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SHARING THE ZEROS OF POLYNOMIALS WITH REVERSE INDEX FOUR IN WEIGHTED WIDER SENSE

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Abstract. In this paper, on the basis of zero set of reverse indexed polynomial, first introduced by us, we have investigated the uniqueness of meromorphic functions under weighted sharing in wider sense criteria [4], which in turn extend some earlier results in different directions. We have succeeded to identify a subclass of meromorphic functions for which uniqueness property exists for higher reverse indexed polynomial in literature. In the last section, we have presented the application of our results in case of derivatives of the concerned functions accompanied by series of examples.

1. Introduction

At the outset, we assume that the readers are acquainted with the conventional notations of value distribution theory such as $N(r,f)$, $T(r,f)$, $S(r,f)$, etc, as outlined in [11]. So we refrain from providing detail explanations.

In value distribution theory we generally concern about the distribution of the zeros of the function $f(z)-a$, where $a\in\mathbb{C}$ and form which the idea of sharing of values or sets comes under consideration. For the standard notations of set sharing, we refer the readers to make a glance over the relevant information provided in the second paragraph of [2], which automatically includes the definition of value sharing. Below we invoke it.

Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$. Let us denote by $E_f(S)=\bigcup_{a\in S}\{z: f(z)-a=0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicities then the set $\bigcup_{a\in S}\{z: f(z)-a=0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S)=E_g(S)$ ($\overline{E}_f(S)=\overline{E}_g(S)$) we say that f and g share the set S CM (IM).

If the readers need further information or a detailed explanation about these concepts, we recommend referring to the original sources cited in the text: [2] and [11].

In 1976, in connection to the famous question of Gross [10], Lin-Yi posed the question (see *Question B*, p. 74, [15]) pertains to meromorphic functions and their relationships when sharing two sets.

QUESTION 1.1. [15] *Can one find two finite sets S_j $(j=1,2)$ such that any two non constant meromorphic functions f and g satisfying $E_f(S_j)=E_g(S_j)$ for $j=1,2$ must be identical?*

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In 2001, the notion of weighted sharing of sets was appeared in the literature, which explores the interconnections of sharing of sets by two meromorphic functions based on specific weighted criteria. This concept immensely contributed to various domains of uniqueness theory vis-a-vis value distribution theory. The specific details and implications of this notion can be found in the paper by Lahiri [12]. The definition is as follows:

DEFINITION 1. [12] Let l be a non negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_l(a;f)$ the set of all a-points of f, where an a-point of multiplicity t is counted t times if $t \leqslant l$ and $l+1$ times if $t > l$. Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$. We denote by $E_f(S,l)$ the set $\bigcup_{a \in S} E_l(a;f)$. If $E_f(S,l) = E_g(S,l)$, we say f and g share the set S with weight l and denote it by (S,l). We say, $E_f(S) = E_f(S,\infty)$ and $\overline{E}_f(S) = E_f(S,0)$.

2. Definitions and background

Recently we have introduced a more comprehensive framework than *Definition* 1 termed as ‘weighted sharing of sets in wider sense’ for meromorphic functions.

DEFINITION 2. [4] Let f and g be two non-constant meromorphic functions and $P(z)$ and $Q(z)$ be two polynomials of degree n without any multiple zero. Let

$$S_P = \{z : P(z) = 0\} \quad \text{and} \quad S_Q = \{z : Q(z) = 0\}.$$

We say that f and g share the sets S_P and S_Q with weight l in the wider sense if $E_f(S_P,l) = E_g(S_Q,l)$ and we denote it by f, g share $(S_P,S_Q;l)$. If $P = Q$, we get the traditional *Definition* 1 of weighted sharing of sets.

Next we slightly modify the definition of [5] in the following manner as the same is necessary in the subsequent stages.

DEFINITION 3. A polynomial

$$P(z) = a_n z^n + a_{n-1} z^{z-1} + \ldots + a_1 z + a_0$$

of degree n is called an initial term non-gap polynomial (ITNGP) if there exist at least one consecutive non-zero term after the first term, i.e., if $a_{n-i} \neq 0$ for some $1 \leqslant i < n$, then $a_{n-1} \neq 0$, $a_{n-2} \neq 0$, $\ldots$ $a_{n-i+1} \neq 0$. Otherwise $P(z)$ is called an initial term gap polynomial (ITGP).

We are now introducing analogues definition of *Definition* 3, in vis-a-vis of terminal term gap and non-gap polynomials.

DEFINITION 4. A polynomial

$$P(z) = a_n z^n + a_{n-1} z^{z-1} + \ldots + a_1 z + a_0$$

with $a_0 \neq 0$ is called a terminal term non-gap polynomial (TTNGP) if before a_0, there exist at least one non-zero consecutive term, i.e., if $a_t \neq 0$, then $a_{t-1} \neq 0$, $a_{t-2} \neq 0, \ldots$, $a_1 \neq 0$ for $t = 1, 2, \ldots, (n-1)$. Otherwise the polynomial is said to be a terminal terms gap polynomials (TTGP).

