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FUNCTIONAL DEUTSCH UNCERTAINTY PRINCIPLE

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Abstract. Entropic uncertainty principle for finite dimensional Hilbert spaces (known as Deutsch uncertainty) obtained by Deutsch *[Phys. Rev. Lett., 1983]* is a foundational result in Mathematics and Physics. We derive the Deutsch uncertainty principle for finite dimensional Banach space and its dual. Our main tool is the notion of Parseval p-frames for Banach spaces. Using the celebrated Buzano inequality in Hilbert spaces, we show that our result reduces to the Deutsch uncertainty principle for Hilbert spaces.

1. Introduction

Let $d \in \mathbb{N}$ and $\widehat{\cdot}: L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be the unitary Fourier transform obtained by extending uniquely the bounded linear operator

$$\widehat{\cdot}: L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \ni f \mapsto \widehat{f} \in C_0(\mathbb{R}^d);$$
$$\widehat{f}: \mathbb{R}^d \ni \xi \mapsto \widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \xi \rangle}\, dx \in \mathbb{C}.$$

The *Shannon entropy* at a function $f \in L^2(\mathbb{R}^d) \setminus \{0\}$ is defined as

$$S(f) := -\int_{\mathbb{R}^d} \left| \frac{f(x)}{\|f\|} \right|^2 \log \left| \frac{f(x)}{\|f\|} \right|^2 dx$$

(with the convention $0 \log 0 = 0$) [12]. In 1957, Hirschman proved the following result [11].

THEOREM 1. [11] (Hirschman Inequality) *For all* $f \in L^2(\mathbb{R}^d) \setminus \{0\}$,

$$S(f) + S(\widehat{f}) \geqslant 0. \tag{1}$$

In the same paper [11] Hirschman conjectured that Inequality (1) can be improved to

$$S(f) + S(\widehat{f}) \geqslant d(1 - \log 2), \quad \text{for all } f \in L^2(\mathbb{R}^d) \setminus \{0\}. \tag{2}$$

Inequality (2) was proved independently in 1975 by Beckner [2] and Bialynicki-Birula and Mycielski [4].

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THEOREM 2. [2, 4] (Hirschman-Beckner-Bialynicki-Birula-Mycielski uncertainty principle) *For all* $f \in L^2(\mathbb{R}^d) \setminus \{0\}$,

$$S(f) + S(\widehat{f}) \geqslant d(1 - \log 2).$$

Now one naturally asks whether there is a finite dimensional version of Shannon entropy and uncertainty principle. Let $\mathscr{H}$ be a finite dimensional Hilbert space. Given an orthonormal basis $\{\tau_j\}_{j=1}^n$ for $\mathscr{H}$, the *(finite) Shannon entropy* at a point $h \in \mathscr{H}_\tau$ is defined as

$$S_\tau(h) := -\sum_{j=1}^{n} \left|\left\langle \frac{h}{\|h\|}, \tau_j \right\rangle\right|^2 \log \left|\left\langle \frac{h}{\|h\|}, \tau_j \right\rangle\right|^2 \geqslant 0,$$

where $\mathscr{H}_\tau := \{h \in \mathscr{H} : \langle h, \tau_j \rangle \neq 0, 1 \leqslant j \leqslant n\}$ [9]. In 1983, Deutsch [9] derived following uncertainty principle for Shannon entropy which is fundamental to several developments in Mathematics and Physics [3, 8, 13, 17].

THEOREM 3. [9] (Deutsch uncertainty principle) *Let* $\{\tau_j\}_{j=1}^n$, $\{\omega_j\}_{j=1}^n$ *be two orthonormal bases for a finite dimensional Hilbert space* $\mathscr{H}$. *Then*

$$2\log n \geqslant S_\tau(h) + S_\omega(h) \geqslant -2\log\left(\frac{1 + \max\limits_{1 \leqslant j,k \leqslant n} |\langle \tau_j, \omega_k \rangle|}{2}\right) \geqslant 0, \quad \textit{for all } h \in \mathscr{H}_\tau. \tag{3}$$

We naturally ask what is the Banach space version of Inequality (3)? In this paper, we are going to answer this question.

