Journal of
Classical
Analysis
Volume 23, Number 2 (2024), 79–94
doi:10.7153/jca-2024-23-07

UNICITY RELATION TO ENTIRE FUNCTIONS AND THEIR DIFFERENTIAL DIFFERENCE POLYNOMIALS

HARINA P. WAGHAMORE* AND B. E. MANJUNATH

Abstract. In this article, we investigate the problem of sharing values between entire functions $f(z)$ and $f_1(z) = b_{-1} + \sum_{i=0}^{n} b_i f^{(k_i)}(z+i\eta)$ share two distinct values a with counted and b with ignoring multiplicities, where b_{-1} and b_i $(i = 0,1\cdots,n)$ are small meromorphic functions of $f(z)$, $k_i \geqslant 0$ $(i = 0,1,\cdots,n)$ are integers. In relation to previous research, we obtain results that improve and generalise the findings conducted by Yang and Qi [CMFT, 20.1 (2020): 159–178].

1. Background information & main result

In this paper, we assume that the reader is familiar with the basic terminology and notations of the Nevanlinna Theory [10]. Meromorphic functions are analytic in the complex plane except at isolated poles; if there are no poles, $f(z)$ reduces to an entire function. We denote any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$, outside of an exceptional set of finite linear or logarithmic measure.

The order of f is indicated by

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}.$$

Let a be a complex number, we say that two meromorphic functions $f_1(z)$ and $f_2(z)$ share the value a CM(IM) if $f_1(z) - a$ and $f_2(z) - a$ have same zeros with counting multiplicities (ignoring multiplicities).

In 1929, Nevanlinna [19] proved the following celebrated five-value theorem, which stated that two nonconstant meromorphic functions must be identically equal if they share five distinct values in the extended complex plane.

Throught this paper we use $N_{f(z)}(r,\infty)$ to represent counting function of poles of $f(z)$ and $N_{f(z)}(r,0)$ to denote the counting function of zeros of $f(z)$.

We begin our discussion recalling the following famous result of Rubel and Yang [18].

THEOREM 1. *Let $f(z)$ be a nonconstant entire function of finite order, let a, b be two finite distinct complex values. If $f(z)$ and $f'(z)$ share a, b CM, then $f(z) \equiv f'(z)$.*

Mathematics subject classification (2020): 30D35, 39A32.
Keywords and phrases: Difference operator, uniqueness, entire function, weighted sharing.
* Corresponding author.

Li and Yang [15] considered 1 CM and 1 IM instead of 2 CM in Theorem 1 and proved the following

THEOREM 2. *Let $f(z)$ be a nonconstant entire function of finite order, let a, b be two finite distinct complex values. If $f(z)$ and $f^{(k)}(z)$ share a CM and b IM. Then $f(z) \equiv f^{(k)}(z)$.*

Let η be a non-zero complex constant. For a meromorphic function f, we denote its shift and difference operators by $f(z+\eta)$ and $\Delta_\eta = f(z+\eta) - f(z)$ respectively. Next we define

$$\Delta_\eta^n f = \Delta_\eta^{n-1}(\Delta_\eta f),\ \forall\, n \in \mathbb{N} - \{1\} \text{ and } \Delta_\eta^n f = \Delta_\eta f \text{ for } n = 1.$$

The difference analouge of the lemma on the logarithmic derivative and Nevanlinna theory for the difference operator have been founded. In this regard there has been many papers (see [7, 8, 11, 16, 20]).

Heittokangas et al. [11] proved a similar result analogue of Theorem 1 concerning shift.

THEOREM 3. *Let $f(z)$ be a non constant entire function of finite order, let η be any nonzero finite complex value, let a,b be two finite distinct complex values. If $f(z)$ and $f(z+\eta)$ share a,b CM then $f(z) \equiv f(z+\eta)$.*

Concerning the uniqueness of $f(z)$ and $\Delta_\eta f$ while sharing a and b Counting multiplicities in 2013, Chen and Yi [6] proved

THEOREM 4. *Let $f(z)$ be a transcendental entire function of finite non integer order, let η be a nonzero complex number and let a and b be two distinct complex values. If $f(z)$ and $\Delta_\eta f(z)$ share a,b CM then $f(z) \equiv \Delta_\eta f(z)$.*

They conjectured that the condition "non-integer" of Theorem 4 can be removed. Zhang and Liao [20] and Liu et al. [13] confirmed the conjecture. They proved

THEOREM 5. *Let $f(z)$ be a transcendental entire function of finite order, let η be a nonzero complex number, n be a positive integer, and let a and b be two distinct complex values. If $f(z)$ and $\Delta_\eta^n f(z)$ share a,b CM then $f(z) \equiv \Delta_\eta^n f(z)$.*

Li et al. [15] proved

THEOREM 6. *Let $f(z)$ be a transcendental entire function of finite order, let η be a nonzero complex number, n be a positive integer, and let a complex number. If $f(z)$ and $\Delta_\eta^n f(z)$ share 0 CM and share a IM, then $f(z) \equiv \Delta_\eta^n f(z)$.*

The authors posed a question:

QUESTION 1. Let $f(z)$ be a transcendental entire function of finite order, let $\eta(\neq 0) \in \mathbb{C}$, n be a positive integer and let a, b be two finite distinct complex values. If $f(z)$ and $\Delta_\eta^n f(z)$ share a CM and share b IM, is $f(z) \equiv \Delta_\eta^n f(z)$?

Recently, Liu and Dong [14] first studied the complex differential difference equation $f'(z) = f(z+\eta)$, where $\eta \neq 0$ is a finite constant.

Qi et al. [17] investigated the value sharing problem related to $f'(z)$ and $f(z+\eta)$ and proved

THEOREM 7. *Let $f(z)$ be a nonconstant entire function of finite order, and let a, η be two nonzero finite complex values. If $f'(z)$ and $f(z+\eta)$ share $0, a$ CM then $f'(z) = f(z+\eta)$.*

Recently, Yang and Qi [16] improved Theorem 7 and proved the following result.