Next, in view of *Definitions* 3 and 4, we want to propose the definitions of Index and Reserve index of a polynomial.

DEFINITION 5. Let us consider the polynomial: $P^{[s]}(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$.

A. If $P^{[s]}(z)$ is an ITNGP, then it is said to be initial term non-gap polynomial of index s (ITNGP$_s$ in short) if one of the followings is satisfied:

i) $a_{n-s+1} \neq 0$ but $a_0 = 0$ $(1 \leqslant s \leqslant n)$

ii) $a_i \neq 0$, for $i = 0, 1, 2, \ldots, n$; then $s = n+1$.

B. If $P^{[s]}(z)$ is an ITGP, then it is of index 1.

Note that any polynomial of degree n is of index $s \geqslant 1$.

DEFINITION 6. Suppose $P_{[\hat{s}]}(z) = b_n z^n + b_{n-1} z^{n-1} + \ldots + b_1 z + b_0$, $b_0 \neq 0$.

A. If $P_{[\hat{s}]}(z)$ is an TTNGP, then it is said to be terminal term non-gap polynomial of reverse index $\hat{s}$ (TTNGP$_{\hat{s}}$ in short) if one of the followings is satisfied:

i) $b_{\hat{s}-1} \neq 0$; $(1 < \hat{s} < n)$

ii) $b_i \neq 0$ for $i = 1, \ldots, n-1$; then $\hat{s} = n+1$

B. If $P_{[\hat{s}]}(z)$ is an TTGP, then reverse index is 1.

NOTE 2.1. *For a polynomial of degree n, n cannot be reverse index of the polynomial.*

For the standard definitions and notations of the value distribution theory we refer to [11] and for the definitions of $N(r,a;f \mid\geqslant s)$, $N(r,a;f \mid= s)$ for $s \geqslant 1$, $\overline{N}_L(r,1;f)$, $\overline{N}_L(r,1;g)$, $N_E^{(k}(r,1;f)$ and $\overline{N}_*(r,a;f,g)$ we refer to [1], [13], [14], [19].

The following polynomial

$$P_{[3]}(z) = az^n - n(n-1)z^2 - 2n(n-2)bz + (n-1)(n-2)b^2,$$

where $\frac{ab^{n-2}}{2} \neq 1$, which is of reverse index 3 have some contribution in uniqueness theory.

In 2002, considering $P_{[3]}(z)$, Yi [20] proved the following results that improve *Question 1.1* affirmatively.

THEOREM A. [20] *Let us take the polynomial with only simple zeros such that Let $S = \left\{z : P_{[3]}(z) = 0\right\}$, where $n \geqslant 8$. Suppose that f and g be two non constant meromorphic functions satisfying $E_f(S,\infty) = E_g(S,\infty)$ and $E_f(\{\infty\},\infty) = E_g(\{\infty\},\infty)$, then $f \equiv g$.*

THEOREM B. [20] *Let* S *be defined as in* Theorem A *with* $n \geqslant 8$ *and* f *and* g *be two non constant meromorphic functions satisfying* $E_f(S,\infty) = E_g(S,\infty)$ *and* $E_f(\{\infty\},0) = E_g(\{\infty\},0)$, *then* $f \equiv g$.

Next, in view of *Question 1.1*, Yi and Lü [21] investigated the problem of the uniqueness of two meromorphic functions f and g when they share the set S as mentioned in *Theorem A*, IM and proved the following theorem.

THEOREM C. [21] *Let* S *be defined as in* Theorem A *and* $n \geqslant 12$. *Suppose that* f *and* g *be two non constant meromorphic functions satisfying* $E_f(S,0) = E_g(S,0)$ *and* $E_f(\{\infty\},\infty) = E_g(\{\infty\},\infty)$, *then* $f \equiv g$.

3. Motivation and main results

In view of definition 6, we see that zeros of polynomial of reverse index higher that 3 did not get any priority for investigation as far as literature of uniqueness theory is concerned. Naturally, it is high time to ponder over the contributions of polynomials of higher reverse index. This is the main motivation in writing this paper. In fact, we are going to tackle this situation more rigorously under the aegis of weighted sharing in wider sense. To this end, let us consider two polynomials of reverse index 4 with simple zeros as

$$P_4^1(z) = az^n - Q(z) \tag{3.1}$$

and

$$P_4^2(z) = az^n - dQ(z),$$

where

$$\begin{aligned} Q(z) &= n(n-1)(n-2)z^3 - 3bn(n-1)(n-3)z^2 + 3b^2n(n-2)(n-3)z \\ &\quad -b^3(n-1)(n-2)(n-3), \end{aligned}$$