Novelty of the paper. Eventhough there is a large body of literature devoted to the improvements and applications of Deutsch uncertainty principle for finite dimensional Hilbert spaces, even finite dimensional Banach space version of Deutsch uncertainty principle is not known. It is well known that for each natural number, there is only one Hilbert space upto unitary equivalence. This is not the case for finite dimensional Banach spaces. Thus we believe that deriving Deutsch uncertainty for Banach spaces will trigger a research of uncertainty principles in Banach spaces. Deutsch uncertainty is continuously in use in the quantum information theory which has real life applications. Thus we believe that deriving uncertainties in Banach spaces will help in understanding quantum information theory better.

Note that there is no traditional orthogonality (which comes from inner product in Hilbert spaces) in Banach spaces. So to derive an uncertainty for Banach spaces, we search for generalization of orthonormal bases. A natural option is the notion of Parseval p-frames. Similar to the fact that an orthonormal basis gives a resolution of identity operator using rank one positive operator, a Parseval p-frame resolves identity into rank one operator without positivity. Our main observation is that positivity is not needed in the derivation of Deutsch uncertainty. We took this observation to Banach spaces and derived the first important uncertainty principle in Theorem 4. Using Buzano inequality (which is a generalization of Cauchy-Schwarz inequality) we showed in Corollary 1 that our result reduces to Deutsch uncertainty and hence is a genuine generalization from Hilbert spaces to Banach spaces. Unlike Hilbert spaces, a Banach space and its dual stand differently. Thus it is a natural observation in Banach space theory that whenever a result holds for a Banach space, it is not guaranteed that the same result holds for its dual. However, using the notion of Parseval p-frame for the dual of a Banach space, we showed in Theorem 5 that there is a version of Deutsch uncertainty even for the dual of a Banach space.

2. Functional Deutsch uncertainty principle

In the paper, $\mathbb{K}$ denotes $\mathbb{C}$ or $\mathbb{R}$ and $\mathscr{X}$ denotes a finite dimensional Banach space over $\mathbb{K}$. Dual of $\mathscr{X}$ is denoted by $\mathscr{X}^*$. We need the notion of Parseval p-frames for Banach spaces.

DEFINITION 1. [1, 7] Let $\mathscr{X}$ be a finite dimensional Banach space over $\mathbb{K}$. A collection $\{f_j\}_{j=1}^n$ in $\mathscr{X}^*$ is said to be a *Parseval p-frame* ($1 \leqslant p < \infty$) for $\mathscr{X}$ if

$$\|x\|^p = \sum_{j=1}^{n} |f_j(x)|^p, \quad \text{for all } x \in \mathscr{X}. \tag{4}$$

Note that (4) says that $\|f_j\| \leqslant 1$ for all $1 \leqslant j \leqslant n$. Given a Parseval p-frame $\{f_j\}_{j=1}^n$ for $\mathscr{X}$, we define the *(finite) p-Shannon entropy* at a point $x \in \mathscr{X}_f$ as

$$S_f(x) := -\sum_{j=1}^{n} \left| f_j\left(\frac{x}{\|x\|}\right)\right|^p \log\left| f_j\left(\frac{x}{\|x\|}\right)\right|^p \geqslant 0,$$

where $\mathscr{X}_f := \{x \in \mathscr{X} : f_j(x) \neq 0, 1 \leqslant j \leqslant n\}$. Following is the fundamental result of this paper.

THEOREM 4. (Functional Deutsch uncertainty principle) *Let $\{f_j\}_{j=1}^n$ and $\{g_k\}_{k=1}^m$ be Parseval p-frames for a finite dimensional Banach space $\mathscr{X}$. Then*

$$\frac{1}{(nm)^{\frac{1}{p}}} \leqslant \sup_{y \in \mathscr{X}, \|y\|=1} \left(\max_{1 \leqslant j \leqslant n, 1 \leqslant k \leqslant m} |f_j(y) g_k(y)| \right)$$

and

$$\begin{aligned} \log(nm) &\geqslant S_f(x) + S_g(x) \\ &\geqslant -p \log\left(\sup_{y \in \mathscr{X}_f \cap \mathscr{X}_g, \|y\|=1} \left(\max_{1 \leqslant j \leqslant n, 1 \leqslant k \leqslant m} |f_j(y) g_k(y)| \right)\right) \\ &> 0, \quad \textit{for all } x \in \mathscr{X}_f \cap \mathscr{X}_g. \end{aligned} \tag{5}$$