THEOREM 8. *Let $f(z)$ be a nonconstant entire function of finite order, and let a, η be two nonzero finite complex values. If $f'(z)$ and $f(z+\eta)$ share 0 CM and a IM then $f'(z) = f(z+\eta)$.*

Sharing value problems are studied, and uniqueness between entire or meromorphic functions are studied in [3], [5], [4], and a question regarding the precise form of the solutions of some difference equations has been posed in [3]. Further, a conjecture of Chen and Yi was studied in [1] for both entire and meromorphic functions when sharing two values, a, b CM. By examples, it has been shown in [1] that 2CM sharing cannot be reduced to 1CM+1IM or 2IM sharing. Recently, in [2], a result on the conjecture is established in [2] answering completely the question posed in [3] finding the solutions of that difference equations completely. A result in [1] for meromorphic functions is improved in [2] by removing a condition.

It is natural to ask the following question:

QUESTION 2. Let $f(z)$ be a transcendental entire function and

$$f_1(z) = b_{-1} + \sum_{i=0}^{n} b_i f^{(k_i)}(z+i\eta), \tag{1}$$

where b_{-1} and b_i $(i = 0, 1\cdots, n)$ are small meromorphic functions of f, $k_i \geqslant 0$ $(i = 0, 1, \cdots, n)$ are integers. Let a, b be two distinct finite complex values. If $f(z)$ and $f_1(z)$ share a CM and b IM, is $f(z) \equiv f_1(z)$?

The problem of sharing values between entire functions is a complex one that has been studied extensively by mathematicians. In this particular article, the focus is on sharing values between two entire functions $f(z)$ and $f_1(z)$.

We give a positive answer to the above question and obtain the following result.

THEOREM 1. (Main) *Let $f(z)$ be a transcendental entire function and $f_1(z)$ be defined as in equation (1). Let a, b be two distinct finite complex values. If $f(z)$ and $f_1(z)$ share a with counted multiplicity and share b with ignoring multiplicity, then either $f(z) \equiv f_1(z)$ or $a = 2b = 2$, $f(z) = e^{2p(z)} - 2(e^p(z) - 1)$ and $f_1(z) = e^{p(z)}$, where $p(z)$ is a non-constant polynomial.*

2. Auxiliary lemmas

We present here some necessary lemmas which will play a key role to prove the main result of the paper.

LEMMA 1. [7] *Let* $f(z)$ *be a transcendental meromorphic function of finite order,* $c\in\mathbb{C}\setminus\{0\}$ *be fixed. Then* $T(r,\ f(z+c)) = T(r,\ f(z)) + S(r,f)$.

LEMMA 2. [7] *Let* $f(z)$ *be a meromorphic function of finite order* σ*, and let* $c\in\mathbb{C}\setminus\{0\}$ *be fixed. Then for each* $\varepsilon>0$*, we have*

$$m\left(r,\frac{f(z+c)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+c)}\right)=O(r^{\sigma-1+\varepsilon})=S(r,f).$$

The following lemma has a few modifications to the original version [7, Corollary 2.5].

LEMMA 3. [19] *Suppose* f *and* g *are two nonconstant meromorphic functions in the complex plane, then*

$$N_{fg}(r,\infty)-N_{fg}(r,0)=N_f(r,\infty)+N_g(r,\infty)-N_f(r,0)-N_g(r,0).$$

LEMMA 4. [19] *Let* $f(z)$ *be a nonconstant meromorphic function, and let* $\mathscr{P}(z)=a_0f^p+a_1f^{p-1}+\cdots+a_p$ $(a_0\neq 0)$ *be a ploynomial of degree* p *with constant coefficients* a_j $(j=0,1,\cdots,p)$*. Suppose that* b_j $(j=0,1,\cdots,q)$ $(q>p)$*. Then*

$$m\left(\frac{\mathscr{P}(f)f'}{(f-b_1)(f-b_2)\cdots(f-b_q)}\right)=S(r,f).$$

LEMMA 5. [19] *Suppose that* $f(z)$ *is meromorphic function in the complex plane and* $P(f)=a_0f^n+a_1f^{n-2}+\cdots+a_n$*, where* $a_0(\equiv 0)$*,* $a_1,a_2,\cdots a_n$ *are small functions of* $f(z)$*. Then*

$$T(r,P(f))=nT(r,f)+S(r,f).$$

LEMMA 6. *Let* $f(z)$ *be a transcendental entire function and* $f_1(z)$ *be defined as in equation* (1)*. Let* a *and* b *be two distinct finite complex value. If* $f(z)$ *and* $f_1(z)$ *share* a *CM and* $N_{f_1-\mathscr{G}}(r,0)=S(r,f)$ *then there is a polynomial* p *such that either* $T(r,e^p)=S(r,f)$ *or* $\mathscr{Q}=\mathscr{H}e^p+\mathscr{G}$*, where* $\mathscr{G}=b_{-1}+\sum_{i=0}^n b_i a_i^{(k_i)}(z+i\eta)$.

Proof. Since $f(z)$ is a transcendental entire function and $f(z)$ and $f_1(z)$ share a CM, then there exist a polynomial p such that

$$f(z)-a=\mathscr{A}e^p(f_1-\mathscr{G})\mathscr{A}e^p(\mathscr{G}-a_1), \tag{2}$$

where the zeros and poles of $\mathscr{A}$ come from the zeros and poles of b_{-1} and b_i $(i=0,1,\cdots,n)$ and $\mathscr{G}=b_{-1}+\sum_{i=0}^n b_i a_1^{(k_i)}(z+\eta i)$. Suppose $T(r,e^p)\neq o(T(r,f))$, set $\mathscr{Q}=f_1-\mathscr{G}$. From (2), we get

$$\mathscr{Q}=\sum_{i=0}^n b_i(\mathscr{A}_{i\eta}e^{p_{i\eta}}\mathscr{Q}_{i\eta})^{(k_i)}+\sum_{i=0}^n b_i(\mathscr{A}_{i\eta}e^{p_{i\eta}}(\mathscr{G}-a_1)_{i\eta})^{(k_i)}+\mathscr{G}. \tag{3}$$

