

$$X = x_0 + ix_1 + jx_2 + kx_3$$

where i, j, k are units such that $i^2 = j^2 = k^2 = -1$. Also, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$.

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* Corresponding author.

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Addition and multiplication on $\mathbb{BC}$ is defined in the similar fashion as is defined on $\mathbb{C}$, it can be hence observed that $\mathbb{BC}$ forms a commutative ring. However due to the presence of zero-divisors, $\mathbb{BC}$ is not a field. The set of zero-divisors in $\mathbb{BC}$ is given as:

$$O = \{z_1 + jz_2 \in \mathbb{BC} : z_1^2 + z_2^2 = 0\} = \{a(1 \pm ij) : a \in \mathbb{C}\}.$$

Now as $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$, we therefore have $Z = z_1 + jz_2 = x_1 + ix_2 + jy_1 + jiy_2$. Thus $\mathbb{BC}$ can be viewed as a real vector space isomorphic to $\mathbb{R}^4$ via the map $x_1 + ix_2 + jy_1 + jiy_2 \to (x_1, x_2, y_1, y_2)$.

1.1.1. Conjugation of bicomplex numbers

As the structure of $\mathbb{BC}$ consists of two imaginary units and one hyperbolic unit in it, therefore there are three possible conjugations on this structure:

1. $\bar{Z} := \bar{z}_1 + j\bar{z}_2$ (the bar-conjugation);
2. $Z^\dagger := z_1 - jz_2$ (the †-conjugation);
3. $Z^* := (\bar{Z}^\dagger) = \bar{Z^\dagger} = \bar{z}_1 - j\bar{z}_2$ (the $*$-conjugation).

1.1.2. Idempotent representation

One of the most important presentation of bicomplex numbers is idempotent representation. The bicomplex number $e = \frac{1+ij}{2}$, $e^\dagger = \frac{1-ij}{2}$ are linearly independent in the linear space $\mathbb{BC}$ over $\mathbb{C}$. From the simple calculations, it can be easily seen that $e + e^\dagger = 1$, $e - e^\dagger = ij$, $e.e^\dagger = 0$, $e^2 = e$ and $(e^\dagger)^2 = e^\dagger$. From the simple calculations again it can be seen that any bicomplex number $Z = z_1 + jz_2$ can be uniquely written as $Z = (z_1 - iz_2)e + (z_1 + iz_2)e^\dagger$ and this unique representation of the bicomplex numbers is known as their idempotent representation.

1.1.3. Norm

If $Z = z_1 + jz_2 = \zeta_1 e + \zeta_2 e^\dagger$, then the norm function $\|\| : \mathbb{BC} \to \mathbb{R}^+(\mathbb{R}^+$ denotes the set of all non-negative real numbers) is defined as:

$$\|Z\| = \{|z_1|^2 + |z_2|^2\}^{1/2} = \left\{\frac{|\zeta_1|^2 + |\zeta_2|^2}{2}\right\}.$$

1.1.4. Auxiliary complex spaces

From the idempotent representation of any bicomplex number $Z = z_1 + jz_2$ as $Z = (z_1 - iz_2)e + (z_1 + iz_2)e^\dagger$, we get the idea of defining two spaces $\mathbb{A} = \{z_1 - iz_2 : z_1, z_2 \in \mathbb{C}\}$ and $\bar{\mathbb{A}} = \{z_1 + iz_2 : z_1, z_2 \in \mathbb{C}\}$, known as auxiliary complex spaces. Though $\mathbb{A}$ and $\bar{\mathbb{A}}$ contain same elements as in $\mathbb{C}$ but this convenient notation are used for special representation of elements in the sense that each $Z = z_1 + jz_2 = (z_1 - iz_2)e + (z_1 + iz_2)e^\dagger \in \mathbb{BC}$ associates the points $(z_1 - iz_2) \in \mathbb{A}$ and $(z_1 + iz_2) \in \bar{\mathbb{A}}$ and also to each point $(z_1 - iz_2, z_1 + iz_2) \in \mathbb{A} \times \bar{\mathbb{A}}$, there is a unique point in $\mathbb{BC}$.