Proof. Let $z \in \mathscr{X}$ be such that $\|z\|=1$. Then

$$\begin{aligned}
1 &= \left(\sum_{j=1}^{n}|f_j(z)|^p\right)\left(\sum_{k=1}^{m}|g_k(z)|^p\right) = \sum_{j=1}^{n}\sum_{k=1}^{m}|f_j(z)g_k(z)|^p\\
&\leqslant \sum_{j=1}^{n}\sum_{k=1}^{m}\left(\sup_{y\in\mathscr{X},\|y\|=1}\left(\max_{1\leqslant j\leqslant n,1\leqslant k\leqslant m}|f_j(y)g_k(y)|\right)\right)^p\\
&= \left(\sup_{y\in\mathscr{X},\|y\|=1}\left(\max_{1\leqslant j\leqslant n,1\leqslant k\leqslant m}|f_j(y)g_k(y)|\right)\right)^p mn
\end{aligned}$$

which gives

$$\frac{1}{mn} \leqslant \left(\sup_{y\in\mathscr{X},\|y\|=1}\left(\max_{1\leqslant j\leqslant n,1\leqslant k\leqslant m}|f_j(y)g_k(y)|\right)\right)^p.$$

Since $1=\sum_{j=1}^{n}\left|f_j\left(\frac{x}{\|x\|}\right)\right|^p$ for all $x\in\mathscr{X}\setminus\{0\}$, $1=\sum_{k=1}^{m}\left|g_k\left(\frac{x}{\|x\|}\right)\right|^p$ for all $x\in\mathscr{X}\setminus\{0\}$ and log function is concave, using Jensen's inequality (see [14]) we get

$$\begin{aligned}
&S_f(x)+S_g(x)\\
&= \sum_{j=1}^{n}\left|f_j\left(\frac{x}{\|x\|}\right)\right|^p \log\left(\frac{1}{\left|f_j\left(\frac{x}{\|x\|}\right)\right|^p}\right) + \sum_{k=1}^{m}\left|g_k\left(\frac{x}{\|x\|}\right)\right|^p \log\left(\frac{1}{\left|g_k\left(\frac{x}{\|x\|}\right)\right|^p}\right)\\
&\leqslant \log\left(\sum_{j=1}^{n}\left|f_j\left(\frac{x}{\|x\|}\right)\right|^p \frac{1}{\left|f_j\left(\frac{x}{\|x\|}\right)\right|^p}\right) + \log\left(\sum_{k=1}^{m}\left|g_k\left(\frac{x}{\|x\|}\right)\right|^p \frac{1}{\left|g_k\left(\frac{x}{\|x\|}\right)\right|^p}\right)\\
&= \log n + \log m = \log(nm), \quad \text{for all } x\in\mathscr{X}_f\cap\mathscr{X}_g.
\end{aligned}$$

Let $x\in\mathscr{X}_f\cap\mathscr{X}_g$. Then

$$\begin{aligned}
&S_f(x)+S_g(x)\\
&= -\sum_{j=1}^{n}\sum_{k=1}^{m}\left|f_j\left(\frac{x}{\|x\|}\right)\right|^p\left|g_k\left(\frac{x}{\|x\|}\right)\right|^p\left[\log\left|f_j\left(\frac{x}{\|x\|}\right)\right|^p+\log\left|g_k\left(\frac{x}{\|x\|}\right)\right|^p\right]\\
&= -\sum_{j=1}^{n}\sum_{k=1}^{m}\left|f_j\left(\frac{x}{\|x\|}\right)\right|^p\left|g_k\left(\frac{x}{\|x\|}\right)\right|^p\log\left|f_j\left(\frac{x}{\|x\|}\right)g_k\left(\frac{x}{\|x\|}\right)\right|^p\\
&= -p\sum_{j=1}^{n}\sum_{k=1}^{m}\left|f_j\left(\frac{x}{\|x\|}\right)\right|^p\left|g_k\left(\frac{x}{\|x\|}\right)\right|^p\log\left|f_j\left(\frac{x}{\|x\|}\right)g_k\left(\frac{x}{\|x\|}\right)\right|\\
&\geqslant -p\sum_{j=1}^{n}\sum_{k=1}^{m}\left|f_j\left(\frac{x}{\|x\|}\right)\right|^p\left|g_k\left(\frac{x}{\|x\|}\right)\right|^p.
\end{aligned}$$