Then, we rewrite (3) as

$$1-\frac{\sum_{i=0}^{n} b_i\left(\mathscr{A}_{i\eta}e^{p_{i\eta}}(\mathscr{G}-a_1)_{i\eta}\right)^{(k_i)}+\mathscr{G}}{\mathscr{Q}}=\mathscr{D}e^{p}, \tag{4}$$

where

$$\mathscr{D}=\frac{\sum_{i=0}^{n} b_i\left(\mathscr{A}_{i\eta}e^{p_{i\eta}}\mathscr{Q}_{i\eta}\right)^{(k_i)}}{\mathscr{Q}e^{p}}. \tag{5}$$

Note that $N_{f_1-\mathscr{G}}(r,0)=N_{\mathscr{Q}}(r,0)=S(r,f)$. Then

$$\begin{aligned}T(r,\mathscr{D}) &\leqslant \sum_{i=0}^{n} T\left(r,\frac{\sum_{i=0}^{n} b_i\left(\mathscr{A}_{i\eta}e^{p_{i\eta}}\mathscr{Q}_{i\eta}\right)^{(k_i)}}{\mathscr{Q}e^{p}}\right)+S(r,f),\\ &\leqslant \sum_{i=0}^{n} m\left(r,\frac{\sum_{i=0}^{n} b_i\left(\mathscr{A}_{i\eta}e^{p_{i\eta}}\mathscr{Q}_{i\eta}\right)^{(k_i)}}{\mathscr{Q}e^{p}}\right)\\ &\quad+\sum_{i=0}^{n} N_{\left(\frac{\sum_{i=0}^{n} b_i\left(\mathscr{A}_{i\eta}e^{p_{i\eta}}\mathscr{Q}_{i\eta}\right)^{(k_i)}}{\mathscr{Q}e^{p}}\right)}(r,\infty)+S(r,f)\\ &\leqslant S(r,e^{p})+S(r,f). \end{aligned} \tag{6}$$

Next we discuss two cases.

Case 1. $e^{-p}-\mathscr{D}\not\equiv 0$. Rewrite (4) as

$$\mathscr{Q}e^{p}\left(e^{-p}-\mathscr{D}\right)=\sum_{i=0}^{n} b_i\left(\mathscr{A}_{i\eta}e^{p_{i\eta}}(\mathscr{G}-a_1)_{i\eta}\right)^{(k_i)}+\mathscr{G}, \tag{7}$$

when $\mathscr{D}\equiv 0$, (7) implies

$$\begin{aligned}\mathscr{Q}&=\sum_{i=0}^{n} b_i\left(\mathscr{A}_{i\eta}e^{p_{i\eta}}(\mathscr{G}-a_1)_{i\eta}\right)^{(k_i)}+\mathscr{G}\\ &=\mathscr{H}e^{p}+\mathscr{G},\end{aligned} \tag{8}$$

where $\mathscr{H}\not\equiv 0$ is a small function of e^{p}. When $\mathscr{D}\not\equiv 0$, it follows from (7) that $N_{(e^{-p}-\mathscr{D})}(r,0)=S(r,f)$. Then using the Second fundamental theorem to e^{p}, we can obtain

$$\begin{aligned}T(r,e^{p})&=T(r,e^{-p})+O(1)\\ &\leqslant \overline{N}_{(e^{-p})}(r,\infty)+\overline{N}_{(e^{-p})}(r,0)+\overline{N}_{(e^{-p}-\mathscr{D})}(r,0)+O(1)=S(r,f).\end{aligned} \tag{9}$$

Case 2. $e^{-p}-\mathscr{D}\equiv 0$. It implies that,

$$T(r,e^{p})=T(r,e^{-p})+O(1)=S(r,f). \qquad \square$$

LEMMA 7. *Let $f(z)$ be a transcendental entire function and $f_1(z)$ be defined as in (1). Let a,b be distinct small function. Suppose $\mathscr{W}(a-b, f-a) = \mathscr{L}(f)$, $\mathscr{W}(a-b, f_1-a) = \mathscr{L}(f_1)$ where $\mathscr{W}(z_1,z_2)$ is Wronskian of z_1 and z_2 and $f(z)$ and $f_1(z)$ share a CM and b IM then $\mathscr{L}(f) \not\equiv 0$ and $\mathscr{L}(f_1) \not\equiv 0$.*

Proof. Suppose that $\mathscr{L}(f)) = \begin{vmatrix} a-b & f-a \\ a'-b' & f'-a' \end{vmatrix} \equiv 0$, then we get

$$(a-b)(f'-a') = (f-a)(a'-b')$$

$$\implies \frac{f'-a'}{f-a} = \frac{a'-b'}{a-b}.$$

Integrating both side, we get

$$f - a = C(a-b),$$

where C is a non zero constant. So we have $T(r,f) = o(T(r,f))$ a contradiction.

Hence $\mathscr{L}(f) \not\equiv 0$. Since $f_1(z)$ and f share a CM and b IM, then by Second fundamental theorem of Nevanlinna, we get

$$\begin{aligned} T(r,f) &\leqslant \overline{N}_{(f-a)}(r,0) + \overline{N}_{(f-b)}(r,0) + o(T(r,f)) \\ &\leqslant \overline{N}_{(f_1-a)}(r,0) + \overline{N}_{(f_1-b)}(r,0) + o(T(r,f)) \\ &\leqslant\leqslant 2T(r,f_1) + O(T(r,f)). \end{aligned} \tag{10}$$

Hence a and b are small function of f_1. If $\mathscr{L}(f_1) \equiv 0$, then we can get $f_1 - a = C_1(a-b)$, where C_2 is a non zero constant. And we get $T(r,f_1) = o(T(r,f))$.

Therefore $T(r,f) = o(T(r,f))$, which is a contradiction. Hence $\mathscr{L}(f_1) \not\equiv 0$. □

LEMMA 8. *Let $f(z)$ be a transcendental meromorphic function and $f_1(z)$ be defined as in (1). Let a and b be two finite distinct complex number. Then $m\left(r, \frac{\mathscr{L}(f)f(z)}{f-a}\right) = S(r,f) = m\left(r, \frac{\mathscr{L}(f)f(z)}{f-b}\right)$. And*

$$m\left(r, \frac{\mathscr{L}(f)f(z)}{(f-z_1)(f-z_2)\cdots(f-z_m)}\right) = o(T(r,f)),$$

where $\mathscr{L}(f)$ is defined as in Lemma 7 and $2 \leqslant m \leqslant q$ where $d_j = a - l_j(a-b)$ $(j = 1,2,\cdots,q)$.