1.1.5. Cartesian product in $\mathbb{BC}$

$\mathbb{BC}$-cartesian set determined by $X_1 \subset \mathbb{A}$ and $X_2 \subset \bar{\mathbb{A}}$ is defined as $X_1 \times_e X_2 := \{z_1 + jz_2 \in \mathbb{BC} : z_1 + jz_2 = \omega_1 e + \omega_2 e^\dagger, (\omega_1, \omega_2) \in X_1 \times X_2\}$.

1.1.6. $\mathbb{BC}$-discus

An open discus $D(a; r_1, r_2)$ with centre $a = a_1 e + a_2 e^\dagger$ and radii $r_1 > 0$, $r_2 > 0$ is defined as

$$D(a;r_1,r_2) = B(a_1,r_1) \times_e B(a_2,r_2) = \{w_1 e + w_2 e^\dagger \in \mathbb{BC} : |w_1 - a_1| < r_1, |w_2 - a_2| < r_2\}$$

and a closed discus $\bar{D}(a; r_1, r_2)$ with centre $a = a_1 e + a_2 e^\dagger$ and radii $r_1 > 0$, $r_2 > 0$ is defined as

$$\bar{D}(a;r_1,r_2) = \bar{B}(a_1,r_1) \times_e \bar{B}(a_2,r_2) = \{w_1 e + w_2 e^\dagger \in \mathbb{BC} : |w_1 - a_1| \leqslant r_1, |w_2 - a_2| \leqslant r_2\},$$

where $B(z,r)$ and $\bar{B}(z,r)$ respectively represent open and closed ball with centre z and radius r.

It is worth here to mention that $D(a; r_1, r_2)$, the product of two disks of respectively radii r_1 and r_2, geometrically represents a duocylinder or double cylinder in 4-dimensional Euclidean space. This duocylinder or double cylinder in 4-dimensional Euclidean space is analogous to a cylinder in 3-dimensional Euclidean space, which is the cartesian product of a disk with a line segment.

1.1.7. $\mathbb{BC}$-disc

If both $r_1 > 0$ and $r_2 > 0$ are equal to r, then the discus is called a $\mathbb{BC} - Disc$ and is denoted by $D(a;r,r) = D(a;r)$.

1.2. Polynomial of a bicomplex variable

A bicomplex polynomial of degree n is a function of the form

$$P(Z) = \sum_{i=0}^{n} A_i Z^i, \; A_n \neq 0,$$

where A_i for all $i = 0, 1, 2, \ldots, n$ are bicomplex numbers and Z is a bicomplex variable. Now if we write $Z = z_1 + jz_2 = \zeta_1 e + \zeta_2 e^\dagger$ and $A_i = \alpha_i e + \beta_i e^\dagger$ for all $i = 0, 1, 2, \ldots, n$, then $Z^i = \zeta_1^i e + \zeta_2^i e^\dagger$ and we can re-write our polynomial in the idempotent representation as

$$P(Z) = \sum_{i=0}^{n} (\alpha_i \zeta_1^i) e + \sum_{i=0}^{n} (\beta_i \zeta_2^i) e^\dagger = f_1(\zeta_1) e + f_2(\zeta_2) e^\dagger.$$

Now if we denote the set of distinct zeros of f_1 and f_2 by S_1 and S_2, and if S denotes the set of distinct zeros of the polynomial P, then

$$S = S_1 e + S_2 e^\dagger,$$

and thus the following three cases fully describe the structure of the null-set of the polynomial $P(Z)$ of degree n:

1. If both polynomials f_1 and f_2 are of degree at least one, and if $S_1=\{\mathfrak{z}_{1,1},\mathfrak{z}_{1,2},\ldots,\mathfrak{z}_{1,k}\}$ and $S_2=\{\mathfrak{z}_{2,1},\mathfrak{z}_{2,2},\ldots,\mathfrak{z}_{2,l}\}$, then the set of distinct zeros of the polynomial $P(z)$ is given by

$$S=\{Z_{s,t}=\mathfrak{z}_{1,s}e+\mathfrak{z}_{2,t}e^{\dagger}:s=1,\ldots,k,t=1,\ldots,l\}.$$

2. If f_1 is identically zero, then $S_1=\mathbb{C}$ and $S_2=\{\mathfrak{z}_{2,1},\mathfrak{z}_{2,2},\ldots,\mathfrak{z}_{2,l}\}$, with $l\leqslant n$. Hence

$$S=\{Z_t\lambda+\mathfrak{z}_{2,t}e^{\dagger}:\lambda\in\mathbb{C},t=1,\ldots,l\}.$$