d is non zero complex number. Let us denote the simple zeros of $Q(z)$ by α_i for $i = 1,2,3$. Set

$$R_1(z) = \frac{az^n}{Q(z)}, \quad R_2(z) = \frac{1}{d}R_1(z). \tag{3.2}$$

We have from (3.2),

$$R_1'(z) = -\frac{a(n-3)z^{n-1}(z-b)^3}{n(n-1)(n-2)(z-\alpha_1)^2(z-\alpha_2)^2(z-\alpha_3)^2},$$

satisfying $R_1(b) \neq 1$, so that b is not a zero of $P_4^1(z)$. Similarly we can show that b is not a zero of $P_4^2(z)$. Hence it is clear that 0 is a zero of $R_1(z)$ of multiplicity n and b is a zero of $(R_1(z) - R_1(b))$ of multiplicity 4.

With respect to the above defined polynomial, in view of the definition of weighted sharing in wider sense, we would like to state our main results of this paper.

THEOREM 1. *Let* $S_i = \{z \mid P_4^i(z) = 0\}$ *for* $i = 1,2$, *where* $P_4^i(z)$ *is given by* (3.1). *Suppose* f *and* g *are two non-constant non entire meromorphic functions satisfying one of the following conditions:*

i) f *and* g *share* $(S_1,S_2;2)$, $(\infty,0)$ *and* $n \geqslant 8$

ii) f *and* g *share* $(S_1,S_2;0)$, (∞,∞) *and* $n \geqslant 9$,

then $f \equiv g$.

THEOREM 2. *Under the same situation as in* Theorem 1, *if* f, g *be two non-constant entire functions sharing* $(S_1,S_2;2)$ *and* $n \geqslant 7$, *then*

$$n(n-1)(n-2)\{df^ng^3 - g^nf^3\} - 3bn(n-1)(n-3)\{df^ng^2 - g^nf^2\}$$
$$+3b^2n(n-2)(n-3)\{df^ng - g^nf\} - b^3(n-1)(n-2)(n-3)\{df^n - g^n\} \equiv 0.$$

The following example shows that for any non constant entire function the set S_1 in *Theorem* 2 can not be replaced by any arbitrary set consisting 7 elements.

EXAMPLE 1. Take a set consisting 7 elements as follows:

$$S = \left\{i, i\sqrt{i}, -1, -i, -i\sqrt{i}, 1, 0\right\}.$$

For some non zero complex number α, choose two functions $f(z) = e^{\alpha z}$ and $g(z) = e^{-\alpha z}$. It is easy to see that f and g share (S,∞), but for $n = 7$, $d = 1$, f, g do not satisfy the relation between f and g as demonstrated in *Theorem* 2 i.e.,

$$35(e^{4\alpha z} - e^{-4\alpha z}) - 84b(e^{5\alpha z} - e^{-5\alpha z}) + 70b^2(e^{6\alpha z} - e^{-6\alpha z}) - 20b^3(e^{7\alpha z} - e^{-7\alpha z}) \not\equiv 0.$$

THEOREM 3. *Let* S_i $i = 1,2$ *be defined as in* Theorem 1. *If* f *and* g *are two non-constant non-entire meromorphic functions that share* $(S_1,S_2;3)$, $(\{b,\infty\},0)$ *and* $n \geqslant 7$ *with* $R_2(b) \neq \frac{1}{d+1}$, *then* $f \equiv g$.

4. Lemmas

Let us define two meromorphic functions F_i and G_i as follows:

$$F \equiv R_1(f), \qquad G \equiv R_2(g). \tag{4.1}$$

On the basis of the two functions in (4.1), we now define the following two auxiliary functions H and V as follows:

$$H \equiv \left[\frac{F''}{F'} - \frac{2F'}{F-1}\right] - \left[\frac{G''}{G'} - \frac{2G'}{G-1}\right] \tag{4.2}$$

and

$$V \equiv \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}. \tag{4.3}$$

The following lemmas will play key roles in proving our results.

LEMMA 1. [19] *If F, G be two non constant meromorphic functions sharing* (1,1) *and* $H \not\equiv 0$, *then*

$$N(r,1;F \mid = 1) = N(r,1;G \mid = 1) \leqslant N(r,H) + S(r,F) + S(r,G).$$

LEMMA 2. [3] *Let f and g be two non constant meromorphic functions sharing* $(1,m)$, *where* $0 \leqslant l < \infty$. *Then*

$$\overline{N}(r,1;f) + \overline{N}(r,1;g) - N(r,1;f \mid = 1) + \left(m - \frac{1}{2}\right) \overline{N}_*(r,1;f,g)$$
$$\leqslant \frac{1}{2}[N(r,1;f) + N(r,1;g)].$$

LEMMA 3. [17] *Let f be a non-constant meromorphic function and let*

$$R_1(f) = \frac{\sum_{k=0}^{n} a_k f^k}{\sum_{j=0}^{m} b_j f^j}$$

be an irreducible function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_n \neq 0$ and $b_m \neq 0$. Then $T(R_1(f)) = dT(r,f) + S(r,f)$, where $d = max\{n,m\}$.