Proof. Obviously, we have

$$\begin{aligned} m\left(r, \frac{\mathscr{L}(f)f(z)}{f-a}\right) &\leqslant m\left(r, -\frac{(a'-b')(f-a)}{f-a}\right) + m\left(r, \frac{(af\prime - a')(a-b)}{f-a}\right) \\ &= o(T(r,f)) \end{aligned}$$

and

$$\frac{\mathscr{L}(f)f(z)}{(f-z_1)(f-z_2)\cdots(f-z_q)} = \sum_{i=1}^{q}\frac{\mathscr{C}_i\mathscr{L}(f)}{f-d_i},$$

where $\mathscr{C}_i$ $(i=1,2,\cdots,q)$ are small function of f. By Lemma 2, we have

$$m\left(r, \frac{\mathscr{L}(f)f(z)}{(f-z_1)(f-z_2)\cdots(f-z_q)}\right) = m\left(r, \sum_{i=1}^{q}\frac{\mathscr{C}_i\mathscr{L}(f)}{f-d_i}\right)$$
$$\leqslant \sum_{i=1}^{q} m\left(r, \frac{\mathscr{L}(f)}{f-d_i},\right) = o(T(r,f)). \quad \square$$

LEMMA 9. *Let f be non constant entire function and $f_1(z)$ be defined as in (1). Let a and b be two distinct small function. If*

$$\mathbb{H} = \frac{\mathscr{L}(f)}{(f-a)(f-b)} - \frac{\mathscr{L}(f_1)}{(f_1-a)(f_1-b)} \equiv 0,$$

and $f(z)$ and $f_1(z)$ share a CM and b IM, then either $f \equiv f_1$ or

$$2T(r,f) \leqslant \overline{N}_{(f-a)}(r,0) + \overline{N}_{(f-b)}(r,0) + S(r,f).$$

Proof. Integrating $\mathbb{H}$ which leads to

$$\frac{f_1-b}{f_1-a} = C\frac{f-b}{f-a}$$

C is a non zero constant. If $C=1$, then $f \equiv f_1$.

If $C \neq 1$, then from above we have

$$\frac{a-b}{f_1-a} \equiv \frac{(C-1)f-Cb+a}{f-a} \quad \text{and} \quad T(r,f) = T(r,f_1) + o(T(r,f)).$$

Obviously, $\frac{Cb-a}{C-1} \neq a$ and $\frac{Cb-a}{C-1} \neq b$. It follows that

$$N_{\left(f-\frac{Cb-a}{C-1}\right)}(r,0) = N_{(a-b)}(r,0) = o(T(r,f)).$$

Then by the Second fundamental theorem

$$2T(r,f) \leqslant \overline{N}_f(r,\infty) + \overline{N}_{(f-a)}(r,0) + \overline{N}_{(f-b)}(r,0) + \overline{N}_{\left(f-\frac{Cb-a}{C-1}\right)}(r,0),$$
$$\leqslant \overline{N}_{(f-a)}(r,0) + \overline{N}_{(f-b)}(r,0) + S(r,f). \quad \square$$

LEMMA 10. [19] *Let f be a non constant meromorphic function and $\mathscr{R}(f) = \frac{P(f)}{Q(f)}$ where $P(f) = \sum_{k=0}^{p} a_k f^k$ and $Q(f) = \sum_{j=0}^{q} a_j f^q$ are two mutually prime polynomials in f. If the coefficient a_k and b_j are small functions of f and $a_p \not\equiv 0$, $b_q \not\equiv 0$, then*

$$T(r,\mathscr{R}(f)) = \max\{p,q\}T(r,f) + S(r,f).$$

LEMMA 11. [19] *Suppose* $f_2, f_3, \cdots, f_n$ $(n \neq 3)$ *are meromorphic functions and* $f_2^*, f_3^*, \cdots, f_n^*$ $(n \neq 3)$ *are entire functions such that*

1. $\sum_{j=2}^{n} f_j e^{f_j^*} = 0,$
2. $f_j^* - f_k^*$ *are not constants for* $2 \leqslant j < k \leqslant n$,
3. *For* $2 \leqslant j \leqslant n$ *and* $2 \leqslant h < k \leqslant n$, $T(r, f_j) = S(r, e^{f_j^* - f_k^*})\{r \to \infty, r \notin E\}$.

   Then $f_j \equiv 0$ *for all* $1 \leqslant j \leqslant n$.

3. Proof of the theorem

If $f(z) \equiv f_1(z)$, where $f_1(z) = b_{-1} f^{(k_i)}(z + \eta i)$, then there is nothing to prove. Suppose $f(z) \not\equiv f_1(z)$, since $f(z)$ and $f_1(z)$ share a CM, then we get

$$\frac{f_1(z) - a}{f - a} = Be^{v_1}, \tag{11}$$

where v_1 is entire function and (11) implies $v_1 = -p$, $B = \frac{1}{\mathscr{A}}$.

Since $f(z)$ and $f_1(z)$ share a CM and b IM, then by Second fundamental theorem of Nevanlinna and Lemma 2, we have

$$\begin{aligned} T(r,f) &\leqslant \overline{N}_{(f-a)}(r,0) + \overline{N}_{(f-b)}(r,0) + S(r,f) \\ &= \overline{N}_{(f_1-a)}(r,0) + \overline{N}_{(f_1-b)}(r,0) + S(r,f) \\ &\leqslant N_{(f-f_1)}(r,0) + S(r,f) \leqslant T(r, f - f_1) + S(r,f) \\ &\leqslant m(r,f) + m\left(r, 1 - \frac{\sum_{i=0}^{n} b_i f^{(k_i)}(z + \eta_i)}{f}\right) + S(r,f) \\ &\leqslant T(r,f) + S(r,f). \end{aligned}$$

That is

$$T(r,f) = \overline{N}_{(f-a)}(r,0) + \overline{N}_{(f-b)}(r,0) + S(r,f). \tag{12}$$