   Similarly, if f_2 is identically zero, then $S_2=\mathbb{C}$ and $S_1=\{\mathfrak{z}_{1,1},\mathfrak{z}_{1,2},\ldots,\mathfrak{z}_{1,k}\}$, with $k\leqslant n$. Hence

$$S=\{Z_s=\mathfrak{z}_{1,s}e+\lambda e^{\dagger}:\lambda\in\mathbb{C},s=1,\ldots,k\}.$$

3. If all the coefficients A_i with the exception $A_0=\alpha_0e+\beta_0e^{\dagger}$ are complex multiples of e (respectively of $e^{\dagger}$), but $\beta_0\neq 0$ (respectively $\alpha_0\neq 0$), then polynomial P has no zeros.

1.3. Apolarity for bicomplex polynomials

Two bicomplex polynomials

$$P(Z)=\sum_{j=0}^{n}\binom{n}{j}A_jZ^j=\left(\sum_{j=0}^{n}\binom{n}{j}\alpha_j\zeta_1^j\right)e+\left(\sum_{j=0}^{n}\binom{n}{j}\beta_j\zeta_2^j\right)e^{\dagger}=p_1(\zeta_1)e+p_2(\zeta_2)e^{\dagger}$$

and

$$Q(Z)=\sum_{j=0}^{n}\binom{n}{j}B_jZ^j=\left(\sum_{j=0}^{n}\binom{n}{j}\omega_j\zeta_1^j\right)e+\left(\sum_{j=0}^{n}\binom{n}{j}\nu_j\zeta_2^j\right)e^{\dagger}=q_1(\zeta_1)e+q_2(\zeta_2)e^{\dagger}$$

are said to be apolar if

$$\begin{aligned}
&A_0B_n-\binom{n}{1}A_1B_{n-1}+\binom{n}{2}A_2B_{n-2}+\cdots+(-1)^nA_nB_0\\
&=(\alpha_0e+\beta_0e^{\dagger})(\omega_ne+\nu_ne^{\dagger})-\binom{n}{1}(\alpha_1e+\beta_1e^{\dagger})(\omega_{n-1}e+\nu_{n-1}e^{\dagger})\\
&\quad+(\alpha_2e+\beta_2e^{\dagger})(\omega_{n-2}e+\nu_{n-2}e^{\dagger})+\cdots+(-1)^n(\alpha_ne+\beta_ne^{\dagger})(\omega_0e+\nu_0e^{\dagger})\\
&=\left(\alpha_o\omega_n-\binom{n}{2}\alpha_1\omega_{n-1}+\binom{n}{1}\alpha_2\omega_{n-2}+\cdots+(-1)^n\alpha_n\omega_0\right)e\\
&\quad+\left(\beta_o\nu_n-\binom{n}{2}\beta_1\nu_{n-1}+\binom{n}{1}\beta_2\nu_{n-2}+\cdots+(-1)^n\beta_n\nu_0\right)e^{\dagger}\\
&=0
\end{aligned}$$

that is, if

$$\alpha_o \omega_n - \binom{n}{1} \alpha_1 \omega_{n-1} + \binom{n}{2} \alpha_2 \omega_{n-2} + \cdots + (-1)^n \alpha_n \omega_0 = 0 \ , \qquad (1)$$

$$\beta_o v_n - \binom{n}{1} \beta_1 v_{n-1} + \binom{n}{2} \beta_2 v_{n-2} + \cdots + (-1)^n \beta_n v_0 = 0. \qquad (2)$$

From (1) and (2), it is clear that $P(Z)$ and $Q(Z)$ are said to be apolar, if the coefficients of their corresponding idempotent parts satisfy the following two conditions simultaneously

$$\alpha_o \omega_n - \binom{n}{1} \alpha_1 \omega_{n-1} + \binom{n}{2} \alpha_2 \omega_{n-2} + \cdots + (-1)^n \alpha_n \omega_0 = 0$$

and

$$\beta_o v_n - \binom{n}{1} \beta_1 v_{n-1} + \binom{n}{2} \beta_2 v_{n-2} + \cdots + (-1)^n \beta_n v_0 = 0.$$

Let us state three well known results on polynomials, whose analogues for bicomplex polynomials will be established in this paper.