LEMMA 4. *Let f and g be two non constant meromorphic functions and F and G be defined by* (4.1) *such that f and g share $(S_1,S_2;0)$ and (∞,p) $0 \leqslant p < \infty$, $H \not\equiv 0$. Then*

$$N(r,\infty;H) \leqslant \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,b;f) + \overline{N}(r,b;g) + \overline{N}_*(r,1;F,G)$$
$$+ \overline{N}_*(r,\infty;f,g) + \overline{N}_0(r,0;f') + \overline{N}_0(r,0;g'),$$

where $\overline{N}_0(r,0;f')$ is the reduced counting function of those zeros of f which are not zeros of $f(f-b)(F-1)$ and $\overline{N}_0(r,0;g')$ is similarly defined.

Proof. Since f and g share $(S_1,S_2;0)$, it follows that F and G share $(1,0)$. We can easily verify that possible poles of H occur at (i) zeros of f, (ii) b- points of f, (iii) those poles of f and g whose multiplicities are distinct from the multiplicities of the corresponding poles of g and f respectively, (iv) those 1-points of F and G with different multiplicities, (v) zeros of f' which are not the zeros of $f(f-b)(F-1)$, (v) zeros of g' which are not zeros of $g(g-b)(G-1)$. Since H has only simple poles, the lemma follows from above. □

LEMMA 5. *Let f and g be two non constant meromorphic functions and F and G be defined by* (4.1) *such that f and g share $(S_1,S_2;0)$ and $(\{b,\infty\},p)$ $0 \leqslant p < \infty$, $H \not\equiv 0$. Then*

$$N(r,\infty;H) \leqslant \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,b;f) + \overline{N}(r,\infty;f) + \overline{N}_*(r,1;F,G)$$
$$+ \overline{N}_0(r,0;f') + \overline{N}_0(r,0;g'),$$

where $\overline{N}_0(r,0;f')$ is the reduced counting function of those zeros of f which are not zeros of $f(f-\delta)(F-1)$ and $\overline{N}_0(r,0;g')$ is similarly defined.

Proof. Proof of the lemma can be carried out in the line of the proof of *Lemma* 4. □

LEMMA 6. *Let F and G be given by* (4.1) *and* $H \not\equiv 0$. *Consider F and G share* $(1,m)$ *and f and g share* (∞, p), *where* $0 \leqslant m, p < \infty$.

$$\begin{aligned}
&[(n-3)p+n-4]\overline{N}(r,\infty;f \mid \geqslant p+1) \\
&= [(n-3)p+n-4]\overline{N}(r,\infty;g \mid \geqslant p+1)] \\
&\leqslant \overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).
\end{aligned}$$

Proof. Proof of the theorem can be carried out in the line of the proof of the *Lemma 2.16* of [2]. □

LEMMA 7. *Let F, G be given by* (4.1) *and F, G share* $(1,m)$. *If f and g share* $(\{b,\infty\}, p)$, *where* $0 \leqslant m, p < \infty$ *then*

$$\begin{aligned}
&(4p+3)\left[\overline{N}(r,b;f \mid \geqslant p+1)+\overline{N}(r,\infty;f \mid \geqslant p+1)\right] \\
&\leqslant \overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).
\end{aligned}$$

Proof. Proof of the lemma can be carried out in the line of the proof of *Lemma 2.5* in [6]. □

5. Proofs of the theorems

Proof of Theorem 1. *(i):* Let F and G be given by (4.1). Since f and g share $(S_1,S_2;2)$, from (4.1) it follows that F and G share $(1,2)$. Suppose $H \not\equiv 0$.

Using *Lemma* 2 for $m=2$, *Lemma* 4 for $p=0$, *Lemma* 6 for $p=0$, *Lemma* 3 we get from the Second Fundamental Theorem,

$$\begin{aligned}
&(n+1)\{T(r,f)+T(r,g)\} \\
&\leqslant \overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,\infty;f)+\overline{N}(r,1;F)+\overline{N}(r,0;g) \\
&\quad +\overline{N}(r,b;g)+\overline{N}(r,\infty;g)+\overline{N}(r,1;G)-N_0(r,0;f')-N_0(r,0;g') \\
&\quad +S(r,f)+S(r,g) \\
&\leqslant N(r,1;F \mid = 1)+\left(\frac{n}{2}+2\right)\{T(r,f)+T(r,g)\}+2\overline{N}(r,\infty;f)-\left(2-\frac{1}{2}\right) \\
&\quad \overline{N}_*(r,1;F,G)-N_0(r,0;f')-N_0(r,0;g')+S(r,f)+S(r,g) \\
&\leqslant \left(\frac{n}{2}+4+\frac{3}{n-4}\right)\{T(r,f)+T(r,g)\}-\left(2-\frac{3}{2}-\frac{3}{n-4}\right)\overline{N}_*(r,1;F,G) \\
&\quad +S(r,f)+S(r,g),
\end{aligned}$$

which is a contradiction $n \geqslant 8$.