According to Lemma 2, (11) and (12)

$$T(r,f) = T(r, f - f_1) + S(r,f) = N_{(f-f_1)}(r,0) + S(r,f) \tag{13}$$

$$T(r, Be^{v_1}) = m(r, Be^{v_1}) \leqslant m\left(r, \frac{1}{f-a}\right) + S(r,f). \tag{14}$$

Then it follows from (11) and (13) that

$$\begin{aligned} m\left(r, \frac{1}{f-a}\right) &= m\left(r, \frac{Be^{v_1} - 1}{f - f_1}\right) \\ &\leqslant m\left(r, \frac{1}{f - f_1}\right) + m\left(r, Be^{v_1} - 1\right) \\ &\leqslant T(r, e^{v_1}) + S(r,f). \end{aligned} \tag{15}$$

Then by (14) and (15),

$$T(r, e^{\nu_1}) = m\left(r, \frac{1}{f-a}\right) + S(r, f). \tag{16}$$

We rewrite (11), as

$$\frac{f_1 - f}{f-a} = Be^{\nu_1} - 1 \tag{17}$$

and it follows that,

$$\overline{N}_{(f-b)}(r,0) \leqslant \overline{N}_{(Be^{\nu_1}-1)}(r,0) = T(r,e^{\nu_1}) + S(r,f). \tag{18}$$

Thus by (12), (16) and (18)

$$\begin{aligned} m\left(r, \frac{1}{f-a}\right) + N_{(f-a)}(r,0) &= \overline{N}_{(f-a)}(r,0) + \overline{N}_{(f-b)}(r,0) + S(r,f) \\ &\leqslant \overline{N}_{(f-a)}(r,0) + \overline{N}_{(Be^{\nu_1}-1)}(r,0) + S(r,f) \end{aligned}$$

i.e.,

$$\overline{N}_{(f-a)}(r,0) = \overline{N}_{(f-b)}(r,0) + S(r,f). \tag{19}$$

And then, we have

$$\overline{N}_{(f-b)}(r,0) = T(r,e^{\nu_1}) + S(r,f). \tag{20}$$

Set

$$\Phi = \frac{\mathscr{L}(f)(f-f_1)}{(f-a)(f-b)} \tag{21}$$

$$\Psi = \frac{\mathscr{L}(f_1)(f-f_1)}{(f-_1 a)(f_1-b)}. \tag{22}$$

It is easy to see that $\Phi \not\equiv 0$ because of $f \not\equiv f_1$ and Φ is an entire function. By Lemma 2 and Lemma 8, we have

$$\begin{aligned} T(r,\Phi) = m(r,\Phi) &= m\left(r, \frac{f'(f-f_1)}{(f-a)(f-b)}\right) + S(r,f) \\ &\leqslant m\left(r, \frac{\mathscr{L}(f)f}{(f-a)(f-b)}\right) + m\left(r, 1 - \frac{f_1}{f}\right) = S(r,f) \end{aligned}$$

$$\text{i.e., } T(r,\Phi) = S(r,f). \tag{23}$$

Let $s = a - j(a-b)$, $(j \neq 0,1)$. Obviously by Lemma 2 and the first fundamental theorem of Nevanlinna, we obtain

$$m(r,1/f) = m\left(r, \frac{\mathscr{L}(f)f}{(f-a)(f-b)} \frac{f - \sum_{i=0}^{n} b_i f^{(k_i)}(z)}{f}\right) \tag{24}$$

and

$$m\left(r,\ \frac{1}{f-d}\right)=m\left(r,\ \frac{\mathscr{L}(f)(f-f_1)}{\Phi(f-a)(f-b)(f-d)}\right)$$
$$\leqslant m\left(r,\ \frac{\mathscr{L}(f)f}{(f-a)(f-b)(f-d)}\right)+m\left(r,\ 1-\frac{f_1}{f}\right)+S(r,f)=S(r,f). \tag{25}$$

Set

$$\phi=\frac{\mathscr{L}(f_1)}{(f_1-a)(f_1-b)}-\frac{\mathscr{L}(f)}{(f-a)(f-b)} \tag{26}$$

we discuss two cases.

Case 1. Suppose that $\phi\equiv 0$. By (26), we have

$$\frac{f-b}{f-a}=c\frac{f_1-b}{f_1-a} \tag{27}$$

where c is a non zero constant, then by Lemma 9

$$2T(r,f)\leqslant \overline{N}_{(f-a)}(r,0)+\overline{N}_{(f-b)}(r,0)+S(r,f) \tag{28}$$

which contradiction with (12).

Case 2. $\phi\not\equiv 0$. By (13) and (23), (26), we can obtain

$$\begin{aligned}m(r,f)&=m(r,f-f_1)+S(r,f)\\&\leqslant m\left(r,\ \frac{\Psi-\Phi}{\phi}\right)+S(r,f)\leqslant T\left(r,\ \frac{\phi}{\Psi-\Phi}\right)+S(r,f)\\&\leqslant T(r,\Psi-\Phi)+T(r,\phi)+S(r,f)\\&\leqslant T(r,\ \Psi)+\overline{N}_{(f-b)}(r,\ 0)+S(r,f).\end{aligned} \tag{29}$$

On the otherhand,

$$T(r,\ \Phi)=T\left(r,\ \frac{\mathscr{L}(f_1)(f-f_1)}{(f_1-a)(f_1-b)}\right)=m\left(r,\ \frac{\mathscr{L}(f_1)(f-f_1)}{(f_1-a)(f_1-b)}\right)+S(r,f)$$
$$\leqslant m\left(r,\ \frac{1}{f-a}\right)+S(r,f)=\overline{N}_{(f-b)}(r,\ 0)+S(r,f). \tag{30}$$

Combining (29) and (30), we obtain

$$T(r,\ f)\leqslant 2\overline{N}_{(f-b)}(r,\ 0)+S(r,f). \tag{31}$$

Next, case 2 is divided into two subcases.

Subcase 1. Let $a=\mathscr{G}$, where $\mathscr{G}$ is defined as $\mathscr{G}=b_{-1}+b_0a_1$. Then by (11) and Lemma 2, we can get

$$m(r,\ e^{v_1})=m\left(r,\ \frac{f_1-\mathscr{G}}{f-a}\right)=S(r,\ f). \tag{32}$$

Then by (20), (31) and (32) we can have $T(r,f)=S(r,f)$, a contradiction.