THEOREM 1.1. (Grace's theorem [2]) *If* $P(z) = \sum_{j=0}^{n} \binom{n}{j} A_j z^j$ *and* $Q(z) = \sum_{j=0}^{n} \binom{n}{j} B_j z^j$, $A_n B_n \neq 0$ *are apolar polynomials and if one of them has all its zeros in a circular region* $\mathscr{C}$, *then the other will also have atleast one zero in* $\mathscr{C}$.

THEOREM 1.2. (Schur-Szegö composition theorem [2]) *Let* $P(z) = \sum_{j=0}^{n} \binom{n}{j} A_j z^j$ *and* $Q(z) = \sum_{j=0}^{n} \binom{n}{j} B_j z^j$ *be two polynomials of degree* n *and the composite polynomial be* $R(z) = \sum_{j=0}^{n} \binom{n}{j} A_j B_j z^j$. *If all the zeros of* $P(z)$ *lie in a circular region* $\mathscr{C}$, *then every zero* γ *of* $R(z)$ *has the form* $\gamma = -\alpha\beta$ *where* α *is suitably chosen point in* $\mathscr{C}$ *and* β *is a zero of* $Q(z)$.

THEOREM 1.3. (Cohn-Egerváry theorem [2]) *If all the zeros of* $P(z) = \sum_{j=0}^{n} \binom{n}{j} A_j z^j$ *lie in the circle* $|z| < r$ *and if all the zeros of* $Q(z) = \sum_{j=0}^{n} \binom{n}{j} B_j z^j$ *lie in* $|z| < s$, *then all the zeros of* $R(z) = \sum_{j=0}^{n} \binom{n}{j} A_j B_j z^j$ *lie in* $|z| < rs$.

2. Main results

In this paper we prove the analogue of Grace's theorem, Schur-Szegö composition and Cohn-Egerváry theorem for bicomplex polynomials as follows:

THEOREM 1. *If two bicomplex polynomials* $P(Z) = \sum_{j=0}^{n} \binom{n}{j} A_j Z^j$ *and* $Q(Z) = \sum_{j=0}^{n} \binom{n}{j} B_j Z^j$ *are apolar and if any one of them has all its zeros in a closed discus* $\bar{D}(c; r_1, r_2)$, *where* $c = c_1 e + c_2 e^\dagger$ *then other will have atleast one zero in* $\bar{D}(c; r_1, r_2)$.

THEOREM 2. *Let* $P(Z) = \sum_{j=0}^{n} \binom{n}{j} A_j Z^j$ *and* $Q(Z) = \sum_{j=0}^{n} \binom{n}{j} B_j Z^j$ *be two bicomplex polynomials and the composite polynomial be* $R(Z) = \sum_{j=0}^{n} \binom{n}{j} A_j B_j Z^j$. *If all the zeros of* $P(Z)$ *lie in a closed discus* $\bar{D}(c; r_1, r_2)$, *then every zero* $\gamma = \gamma_1 e + \gamma_2 e^\dagger$ *of* $R(z)$ *has the form* $\gamma = -\mu\delta$ *where* $\mu = \mu_1 e + \mu_2 e^\dagger$ *is suitably chosen point in* $\bar{D}(c; r_1, r_2)$ *and* $\delta = \delta_1 e + \delta_2 e^\dagger$ *is a zero of* $Q(Z)$.

THEOREM 3. *If all the zeros of a polynomial* $P(Z) = \sum_{j=0}^{n} \binom{n}{j} A_j Z^j$ *lie in open discus* $D(c; r_1, r_2)$ *and if all the zeros of the polynomial* $Q(Z) = \sum_{j=0}^{n} \binom{n}{j} B_j Z^j$ *lie in a closed discus* $\bar{D}(c; s_1, s_2)$, *then all the zeros of the composite polynomial* $R(Z) = \sum_{j=0}^{n} \binom{n}{j} A_j B_j Z^j$ *lie in an open discus* $D(c; r_1 s_1, r_2 s_2)$.