$$\log\left(\sup_{y\in\mathscr{X}_f\cap\mathscr{X}_g,\|y\|=1}\left(\max_{1\leqslant j\leqslant n,1\leqslant k\leqslant m}|f_j(y)g_k(y)|\right)\right)$$
$$=-p\log\left(\sup_{y\in\mathscr{X}_f\cap\mathscr{X}_g,\|y\|=1}\left(\max_{1\leqslant j\leqslant n,1\leqslant k\leqslant m}|f_j(y)g_k(y)|\right)\right)\cdot$$
$$\sum_{j=1}^{n}\sum_{k=1}^{m}\left|f_j\left(\frac{x}{\|x\|}\right)\right|^p\left|g_k\left(\frac{x}{\|x\|}\right)\right|^p$$
$$=-p\log\left(\sup_{y\in\mathscr{X}_f\cap\mathscr{X}_g,\|y\|=1}\left(\max_{1\leqslant j\leqslant n,1\leqslant k\leqslant m}|f_j(y)g_k(y)|\right)\right). \quad \square$$

COROLLARY 1. *Theorem 3 follows from Theorem 4.*

Proof. Let $\{\tau_j\}_{j=1}^n$, $\{\omega_j\}_{j=1}^n$ be two orthonormal bases for a finite dimensional Hilbert space $\mathscr{H}$. Define

$$f_j:\mathscr{H}\ni h\mapsto\langle h,\tau_j\rangle\in\mathbb{K};\quad g_j:\mathscr{H}\ni h\mapsto\langle h,\omega_j\rangle\in\mathbb{K},\quad \text{for all } 1\leqslant j\leqslant n.$$

Now by using Buzano inequality (see [5, 10]) we get

$$\sup_{h\in\mathscr{H},\|h\|=1}\left(\max_{1\leqslant j,k\leqslant n}|f_j(h)g_k(h)|\right)=\sup_{h\in\mathscr{H},\|h\|=1}\left(\max_{1\leqslant j,k\leqslant n}|\langle h,\tau_j\rangle||\langle h,\omega_k\rangle|\right)$$
$$\leqslant\sup_{h\in\mathscr{H},\|h\|=1}\left(\max_{1\leqslant j,k\leqslant n}\left(\|h\|^2\frac{\|\tau_j\|\|\omega_k\|+|\langle\tau_j,\omega_k\rangle|}{2}\right)\right)$$
$$=\frac{1+\max_{1\leqslant j,k\leqslant n}|\langle\tau_j,\omega_k\rangle|}{2}. \quad \square$$

Theorem 4 brings the following question.

QUESTION 1. Given p, m, n and a Banach space $\mathscr{X}$, for which pairs of Parseval p-frames $\{f_j\}_{j=1}^n$ and $\{g_k\}_{k=1}^m$ for $\mathscr{X}$, we have equality in Inequality (5)?

Next we derive a dual inequality of (5). For this we need dual of Definition 1.

