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POINTWISE CONVERGENCE OF THE DOUBLE FOURIER–LEGENDRE SERIES OF FUNCTIONS OF GENERALIZED BOUNDED VARIATION

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Abstract. In this paper, we have studied the pointwise convergence of the double Fourier-Legendre series of functions of the generalized bounded variation. In particular, we also have the convergence of double Fourier-Legendre series of functions of (p,q)-bounded variation.

1. Preliminaries

The convergence of Fourier-Legendre series is useful in several areas of mathematics, physics, and engineering. For example in Approximation Theory, Boundary Value Problems, Quantum Mechanics, and Statistical Analysis. The Dirichlet-Jordan theorem (see [8] or [14, p. 57]) asserts that the Fourier series of a 2π-periodic function f of bounded variation on $[-\pi,\pi]$ converges at each point and the convergence is uniform on closed intervals of continuity of f. A similar theorem for the Fourier-Legendre series of a function of bounded variation on $I := [-1,1]$ was proved by Hobson [6]. Many authors have studied pointwise convergence, rate of convergence, and uniform convergence of Fouier-Legendre series of functions of certain classes (see, e.g., [3], [4], [5], [10]). In this paper we will discuss pointwise convergence of double Fourier-Legendre series of functions of the class $(m^\alpha,n^\beta)\mathrm{BV}^{(p,q)}(I^2)$ (in particular, for $\mathrm{BV}_H^{(p,q)}(I^2)$, $\mathrm{BV}_H^p(I^2)$, $(m^\alpha,n^\beta)\mathrm{BV}(I^2)$).

Let $P_n(x)$ be the Legendre polynomial of degree n normalized so that $P_n(1)=1$. If f is an integrable function on $I := [-1,1]$, then the Fourier-Legendre series (see, e.g., [11, p. 237, section 8.3]) of f is the series

$$\sum_{k=0}^{\infty} a_k(f)P_k(x)$$

where

$$a_k(f) = \left(k+\frac{1}{2}\right)\int_{-1}^{1} f(t)P_k(t)dt, \quad k=0,1,2,\ldots.$$

The n^{th} partial sum of the Fourier-Legendre series of f, denoted by $S_n(f,x)$, is defined as

$$S_n(f,x) = \sum_{k=0}^{n} a_k(f)P_k(x), \quad n=0,1,2,\ldots,$$

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which can be written as

$$S_n(f,x) = \int_{-1}^{1} f(t)K_n(x,t)dt,$$

where

$$K_n(x,t) = \sum_{k=0}^{n}\left(k+\frac{1}{2}\right)P_k(x)P_k(t) \text{ or } \frac{n+1}{2}\left(\frac{P_{n+1}(x)P_n(t)-P_n(x)P_{n+1}(t)}{x-t}\right). \quad (1)$$

DEFINITION 1. The (ordinary) oscillation of a function $h:[a,b]\to\mathbb{C}$ over a subinterval J of $[a,b]$ is defined as

$$\operatorname{osc}_1(h,J) = \sup\{|h(t)-h(t')| : t,t' \in J\}.$$

In 1980, Shiba [12] introduced the following class $\Lambda\mathrm{BV}^p$ of functions of p-Λ-bounded variation.

DEFINITION 2. Given a function $f:[a,b]\to\mathbb{R}$, a sequence $\Lambda=\{\lambda_k\}_{k\in\mathbb{N}}$ of non-decreasing of positive numbers such that $\sum\frac{1}{\lambda_k}$ diverges and a real number p, $1\leqslant p<\infty$, we say that $f\in\Lambda\mathrm{BV}^p[a,b]$ (that is, f is of p-Λ-bounded variation over $[a,b]$) if

$$V_{p\Lambda}(f,[a,b]) = \sup\left\{\sum_{k=1}^{n}\frac{|f(a_k)-f(b_k)|^p}{\lambda_k}\right\}^{1/p} < \infty,$$

where the supremum is extended over all sequences $\{I_k\}$ of non-overlapping intervals with $I_k=[a_k,b_k]\subset[a,b]$, $k=1,\ldots,n$.

When $\Lambda=\{1\}$ and $p=1$, the class is referred to as the class of functions of bounded variation (BV) and we denote the variation of any $f\in\mathrm{BV}$ by $V(f,[a,b])$. When $\Lambda=\{n^\alpha\}$, $0<\alpha<1$ and $p=1$, we denote this class by $(n^\alpha)\mathrm{BV}$ and the variation for any f in this class by $V_{n^\alpha}(f,[a,b])$. When $\Lambda=\{1\}$, the class is referred to as the class of functions of p-bounded variation (BV^p) (that is, Wiener class) and we denote the variation of any $f\in\mathrm{BV}^p$ by $V_p(f,[a,b])$. When $\Lambda=\{n^\alpha\}$, $0<\alpha<1$, we denote this class by $(n^\alpha)\mathrm{BV}^p$ and the variation for any f in this class by $V_{pn^\alpha}(f,[a,b])$.

We note that if f is of p-Λ-bounded variation, then right-hand limit $f(x+0)$ and left-hand limit $f(x-0)$ exist at every point x of $[a,b]$ (see [13, Theorem 2]). Also, M. Hormozi et. al. [7, Lemma 2.2] proved the following lemma.