Hence $H \equiv 0$. So for two constants $A(\neq 0)$, B we get

$$\frac{1}{F-1} \equiv \frac{A}{G-1} + B \tag{5.1}$$

and

$$T(r,f) = T(r,g) + S(r,g). \tag{5.2}$$

Case 1: ∞ is an e.v.P of both f and g.

Subcase 1.1: Let us assume that $B \neq 0$. Then by (5.1) we can have

$$F \equiv \frac{(B+1)G + (A-B-1)}{A+B(G-1)}. \tag{5.3}$$

Subcase 1.1.1: Let $(B+1) \neq 0$.
First we assume that $(A-B-1) \neq 0$, then we say from (5.3),

$$\overline{N}(r,0;f) = \overline{N}\left(r, \frac{B+1-A}{B+1}; G\right).$$

Using the Second Fundamental theorem and by the above fact we can write,

$$\begin{aligned} nT(r,g) &= T(r,G) + S(r,G) \leqslant \overline{N}(r,0;G) + \overline{N}\left(r, \frac{B+1-A}{B+1}; G\right) \\ &\quad + \overline{N}(r,\infty;G) + S(r,G) \\ &\quad \leqslant \overline{N}(r,0;f) + \overline{N}(r,0;g) + \sum_{i=1}^{3} \overline{N}(r,\alpha_i;g) + S(r,g) \leqslant 5T(r,g) + S(r,g), \end{aligned}$$

which is a contradiction for $n \geqslant 8$.
Next we assume that $(A-B-1) = 0$. Then (5.3) yields

$$F \equiv \frac{AG}{BG+1}$$

and this implies that

$$\overline{N}\left(r, 0; G + \frac{1}{B}\right) = \overline{N}(r,\infty;F).$$

By the Second Fundamental Theorem and the above stated fact using the previous arguments we can write

$$nT(r,g) \leqslant \sum_{i=1}^{3} \overline{N}(r,\alpha_i;f) + \overline{N}(r,0;g) + \sum_{i=1}^{3} \overline{N}(r,\alpha_i;g) + S(r,g) \leqslant 7T(r,g) + S(r,g),$$

a contradiction for $n \geqslant 8$.

Subcase 1.1.2: Let $(B+1)=0$. Then again from (5.3) we obtain

$$F \equiv \frac{A}{(A+1)-G}.$$

First let us suppose that $(A+1)\neq 0$. Then

$$\overline{N}(r,\infty;F)=\overline{N}(r,(A+1);G).$$

By using the similar arguments as made in the above subcase we again obtain a contradiction of $n \geqslant 8$. Next suppose that $A=-1$. Then we have

$$FG \equiv 1,$$

which implies

$$a^2(fg)^n \equiv \{n(n-1)(n-2)\}^2 \prod_{i=1}^{3}(f-\alpha_i)\prod_{i=1}^{3}(g-\alpha_i).$$

By the Second Fundamental theorem and the above equation we can write

$$\begin{aligned}
T(r,f)+T(r,g) &\leqslant \sum_{i=1}^{3}\overline{N}(r,\alpha_i;f)+\sum_{i=1}^{3}\overline{N}(r,\alpha_i;g)+S(r,f)+S(r,g)\\
&\leqslant \frac{3}{n}\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),
\end{aligned}$$

a contradiction for $n \geqslant 8$.

Case 2: ∞ is not an e.v.P. of both f and g, i.e. there exists a complex number z_0 such that $f(z_0)=g(z_0)=\infty$. From (5.1) we can say $B=0$. Hence we have

$$(G-1)=A(F-1)$$

i.e.,

$$G=A\left[F-\frac{A-1}{A}\right].$$

Let us assume that $A\neq 1$. From the above equation it is clear that

$$\overline{N}\left(r,\frac{A-1}{A};F\right)=\overline{N}(r,0;g).$$

Using the Second Fundamental Theorem we obtain

$$\begin{aligned}
nT(r,f) &\leqslant \overline{N}(r,0;F)+\overline{N}\left(r,\frac{A-1}{A};F\right)+\overline{N}(r,\infty;F)+S(r,g)\\
&\leqslant \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\sum_{i=1}^{3}\overline{N}(r,\alpha_i;f)+\overline{N}(r,0;g)+S(r,f)+S(r,g)\\
&\leqslant 6T(r,f)+S(r,f),
\end{aligned}$$

which is a contradiction. Therefore $A = 1$, which shows that

$$F \equiv G.$$

By a simple computation we can write

$$\begin{aligned} & n(n-1)(n-2)\{df^n g^3 - g^n f^3\} - 3bn(n-1)(n-3)\{df^n g^2 - g^n f^2\} \\ & +3b^2 n(n-2)(n-3)\{df^n g - g^n f\} - b^3(n-1)(n-2)(n-3)\{df^n - g^n\} \equiv 0. \end{aligned} \tag{5.4}$$