Subcase 2. Let $b=\mathscr{G}$. Then by (16), (20) and (31), we get

$$\begin{aligned} T(r,\ f) &\leqslant m\left(r,\ \frac{1}{f-a}\right)+\overline{N}_{(f_1-\mathscr{G})}(r,\ 0)+S(r,f) \\ &\leqslant m\left(r,\ \frac{1}{f_1-\mathscr{G}}\right)+\overline{N}_{(f_1-\mathscr{G})}(r,\ 0)+S(r,f) \\ &\leqslant T(r,\ f_1)+S(r,f). \end{aligned} \tag{33}$$

From the fact that

$$T(r,f_1)\leqslant T(r,f)+S(r,f). \tag{34}$$

From (33), we

$$T(r,f)=T(r,f_1)+S(r,f). \tag{35}$$

By the Second fundamental theorem of Nevanlinna, Lemma 2, (12), (35), we get

$$\begin{aligned} 2T(r,\ f) &\leqslant 2T(r,\ f_1)+S(r,f) \\ &\leqslant \overline{N}_{f_1}(r,\ \infty)+\overline{N}_{(f_1-\mathscr{G})}(r,\ 0)+\overline{N}_{(f_1-a)}(r,\ 0)+\overline{N}_{(f_1-s)}(r,\ 0)+S(r,f) \\ &\leqslant T(r,\ f)+T(r,\ f_1)-m\left(r,\ \frac{1}{f_1-s}\right)+S(r,\ f) \\ &\leqslant 2T(r,f)-m\left(r,\ \frac{1}{f_1-s}\right)+S(r,f). \end{aligned}$$

Thus

$$m\left(r,\ \frac{1}{f_1-s}\right)=S(r,f). \tag{36}$$

From the first fundamental theorem of Nevanlinna, Lemma 2, Equations (24), (25), (35), (36) and that f is transcendental entire function, we obtain

$$\begin{aligned} m\left(r,\frac{f-s}{f_1-s}\right) &\leqslant m\left(r,\frac{f}{f_1-s}\right)+m\left(r,\frac{s}{f_1-s}\right)+S(r,f) \\ &\leqslant T\left(r,\frac{f}{f_1-s}\right)-N_{\left(\frac{f}{f_1-s}\right)}(r,0)+S(r,f) \\ &\leqslant N_f(r,0)-N_{(f_1-s)}(r,0)+S(r,f) \\ &\leqslant T(r,f)-T(r,f_1)+S(r,f)=S(r,f). \end{aligned}$$

Thus we get,

$$m\left(r,\frac{f-s}{f_1-s}\right)=S(r,f). \tag{37}$$

It's easy to see that $N_\Psi(r,\infty)=S(r,f)$ and (22) can be rewritten as

$$\Psi=\left[\frac{a-s}{a}\frac{\mathscr{L}(f_1)}{f_1-b}-\frac{s}{a}\frac{\mathscr{L}(f_1)}{f_1-a}\right]\left[\frac{f-s}{f_1-s}-1\right]. \tag{38}$$

Then by (37) and (38) we can get

$$T(r,\Psi) = m(r,\Psi) + N_{\Psi}(r,\infty) = S(r,f). \tag{39}$$

By (12), (29) and (39), we get

$$\overline{N}_{(f-a)}(r,0) = S(r,f). \tag{40}$$

Moreover by (12), (35) and (40), we have

$$m\left(r, \frac{1}{f_1 - \mathscr{G}}\right) = S(r,f) \tag{41}$$

which implies

$$\overline{N}_{(f-b)}(r,0) = m\left(r, \frac{1}{f-b}\right) \leqslant m\left(r, \frac{1}{f_1 - \mathscr{G}}\right) = S(r,f) \tag{42}$$

then by (12) we obtain $T(r,f) = S(r,f)$ a contradiction.

Subcase 3. Suppose that $a \not\equiv \mathscr{G}$ and $b \not\equiv \mathscr{G}$. So by (16), (20), (31) and Second fundamental theorem of Nevanlinna, we can get

$$\begin{aligned}
T(r,f) &\leqslant 2m\left(r, \frac{1}{f-a}\right) + S(r,f) \leqslant 2m\left(r, \frac{1}{f_1 - \mathscr{G}}\right) + S(r,f) \\
&\leqslant 2\left[T(r,f_1) - N_{(f_1-\mathscr{G})}(r,0)\right] + S(r,f) \\
&\leqslant \overline{N}_{(f_1-a)}(r,0) + \overline{N}_{(f_1-b)}(r,0) + \overline{N}_{(f_1-\mathscr{G})}(r,0) - 2N_{(f_1-\mathscr{G})} + S(r,f) \\
&\leqslant T(r,f) - N_{(f_1-\mathscr{G})}(r,0) + S(r,f)
\end{aligned}$$

$$\Longrightarrow N_{(f_1-\mathscr{G})}(r,0) = S(r,f). \tag{43}$$

It follows from (43) and Second fundamental theorem that

$$\begin{aligned}
T(r,f_1) &\leqslant \overline{N}_{(f_1-\mathscr{G})}(r,0) + \overline{N}_{(f_1-a)}(r,0) + S(r,f) \\
&\leqslant T(r,f_1) + S(r,f)
\end{aligned}$$

$$\Longrightarrow T(r,f_1) = \overline{N}_{(f_1-a)}(r,0) + S(r,f). \tag{44}$$

Similarly,

$$T(r,f_1) = \overline{N}_{(f_1-b)}(r,0) + S(r,f). \tag{45}$$

Then by (12), (44), (45) and the fact that f and f_1 share a CM and b IM, we get

$$T(r,f) = 2T(r,f_1) + S(r,f). \tag{46}$$

Easy to see from (26) that

$$T(r,\Phi) = N_{\Phi}(r,\infty) + S(r,f) \leqslant \overline{N}_{(f_1-b)}(r,0) + S(r,f). \tag{47}$$