3. Lemmas

Before presenting the main results, here we present a lemma [4] that is required for the proofs of the theorems:

LEMMA 1. *Let* $F(z)$ *be a bicomplex holomorphic function defined in a domain* $X = X_1 e + X_2 e^\dagger := \{\zeta_1 e_1 + \zeta_2 e^\dagger : \zeta_1 \in X_1, \zeta_2 \in X_2\}$ *such that* $F(z) = f_1(\zeta_1) e + f_2(\zeta_2) e^\dagger$, *for all* $z = \zeta_1 e + \zeta_2 e^\dagger \in X$. *Then* $F(z)$ *has zero on* X *if and only if* $f_1(\zeta_1)$ *and* $f_2(\zeta_2)$ *both have zero at* ζ_1 *in* X_1 *and at* ζ_2 *in* X_2 *respectively.*

4. Proofs of theorems

Proof of Theorem 1. Consider the two bicomplex polynomials in their idempotent representation as

$$P(Z) = \sum_{j=0}^{n} \binom{n}{j} A_j Z^j = p_1(\zeta_1)e + p_2(\zeta_2)e^{\dagger}$$

and

$$Q(Z) = \sum_{j=0}^{n} \binom{n}{j} B_j Z^j = q_1(\zeta_1)e + q_2(\zeta_2)^{\dagger}.$$

Without loss of generality, let us suppose that all the zeros of $P(Z) = p_1(\zeta_1)e + p_2(\zeta_2)e^{\dagger}$ lie in a discus $\bar{D}(c; r_1, r_2)$, where $c = c_1 e + c_2 e^{\dagger}$. Therefore by Lemma 1, $p_1(\zeta_1)$ and $p_2(\zeta_2)$ have all their zeros in

$X_1 = \{\zeta_1 \in A : |\zeta_1 - c_1| \leqslant r_1\} \subset \mathbb{C}$ and $X_2 = \{\zeta_2 \in A : |\zeta_2 - c_2| \leqslant r_2\} \subset \mathbb{C}$ respectively.

Since $P(Z)$ and $Q(Z)$ are apolar bicomplex polynomials which implies by section 1.3 that $p_1(\zeta_1)$ and $q_1(\zeta_1)$, $p_2(\zeta_2)$ and $q_2(\zeta_2)$ are apolar simultaneously. Therefore by Theorem 1.3, we conclude that atleast one zero of $q_1(\zeta_1)$ and $q_2(\zeta_2)$ lie in X_1 and X_2 respectively.

Hence by Lemma 1, $Q(Z) = q_1(\zeta_1)e + q_2(\zeta_2)^{\dagger}$ has atleast one zero in $X = X_1 e + X_2 e^{\dagger}$.

That is, $Q(Z) = q_1(\zeta_1)e + q_2(\zeta_2)^{\dagger}$ has atleast one zero in a discus $\bar{D}(c; r_1, r_2)$.

This completes the proof of Theorem 1. □

Proof of Theorem 2. Consider the two bicomplex polynomials in their idempotent representation as

$$P(Z) = \sum_{j=0}^{n} \binom{n}{j} A_j Z^j = p_1(\zeta_1)e + p_2(\zeta_2)e^{\dagger}$$

and

$$Q(Z) = \sum_{j=0}^{n} \binom{n}{j} B_j Z^j = q_1(\zeta_1)e + q_2(\zeta_2)e^{\dagger}.$$

Now we have the composite polynomial

$$\begin{aligned} R(Z) =& P(Z) * Q(Z) \\ =& \sum_{j=0}^{n} \binom{n}{j} A_j B_j Z^j \\ =& R_1(\zeta_1)e + R_2(\zeta_2)e^{\dagger}, \end{aligned}$$

where $R_1(\zeta_1) = (p_1 * q_1)(\zeta_1)$ and $R_2(\zeta_1) = (p_2 * q_2)(\zeta_2)$.