Let us take $f = hg$. Then a simple calculation yields from the above equation that

$$\begin{aligned} & n(n-1)(n-2)(gh)^3\left[h^{n-3} - \frac{1}{d}\right] - 3bn(n-1)(n-3)(gh)^2\left[h^{n-2} - \frac{1}{d}\right] \\ & +3b^2 n(n-2)(n-3)(gh)\left[h^{n-1} - \frac{1}{d}\right] - b^3(n-1)(n-2)(n-3)\left[h^n - \frac{1}{d}\right] \equiv 0. \end{aligned} \tag{5.5}$$

First suppose h is non constant. It is clear from the above facts that h doesn't possess any zeros or poles. Note that none of the zeros of $(h^i - \alpha)$, $i = n, (n-1)$, $(n-2), (n-3)$ is e.v.P. If one of the functions f or g has at least one pole then zeros of $(h^{n-3} - \alpha)$ are of multiplicities at least 3 namely δ_i for $i = 1, 2, \ldots, (n-3)$. Now using the Second Fundamental Theorem we get

$$(n-3)T(r,h) \leqslant \sum_{i=1}^{n-3} \overline{N}(r, \delta_i; h) + \overline{N}(r, 0; h) + \overline{N}(r, \infty; h) + S(r,h) \leqslant \frac{n-3}{3} T(r,h) + S(r,h),$$

which is a contradiction.

Hence, h is constant then we have $h^n \equiv h^{n-1} \equiv h^{n-2} \equiv h^{n-3} \equiv \frac{1}{d}$ which implies $h = 1$, i.e., $f \equiv g$. Proof of the *Theorem* 1 (ii) can be carried out in the line of the proof of *Theorem* 1 (i). □

Proof of Theorem 2. Proof of the theorem can be carried out in the line of the proof of the *Theorem* 1. □

Proof of Theorem 3. Let F and G be given by (4.1). Since f and g share $(S_1, S_2; 3)$, from (4.1) it follows that F and G share $(1, 3)$. Suppose $H \not\equiv 0$.

Using *Lemma* 2 for $m = 3$, *Lemma* 5 for $p = 0$, *Lemma* 7 for $p = 0$, *Lemma* 3 we get from the Second Fundamental Theorem,

$$\begin{aligned} & (n+1)\{T(r,f) + T(r,g)\} \\ \leqslant\; & \overline{N}(r,0;f) + \overline{N}(r,b;f) + \overline{N}(r,\infty;f) + \overline{N}(r,1;F) + \overline{N}(r,0;g) + \overline{N}(r,b;g) + \overline{N}(r,\infty;g) \\ & + \overline{N}(r,1;G) - N_0(r,0;f') - N_0(r,0;g') + S(r,f) + S(r,g) \\ \leqslant\; & \left(\frac{n}{2} + 2\right)\{T(r,f) + T(r,g)\} + 3\{\overline{N}(r,\infty;f) + \overline{N}(r,\delta;f)\} - \left(3 - \frac{3}{2}\right)\overline{N}_*(r,1;F,G) \\ & - N_0(r,0;f') - N_0(r,0;g') + S(r,f) + S(r,g) \\ \leqslant\; & \left(\frac{n}{2} + 3\right)\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g), \end{aligned}$$

a contradiction to the fact $n \geqslant 7$. Hence $H \equiv 0$. Thus, we get (5.1), (5.2) and (5.3). Then using the same arguments that are used in *Theorem* 1 we can reach before *Subcase 1.1.1*. Under the case $B+1 \neq 0$, suppose that $A-B-1 \neq 0$. Here

$$\overline{N}(r,\infty;G) = \overline{N}(r,\infty;g) + \sum_{i=1}^{3} \overline{N}(r,\alpha_i;g).$$