We claim that

$$T(r,\Phi)=\overline{N}_{(f_1-b)}(r,0)+S(r,f). \tag{48}$$

Otherwise,

$$T(r,\Phi)<\overline{N}_{(f_1-b)}(r,0)+S(r,f). \tag{49}$$

We can deduce from (12), (22) and Lemma 3 that

$$\begin{aligned} T(r,\Psi)&=T\left(r,\frac{\mathscr{L}(f_1)(f-f_1)}{(f_1-a)(f_1-b)}\right)=m\left(r,\frac{\mathscr{L}(f_1)(f-f_1)}{(f_1-a)(f_1-b)}\right)+S(r,f)\\ &\leqslant m\left(r,\frac{\mathscr{L}(f_1)}{f_1-a}\right)+m\left(r,\frac{f-b}{f_1-b}-1\right)+S(r,f)\\ &\leqslant m\left(r,\frac{f_1-b}{f-b}\right)+N_{\left(\frac{f_1-b}{f-b}\right)}(r,\infty)-N_{\left(\frac{f-b}{f_1-b}\right)}(r,\infty)+S(r,f)\\ &\leqslant \overline{N}_{(f-a)}(r,0)+S(r,f). \end{aligned} \tag{50}$$

Then combining (12), (49), (50) and the proof of (29), we obtain

$$\begin{aligned} \overline{N}_{(f-a)}(r,0)+\overline{N}_{(f-b)}(r,0)&=T(r,f)+S(r,f)\\ &\leqslant \overline{N}_{(f-a)}(r,0)+T(r,\Phi)+S(rf) \end{aligned}$$

that is

$$\overline{N}_{(f-b)}(r,0)\leqslant T(r,\Phi)+S(r,f) \tag{51}$$

a contradiction. Similarly, we can also obtain

$$T(r,\Psi)=\overline{N}_{(f_1-a)}(r,0)+S(r,f). \tag{52}$$

By Lemma 6, if $T(r,e^p)=S(r,f)$, then we can obtain $T(r,f)=S(r,f)$ from (20) and (31), a contradiction. Hence

$$f_1=\mathscr{H}e^p+\mathscr{G} \tag{53}$$

where $\mathscr{H}\not\equiv 0$ is a small function of e^p.

Rewrite (26) as

$$\Phi=\frac{\mathscr{L}(f_1)(f-a)(f-b)-\mathscr{L}(f)(f_1-a)(f_1-b)}{(f-a)(f-b)(f_1-a)(f_1-b)}. \tag{54}$$

Combining (2) with (48), we can set

$$\begin{aligned} \mathscr{P}&=\mathscr{L}(f_1)(f-a)(f-b)-\mathscr{L}(f)(f_1-a)(f_1-b)\\ &=\sum_{i=0}^{5}\alpha_i e^{ip} \end{aligned} \tag{55}$$

and

$$\begin{aligned} Q&=(f-a)(f-b)(f_1-a)(f_1-b)\\ &=\sum_{l=0}^{6}\beta_l e^{lp} \end{aligned} \tag{56}$$

where α_i and β_l are small functions of e^p and $\alpha_5 \not\equiv 0$, $\beta_6 \not\equiv 0$. If $\mathscr{P}$ and Q are two mutually prime polynomails in e^p, then by Lemma 5 we can get

$$T(r,\Phi) = 6T(r,e^p) + S(r,f).$$

It follows from (20), (48) and (54)–(56) that $T(r,f) = S(r,f)$ a contradiction.

If $\mathscr{P}$ and Q are not two mutually prime polynomials in e^p, it is easy to see that the degree of Q is large than $\mathscr{P}$.

According to (48), (54) and by simple computation, we obtain

$$\Phi = \frac{C}{f_1 - b} \tag{57}$$

where $C \not\equiv 0$ is a small function of e^p.

Put (57) into (26), we obtain

$$\frac{Cf_1 - \mathscr{L}(f_1) - Ca}{(f_1 - a)(f_1 - b)} = -\frac{\mathscr{L}(f)}{(f-a)(f-b)}. \tag{58}$$

We claim that $C\mathscr{H}e^p \equiv (a-b)(\mathscr{H}' + p'\mathscr{H})e^p - (a' - b')\mathscr{H}e^p$. Otherwise, combining (2), (53) and (58) and Lemma 5, we can get $T(r,e^p) = S(r,f)$. It follows from (20) and (31) that $T(r,f) = S(r,f)$, a contradiction.

Then substituting (53) into (26), we have

$$\Psi = \frac{(C\mathscr{H}e^p + F)(\mathscr{A}e^p - 1)}{(\mathscr{H}e^p + \mathscr{G} - b)} \tag{59}$$

where $F = (\mathscr{G}'' - a')(a-b) - (\mathscr{G} - a)(a' - b')$.

Put $\mathscr{R} = \mathscr{A}C\mathscr{H}e^{2p} + (\mathscr{A}F - C\mathscr{H})e^p - F$, $\mathscr{S} = \mathscr{H}e^p + \mathscr{G} - b$.

If $\mathscr{R}$ and $\mathscr{S}$ are two mutually prime polynomials in e^p, then by Lemma 10, we get $T(r,\Psi) = 2T(r,e^p) + S(r,f)$. Then by (19), (30), (31), we can get $T(r,f) = S(r,f)$. Therefore $\mathscr{R}$ and $\mathscr{S}$ are not two mutually prime polynomials in e^p.