DEFINITION 2. [16, 6, 15] Let $\mathscr{X}$ be a finite dimensional Banach space over $\mathbb{K}$. A collection $\{\tau_j\}_{j=1}^n$ in $\mathscr{X}$ is said to be a *Parseval p-frame* ($1\leqslant p<\infty$) for $\mathscr{X}^*$ if

$$\|f\|^p=\sum_{j=1}^{n}|f(\tau_j)|^p,\quad \text{for all } f\in\mathscr{X}^*. \tag{6}$$

Note that (6) says that

$$\|\tau_j\|=\sup_{f\in\mathscr{X}^*,\|f\|=1}|f(\tau_j)|\leqslant\sup_{f\in\mathscr{X}^*,\|f\|=1}\left(\sum_{j=1}^{n}|f(\tau_j)|^p\right)^{\frac{1}{p}}$$
$$=\sup_{f\in\mathscr{X}^*,\|f\|=1}\|f\|=1,\quad \text{for all } 1\leqslant j\leqslant n.$$

Given a Parseval p-frame $\{\tau_j\}_{j=1}^n$ for $\mathscr{X}^*$, we define the *(finite) p-Shannon entropy* at a point $f \in \mathscr{X}_\tau^*$ as

$$S_\tau(f) \coloneqq -\sum_{j=1}^{n} \left|\frac{f(\tau_j)}{\|f\|}\right|^p \log\left|\frac{f(\tau_j)}{\|f\|}\right|^p \geqslant 0,$$

where $\mathscr{X}_\tau^* \coloneqq \{f \in \mathscr{X}^* : f(\tau_j) \neq 0, 1 \leqslant j \leqslant n\}$. We now have the following dual to Theorem 4.

THEOREM 5. (Functional Deutsch uncertainty principle) *Let $\{\tau_j\}_{j=1}^n$ and $\{\omega_k\}_{k=1}^m$ be two Parseval p-frames for the dual $\mathscr{X}^*$ of a finite dimensional Banach space $\mathscr{X}$. Then*

$$\frac{1}{(nm)^{\frac{1}{p}}} \leqslant \sup_{g \in \mathscr{X}^*, \|g\|=1} \left(\max_{1 \leqslant j \leqslant n, 1 \leqslant k \leqslant m} |g(\tau_j) g(\omega_k)| \right)$$

and

$$\begin{aligned} \log(nm) &\geqslant S_\tau(f) + S_\omega(f) \qquad (7)\\ &\geqslant -p \log\left(\sup_{g \in \mathscr{X}_\tau^* \cap \mathscr{X}_\omega^*, \|g\|=1} \left(\max_{1 \leqslant j \leqslant n, 1 \leqslant k \leqslant m} |g(\tau_j) g(\omega_k)| \right)\right)\\ &> 0, \quad \textit{for all } f \in \mathscr{X}_\tau^* \cap \mathscr{X}_\omega^*. \end{aligned}$$

Proof. Let $h \in \mathscr{X}^*$ be such that $\|h\| = 1$. Then

$$\begin{aligned} 1 &= \left(\sum_{j=1}^{n} |h(\tau_j)|^p\right)\left(\sum_{k=1}^{m} |h(\omega_k)|^p\right) = \sum_{j=1}^{n}\sum_{k=1}^{m} |h(\tau_j) h(\omega_k)|^p\\ &\leqslant \sum_{j=1}^{n}\sum_{k=1}^{m} \left(\sup_{g \in \mathscr{X}^*, \|g\|=1} \left(\max_{1 \leqslant j \leqslant n, 1 \leqslant k \leqslant m} |g(\tau_j) g(\omega_k)| \right)\right)^p\\ &= \left(\sup_{g \in \mathscr{X}^*, \|g\|=1} \left(\max_{1 \leqslant j \leqslant n, 1 \leqslant k \leqslant m} |g(\tau_j) g(\omega_k)| \right)\right)^p mn \end{aligned}$$

which gives

$$\frac{1}{mn} \leqslant \left(\sup_{g \in \mathscr{X}^*, \|g\|=1} \left(\max_{1 \leqslant j \leqslant n, 1 \leqslant k \leqslant m} |g(\tau_j) g(\omega_k)| \right)\right)^p.$$