It is given that $\delta = \delta_1 e + \delta_2 e^{\dagger}$ is a zero of $Q(Z) = q_1(\zeta_1)e + q_2(\zeta_2)e^{\dagger}$, therefore δ_1 and δ_2 are the zeros of $q_1(\zeta_1)$ and $q_2(\zeta_2)$ respectively. Also $\nu = \nu_1 e + \nu_2 e^{\dagger}$ is a suitably chosen point in a discus $\bar{D}(c;r_1,r_2)$, therefore $\nu_1 \in X_1 = \{\zeta_1 \in A : |\zeta_1 - c_1| \leqslant r_1\} \subset \mathbb{C}$ and $\nu_2 \in X_2 = \{\zeta_2 \in \bar{A} : |\zeta_2 - c_2| \leqslant r_2\} \subset \mathbb{C}$ respectively. With the help of Theorem 1.3, we conclude that all the zeros of $R_1(\zeta_1) = (p_1 * q_1)(\zeta_1)$ and $R_2(\zeta_1) = (p_2 * q_2)(\zeta_2)$ are of the form $\gamma_1 = -\nu_1\delta_1$ and $\gamma_2 = -\nu_2\delta_2$ respectively. Therefore by Lemma 1, all the zeros of the polynomial $R(Z) = R_1(\zeta_1)e + R_2(\zeta_2)e^{\dagger}$ are of the form

$$\begin{aligned}\gamma =& \gamma_1 e + \gamma_2 e^{\dagger}\\ =&(-\nu_1\delta_1)e + (-\nu_2\delta_2)e^{\dagger}\\ =&-\{\nu_1\delta_1 e + \nu_2\delta_2\}\\ =&-\nu\delta.\end{aligned}$$

This completes the proof of Theorem 2. $\square$

Proof of Theorem 3. Consider the two bicomplex polynomials in their idempotent representation as

$$P(Z) = \sum_{j=0}^{n}\binom{n}{j}A_j Z^j = p_1(\zeta_1)e + p_2(\zeta_2)e^{\dagger}$$

and

$$Q(Z) = \sum_{j=0}^{n}\binom{n}{j}B_j Z^j = q_1(\zeta_1)e + q_2(\zeta_2)e^{\dagger}.$$

Now we have the composite polynomial

$$\begin{aligned}R(Z) =& P(Z) * Q(Z)\\ =&\sum_{j=0}^{n}\binom{n}{j}A_j B_j Z^j\\ =&R_1(\zeta_1)e + R_2(\zeta_2)e^{\dagger},\end{aligned}$$

where $R_1(\zeta_1) = (p_1 * q_1)(\zeta_1)$ and $R_2(\zeta_1) = (p_2 * q_2)(\zeta_2)$.

It is given that $\delta = \delta_1 e + \delta_2 e^{\dagger}$ is a zero of $Q(Z) = q_1(\zeta_1)e + q_2(\zeta_2)e^{\dagger}$, therefore δ_1 and δ_2 are the zeros of $q_1(\zeta_1)$ and $q_2(\zeta_2)$ respectively. Therefore from the proof of the Theorem 2, we conclude that every zero of $R_1(\zeta_1) = (p_1 * q_1)(\zeta_1)$ and $R_2(\zeta_2) = (p_2 * q_2)(\zeta_2)$ are of the form $\gamma_1 = -\nu_1\delta_1$ and $\gamma_2 = -\nu_2\delta_2$.

This implies that

$$\begin{aligned}|\gamma_1| =& |-\nu_1\delta_1|\\ =&|\nu_1||\delta_1|\\ <& r_1 s_1.\end{aligned}$$

Similarly, $|\gamma_2| < r_2 s_2$. Thus we conclude that all the zeros of $R_1(\zeta_1)$ lie in $X_1 = \{\zeta_1 \in A : |\zeta_1 - c_1| < r_1 s_1\} \subset \mathbb{C}$ and all the zeros of $R_2(\zeta_2)$ lie in $X_2 = \{\zeta_2 \in A : |\zeta_2 - c_2| < r_2 s_2\} \subset \mathbb{C}$. Hence by Lemma 1, polynomial $R(Z) = R_1(\zeta_1)e + R_2(\zeta_2)e^{\dagger}$ has all its zeros in $X_1 e + X_2 e^{\dagger} = D(c : r_1 s_1, r_2 s_2)$.

This completes the proof of Theorem 3. □

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Ashish Kumar
Department of Mathematics
University of Kashmir
Srinagar-190006, India
e-mail: ashishsanthal@gmail.com

B. A. Zargar
Department of Mathematics
University of Kashmir
Srinagar-190006, India
e-mail: bazargar@gmail.com