(59) implies

$$\Psi = C\mathscr{A}e^p, \qquad \mathscr{H} \equiv -\mathscr{A}(\mathscr{G} - b). \tag{60}$$

It follows from (58), (59) that

$$N_{(C\mathscr{H}e^p + F)}(r,0) = S(r,f). \tag{61}$$

We claim that $F \equiv 0$. Otherwise, if $F \not\equiv 0$ then by (55), (56) and Second fundamental theorem of Nevanlinna, we get,

$$T(r,e^p) \leqslant \overline{N}_{(e^p)}(r,\infty) + \overline{N}_{(e^p)}(r,0) + \overline{N}_{\left(e^p + \frac{F}{C\mathscr{H}}\right)}(r,0) + S(r,f) = S(r,f). \tag{62}$$

(20) and (30) deduce that $T(r,f) = S(r,f)$, and hence a contradiction. Due to (53), (58) and (60), we get

$$\mathscr{H} \equiv b\mathscr{A}, \qquad \mathscr{G} \equiv 0 \tag{63}$$

and hence

$$f_1 \equiv b\mathscr{A} e^p, \tag{64}$$

$$f_1 - b = b(\mathscr{A} e^p - 1). \tag{65}$$

Furthermore, we can deduce from (64) and (11) that

$$f \equiv (\mathscr{A} e^p)^2 b - a\mathscr{A} e^p + a. \tag{66}$$

Since f and f_1 share b IM and by (45)–(46) and (66), we get

$$f - b \equiv b(\mathscr{A} e^p - 1)^2. \tag{67}$$

It follows from $F \equiv 0$, (66) and (67) that

$$a \equiv 2b. \tag{68}$$

By (68) and the fact that $C\mathscr{H} e^p \equiv (a-b)(\mathscr{H}' + p'\mathscr{H}) e^p - (a' - b\prime)\mathscr{H} e^p$, we get $C = \frac{\mathscr{A}'}{\mathscr{A}} + bp'$.

It follows from (11), (58), (67) and (68) that

$$\mathscr{A} = b = 1, \qquad C = p' \tag{69}$$

and therefore

$$b = 2 \tag{70}$$

$$f_1 = e^p \tag{71}$$

where $C \neq 0$ and a are two finite constants. Thus by (11) and (69), (71), we obtain

$$f(z) = e^{2p} - 2(e^p - 1). \tag{72}$$

If $m(r, e^p) = m\left(r, e^h\right) + O(1) = S(r, f)$.

Then by (20) and (31), we deduce $T(r, f) = S(r, f)$ and thus a contradiction.

REFERENCES

[1] M. B. AHAMED, *An Investigation on the Conjecture of Chen and Yi*, Results. Math. **74**, 122 (2019), https://doi.org/10.1007/s00025-019-1045-4.

[2] M. B. AHAMED, *The class of meromorphic functions sharing values with their difference polynomials*, Indian. J. Pure. Appl. Math. **54**, 1158–1169 (2023), https://doi.org/10.1007/s13226-022-00329-3.

[3] S. MALLICK, M. B. AHAMED, *On uniqueness of a meromorphic function and its higher difference operators sharing two sets*, Anal. Math. Phys. **12**, 78 (2022), https://doi.org/10.1007/s13324-022-00668-8.

[4] A. BANERJEE, M. B. AHAMED, *On some sufficient conditions for periodicity of meromorphic function under new shared sets*, Filomat, **33** (18) (2019), 6055–6072.

[5] A. BANERJEE, M. B. AHAMED, *Uniqueness of meromorphic function with its shift operator under the purview of two or three shared sets*, Mathematica Slovaca, **69** (3) (2019), 557–572.

[6] ZONG-XUAN CHEN, HONG-XUN YI, *On sharing values of meromorphic functions and their differences*, Results Math. **63** (2013), no. 1–2, 557–565.

[7] YIK-MAN CHIANG, SHAO-JI FENG, *On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane*, Ramanujan J. **16** (1) (2008), 105–129.

[8] R. G. HALBURD, R. J. KORHONEN, *Difference analogue of the lemma on the logarithmic derivative with applications to difference equations*, J. Math. Anal. Appl. **314** (2) (2006), 477–487.

[9] R. G. HALBURD, R. J. KORHONEN, *Nevanlinna theory for the difference operator*, Ann. Acad. Sci. Fenn. Math. **31** (2) (2006), 463–478.

[10] W. K. HAYMAN, *Meromorphic functions*, Oxford Math. Monogr. Clarendon Press, Oxford, 1964, xiv+191.

[11] J. HEITTOKANGAS, R. KORHONEN, I. LAINE, J. RIEPPO, *Uniqueness of meromorphic functions sharing values with their shifts*, Complex Var. Elliptic Equ. **56** (2011), no. 1–4, 81–92.

[12] SHENG LI, DUAN MEI, BAOQIN CHEN, *Uniqueness of entire functions sharing two values with their difference operators*, Adv. Difference Equ. (2017), Paper No. 390, 9.

[13] DAN LIU, DEGUI YANG, MINGLIANG FANG, *Unicity of entire functions concerning shifts and difference operators*, Abstr. Appl. Anal. (2014), Art. ID 380910, 5.

[14] KAI LIU, XIANJING DONG, *Some results related to complex differential-difference equations of certain types*, Bull. Korean Math. Soc. **51** (5) (2014), 1453–1467.

[15] PING LI, CHUNG-CHUN YANG, *Value sharing of an entire function and its derivatives*, J. Math. Soc. Japan, **51** (4) (1999), 781–799.

[16] XIAOGUANG QI, LIANZHONG YANG, *Uniqueness of meromorphic functions concerning their shifts and derivatives*, Comput. Methods Funct. Theory, **20** (1) (2020), 159–178.

[17] XIAOGUANG QI, NAN LI, LIANZHONG YANG, *Uniqueness of meromorphic functions concerning their differences and solutions of difference Painlevé equations*, Comput. Methods Funct. Theory, **18** (4) (2018), 567–582.

[18] L. A. RUBEL, CHUNG CHUN YANG, *Values shared by an entire function and its derivative*, Complex analysis (Proc. Conf., Univ. Kentucky, Lexington, Ky., 1976), 101–103.

[19] CHUNG-CHUN YANG, HONG-XUN YI, *Uniqueness theory of meromorphic functions*, Math. Appl. **557**, Kluwer Academic Publishers Group, Dordrecht, 2003, viii+569, ISBN:1-4020-1448-1.

[20] JIE ZHANG, LIANGWEN LIAO, *Entire functions sharing some values with their difference operators*, Sci. China Math. **57** (10) (2014), 2143–2152.

(Received October 11, 2023)

Harina P. Waghamore
Department of Mathematics
Bangalore University
Jnana Bharathi Campus, Bangalore-560 056, India
e-mail: harinapw@gmail.com

B. E. Manjunath
Department of Mathematics
Bangalore University
Jnana Bharathi Campus, Bangalore-560 056, India
e-mail: manjunath.bebub@gmail.com