Since $1=\sum_{j=1}^{n}\left|\frac{f(\tau_j)}{\|f\|}\right|^p$ for all $f\in\mathscr{X}^*\setminus\{0\}$, $1=\sum_{k=1}^{m}\left|\frac{f(\omega_k)}{\|f\|}\right|^p$ for all $f\in\mathscr{X}^*\setminus\{0\}$ and log function is concave, using Jensen's inequality we get

$$\begin{aligned}S_\tau(f)+S_\omega(f)&=\sum_{j=1}^{n}\left|\frac{f(\tau_j)}{\|f\|}\right|^p\log\left(\frac{1}{\left|\frac{f(\tau_j)}{\|f\|}\right|^p}\right)+\sum_{k=1}^{m}\left|\frac{f(\omega_k)}{\|f\|}\right|^p\log\left(\frac{1}{\left|\frac{f(\omega_k)}{\|f\|}\right|^p}\right)\\&\leqslant\log\left(\sum_{j=1}^{n}\left|\frac{f(\tau_j)}{\|f\|}\right|^p\frac{1}{\left|\frac{f(\tau_j)}{\|f\|}\right|^p}\right)+\log\left(\sum_{k=1}^{m}\left|\frac{f(\omega_k)}{\|f\|}\right|^p\frac{1}{\left|\frac{f(\omega_k)}{\|f\|}\right|^p}\right)\\&=\log n+\log m=\log(nm),\quad\text{for all } f\in\mathscr{X}_\tau^*\cap\mathscr{X}_\omega^*.\end{aligned}$$

Let $f\in\mathscr{X}_\tau^*\cap\mathscr{X}_\omega^*$. Then

$$\begin{aligned}&S_\tau(f)+S_\omega(f)\\&\quad=-\sum_{j=1}^{n}\sum_{k=1}^{m}\left|\frac{f(\tau_j)}{\|f\|}\right|^p\left|\frac{f(\omega_k)}{\|f\|}\right|^p\left[\log\left|\frac{f(\tau_j)}{\|f\|}\right|^p+\log\left|\frac{f(\omega_k)}{\|f\|}\right|^p\right]\\&\quad=-\sum_{j=1}^{n}\sum_{k=1}^{m}\left|\frac{f(\tau_j)}{\|f\|}\right|^p\left|\frac{f(\omega_k)}{\|f\|}\right|^p\log\left|\frac{f(\tau_j)}{\|f\|}\frac{f(\omega_k)}{\|f\|}\right|^p\\&\quad=-p\sum_{j=1}^{n}\sum_{k=1}^{m}\left|\frac{f(\tau_j)}{\|f\|}\right|^p\left|\frac{f(\omega_k)}{\|f\|}\right|^p\log\left|\frac{f(\tau_j)}{\|f\|}\frac{f(\omega_k)}{\|f\|}\right|\\&\quad\geqslant-p\sum_{j=1}^{n}\sum_{k=1}^{m}\left|\frac{f(\tau_j)}{\|f\|}\right|^p\left|\frac{f(\omega_k)}{\|f\|}\right|^p\log\left(\sup_{g\in\mathscr{X}^*,\|g\|=1}\left(\max_{1\leqslant j\leqslant n,1\leqslant k\leqslant m}|g(\tau_j)g(\omega_k)|\right)\right)\\&\quad=-p\log\left(\sup_{g\in\mathscr{X}_\tau^*\cap\mathscr{X}_\omega^*,\|g\|=1}\left(\max_{1\leqslant j\leqslant n,1\leqslant k\leqslant m}|g(\tau_j)g(\omega_k)|\right)\right)\sum_{j=1}^{n}\sum_{k=1}^{m}\left|\frac{f(\tau_j)}{\|f\|}\right|^p\left|\frac{f(\omega_k)}{\|f\|}\right|^p\\&\quad=-p\log\left(\sup_{g\in\mathscr{X}_\tau^*\cap\mathscr{X}_\omega^*,\|g\|=1}\left(\max_{1\leqslant j\leqslant n,1\leqslant k\leqslant m}|g(\tau_j)g(\omega_k)|\right)\right).\quad\square\end{aligned}$$

Theorem 5 again gives the following question.

QUESTION 2. Given p, m, n and a Banach space $\mathscr{X}$, for which pairs of Parseval p-frames $\{\tau_j\}_{j=1}^n$ and $\{\omega_k\}_{k=1}^m$ for $\mathscr{X}^*$, we have equality in Inequality (7)?

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