By the Second Fundamental Theorem we have

$$\begin{aligned} nT(r,g) &\leqslant \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \sum_{i=1}^{3} \overline{N}(r,\alpha_i;g) + S(r,f) + S(r,g) \\ &\leqslant 6T(r,g) + S(r,g), \end{aligned}$$

which is a contradiction. Hence we have $A-B-1=0$ and we have

$$F \equiv \frac{AG}{BG+1}. \tag{5.6}$$

Let $-\frac{1}{B} = R_2(b)$. (5.6) gives

$$G \equiv \frac{F}{(B+1)-BF}.$$

Now we claim that $\frac{B+1}{B} \neq R_1(b)$. Because if $\frac{B+1}{B} = R_1(b)$, then we get $\frac{B+1}{B} = -\frac{d}{B}$ i.e., $B=-d-1$ i.e., $R_2(b) = \frac{1}{d+1}$, which is a contradiction. Therefore

$$BF-(B+1) \equiv B\frac{a\prod_{i=1}^{n}(f-\zeta_i)}{n(n-1)(n-2)(f-\alpha_1)(f-\alpha_2)(f-\alpha_3)},$$

where $\zeta_i^{'}$s are distinct zeros of $BR_1(z)-(B+1)$ for $i=1,2,\ldots n$. Since f, g share $(\{b,\infty\},0)$, we get from the above equation ∞ is an e.v.P. of g and

$$\overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g) + \overline{N}(r,\alpha_3;g) = \sum_{i=1}^{n} \overline{N}(r,\zeta_i;f).$$

Using the Second Fundamental Theorem we get

$$\begin{aligned} nT(r,f) &\leqslant \sum_{i=1}^{n} \overline{N}(r,\zeta_i;f) + \overline{N}(r,b;f) + \overline{N}(r,\infty;f) + S(r,f) \\ &\leqslant \overline{N}(r,\alpha_1;g) + \overline{N}(r,\alpha_2;g) + \overline{N}(r,\alpha_3;g) + \overline{N}(r,b;g) + \overline{N}(r,\infty;g) + S(r,f) \\ &\leqslant 4T(r,f) + S(r,f), \end{aligned}$$

which is a contradiction.

If $-\frac{1}{B} \neq R_2(b)$, the using similar arguments as done above we again obtain a contradiction. Hence we have from (5.3)

$$F \equiv \frac{A}{-G+A+1}.$$

Let us suppose that $A \neq -1$. Then we have from the above equation

$$\overline{N}(r,\infty;F) = \overline{N}(r,A+1;G).$$

By the Second Fundamental Theorem using the above stated fact we will get a contradiction. Hence $A = -1$ i.e., $F \equiv \frac{1}{G}$. Clearly, ∞ is an e.v.P. of both f and g. Next by the similar arguments that has done in *Theorem* 1 we will arrive at a contradiction. Hence $B = 0$ and we obtain

$$AF \equiv G + A - 1.$$

next by the similar arguments as made in *Case 2* under *Theorem* 1 we will get the result $f \equiv g$. $\square$

6. Application

In this section, we will discuss about the relation between a meromorphic function and its derivatives sharing sets of different cardinalities. In this regard, initially Mues and Steinmetz [18] stared their investigations corresponding to value sharing to obtain the uniqueness of a non-constant entire function and its derivative sharing two distinct values IM.

Following series of example show that for f and f' two distinct values IM sharing cannot be replaced by a set of two elements CM sharing.

EXAMPLE 2. Let $S = \{a,b\}$, where a and b are any two distinct complex numbers. Let $f(z) = e^{\lambda z} + a + b$, when λ is a complex number such that $\lambda^k = 1$ for a non negative integer k, then f and $f^{(k)}$ share (S,∞) but $f \not\equiv f^{(k)}$.

EXAMPLE 3. Let us take $f(z) = sinz$. Clearly, f and f' share $\left(\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\},\infty\right)$, but $f \not\equiv f'$.

Using Normal families, in 2003, Fang and Zalcman [9] contributed remarkably about the forms of the function, where the function and its derivative share a set $\{0,a,b\}$, with some constraints on a and b, then they are identical. Next, in 2007 Chang, Fang and Zalcman [7] further extended the result of [9] when f and f' share $\{a,b,c\}$ by finding out 3 forms of functions.

In the next year 2008, Chang and Zalcman [8] replaced the entire function by meromorphic function with some constraints to get the following result.

THEOREM D. [8] *Let f be a non-constant meromorphic function with at most finitely many simple poles such that f and f' share $(\{0,a,b\},\infty)$, then either*

1. $f(z) = Ce^z$*; or*

2. $f(z) = Ce^{-z} + \frac{2}{3}(a+b)$ *and either* $(a+b) = 0$ *or* $(2a^2 - 5ab + 2b^2) = 0$*; or*

3. *$f(z) = Ce^{\frac{-1\pm i\sqrt{3}}{2}z} + \frac{3\pm i\sqrt{3}}{6}(a+b)$ and $a^2 - ab + b^2 = 0$, where C is a non-zero constant.*

NOTE 6.1. *We see that though the initial consideration in* Theorem D *about the function f was meromorphic, but the function ultimately becomes an entire one.*

However, for $k \geqslant 2$, the following example shows that there may be other forms of functions satisfying the hypothesis of *Theorem D*.

EXAMPLE 4. Suppose $f(z) = e^{\alpha z}$, where $\alpha^k + 1 = 0$. Evidently, f and $f^{(k)}$ share $(\{0, a, -a\}, \infty)$, but $f \not\equiv f^{(k)}$.

Finally, in 2011, Lü [16] extended *Theorem D* in case of an arbitrary set with three elements to obtain the same result with some additional suppositions. Lü obtained the following result.

THEOREM E. [16] *Let f be a non-constant meromorphic function with at most finitely many simple poles such that f and f' share $(\{a,b,c\}, \infty)$, then either*

1. *$f(z) = Ce^z$; or*

2. *$f(z) = Ce^{-z} + \frac{2}{3}(a+b+c)$ and $(2a-b-c)(2b-c-a)(2c-a-b) = 0$; or*

3. *$f(z) = Ce^{\frac{-1\pm i\sqrt{3}}{2}z} + \frac{3\pm i\sqrt{3}}{6}(a+b+c)$ and $a^2+b^2+c^2-ab-bc-ca = 0$, where C is a non-zero constant.*

In the following we have examples of different cardinalities of sets to show that in general, sharing of any arbitrary set, by a function and its derivative does not imply that they will be identical.

EXAMPLE 5. Let r, n be two natural numbers and $\alpha_s = e^{\frac{2s\pi i}{r}}$, for $s = 0, 1, 2, \ldots,$ $(r-1)$. Choose the set S_n as follows

$$S_n = \left\{ e^{\frac{2\pi i\left(s+s_1 r\right)}{nr}} : s_1 = 0, 1, 2, \ldots, (n-1) \right\}$$

and take a function $f(z) = e^{e^{\frac{2\pi i}{kn}}z}$. Then f and $f^{(k)}$ share (S_n, ∞), but $f \not\equiv f^{(k)}$.

From the theorems and followed by pertinent *Example 6.4*, we see that it is not easy to determine all the forms of the functions when the same share a set with its derivative counter part. In view of the content of the paper, it will be natural to determine the relation between f and $f^{(k)}$ on the basis of sharing of zeros of the polynomial (3.1).

THEOREM 4. *Under the same situation as in the* Theorem 1 *or* Theorem 3*, if* f *is a non constant meromorphic function and* $g = f^{(k)}$*, then* f *becomes entire. Also the relation between* f *and* $f^{(k)}$ *is given by the following equation*

$$\begin{aligned} &n(n-1)(n-2)\{df^n(f^{(k)})^3 - f^3(f^{(k)})^n\} - 3bn(n-1)(n-3) \\ &\{df^n(f^{(k)})^2 - f^2(f^{(k)})^n\} + 3b^2n(n-2)(n-3)\{df^n(f^{(k)}) - f(f^{(k)})^n\} \\ &-b^3(n-1)(n-2)(n-3)\{df^n - (f^{(k)})^n\} \equiv 0. \end{aligned} \tag{6.1}$$

Proof. Let $g = f^{(k)}$. Proceeding in the similar manner as done in the proof of *Theorem* 1 and *Theorem* 3, we can reach up to $F \equiv G$, which implies that f and $f^{(k)}$ share (∞,∞) and that implies f is an entire function. Hence we obtain (5.4) and (5.5). If $h = \frac{f}{f^{(k)}}$ is non constant, using the Second Fundamental Theorem and (5.5) we cannot get any contradiction. Hence, (5.4) demonstrate the relation between f and $f^{(k)}$. □

EXAMPLE 6. Let us take a set

$$S = \left\{i, \sqrt{i}, 1, -i\sqrt{i}, -i, -\sqrt{i}, -1, i\sqrt{i}\right\}$$

and $f(z) = e^{\left(\frac{1+i}{\sqrt{2}}\right)^{\frac{1}{k}} z}$, k is non negative integer. It is easy to say f and $f^{(k)}$ share (S,∞), but $f \not\equiv f^{(k)}$. Now, if we put the expressions of the function f and $f^{(k)}$ in (6.1) for $n = 8$, $d = 1$, by a simple calculation we get

$$\begin{aligned} &336\{f^8(f^{(k)})^3 - f^3(f^{(k)})^8\} - 840b\{f^8(f^{(k)})^2 - f^2(f^{(k)})^8\} \\ &+720b^2\{f^8(f^{(k)}) - f(f^{(k)})^8\} - 210b^3\{f^8 - (f^{(k)})^8\} = 0 \\ &\text{i.e.,}\quad 14i(1+\sqrt{i})e^{2zi^{\frac{1}{2k}}} - 35b\sqrt{i}(1+i)e^{zi^{\frac{1}{2k}}} + 30b^2(1+i\sqrt{i}) = 0, \end{aligned}$$

which is not possible. Hence, if we consider any arbitrary set consisting of 8 elements other than S_1 defined *Theorem* 1, (6.1) will not be satisfied by above defined function f.

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