LEMMA 1. *If f is of p-Λ-bounded variation, then*

(1) $\lim_{\delta\to 0^+} V_{p\Lambda}(f,(a,a+\delta]) = 0 = \lim_{\delta\to 0^+} V_{p\Lambda}(f,[b-\delta,b))$

(2) $\lim_{\delta\to 0^+} V_{p\Lambda}(f,[a,a+\delta]) = \frac{|f(a)-f(a+0)|}{\lambda_1^{1/p}}$.

(3) $\lim_{\delta\to 0^+} V_{p\Lambda}(f,[b-\delta,b]) = \frac{|f(b)-f(b-0)|}{\lambda_1^{1/p}}$.

We define, for $x \in [a,b]$,

$$s(f,x) = \frac{1}{2}(f(x+0)+f(x-0)).$$

Hobson [6] proved the following theorem concerning the pointwise convergence of the Fourier-Legendre series of functions of bounded variation.

THEOREM 1. *If f is of bounded variation on $[-1,1]$, then its Fourier-Legendre series converges to $s(f,x)$ at each point $x \in (-1,1)$, i.e.,*

$$S_n(f,x) \to s(f,x), \ \ as \ \ n \to \infty.$$

Also, in [3], we have derived the rate of convergence of the Fourier-Legendre series of functions belonging to $(n^\alpha)\mathrm{BV}$ class. In a particular case, we have the following theorem.

THEOREM 2. *If $f \in (n^\alpha)\mathrm{BV}(I)$, for $0 < \alpha < 1$, then its Fourier-Legendre series converges to $s(f,x)$ at each point $x \in (-1,1)$.*

In this paper, we extend above theorems for the convergence of double Fourier-Legendre series of functions belonging to classes $(m^\alpha, n^\beta)\mathrm{BV}^{(p,q)}(I^2)$ (in particular, for the class $\mathrm{BV}^{(p,q)}(I^2)$, $(m^\alpha, n^\beta)\mathrm{BV}(I^2)$, $(m^\alpha, n^\beta)\mathrm{BV}^p(I^2)$) by proving analogous result to (1) of above Lemma. We need the following definitions and notations.

If f is an integrable function on I^2, then the Fourier-Legendre series of f is the series

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{j,k} P_j(x) P_k(y),$$

where

$$a_{j,k}(f) = \left(j+\frac{1}{2}\right)\left(k+\frac{1}{2}\right)\int_{-1}^{1}\int_{-1}^{1} f(u,v)P_j(u)P_k(v)dudv, \ \ j,k = 0,1,2,\ldots,$$

is the $(j,k)^{th}$ Fourier-Legendre coefficient of the function f.

The rectangular partial sums of the double Fourier-Legendre series are defined by

$$S_{m,n}(f,x,y) = \sum_{j=0}^{m}\sum_{k=0}^{n} a_{j,k} P_j(x) P_k(y).$$

It is easy to show that

$$S_{m,n}(f,x,y) = \int_{-1}^{1}\int_{-1}^{1} f(u,v) K_m(x,u) K_n(y,v) dudv,$$

where $K_m(x,u)$, $K_n(y,v)$ are in (1) as follows.

DEFINITION 3. Let f be a real valued measurable function defined on the rectangle $R := [a,b] \times [c,d]$ and $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ and $\Lambda' = \{\lambda'_n\}_{n=1}^{\infty}$ be non-decreasing sequences of positive numbers such that $\sum \frac{1}{\lambda_n}$, $\sum \frac{1}{\lambda'_n}$ diverges, and real numbers p and q, $1 \leqslant p,q < \infty$, we say that $f \in (\Lambda,\Lambda')\mathrm{BV}^{(p,q)}(R)$ (that is, f is of (p,q)-(Λ,Λ')-bounded variation over R) if

(1) $f(\cdot,c) \in \Lambda\mathrm{BV}^p[a,b]$ and $f(a,\cdot) \in \Lambda'\mathrm{BV}^q[c,d]$, and

(2) if $\mathscr{I}_1$ and $\mathscr{I}_2$ are the sets of finite collections of non-overlapping intervals $I_j = [a_j,b_j]$, $j = 1,2,\ldots,m$, and $J_k = [c_k,d_k]$, $k = 1,2,\ldots,n$, in $[a,b]$ and $[c,d]$ respectively, and $f(I_j \times J_k) = f(a_j,c_k) - f(a_j,d_k) - f(b_j,c_k) + f(b_j,d_k)$, then

$$\sup_{\mathscr{I}_1,\mathscr{I}_2} \left(\sum_{j=1}^{m} \frac{1}{\lambda_j} \left(\sum_{k=1}^{n} \frac{(|f(I_j \times J_k)|)^q}{\lambda'_k} \right)^{p/q} \right)^{1/p} < \infty. \tag{2}$$

We denote the supremum in (2) by $V_{p\Lambda,q\Lambda'}(f,[a,b],[c,d])$.

When $\Lambda = \{m^{\alpha}\}$ and $\Lambda' = \{n^{\beta}\}$ for $\alpha,\beta \geqslant 0$ with $\alpha+\beta \leqslant 1$, we denote this class by $(m^{\alpha},n^{\beta})\mathrm{BV}^{(p,q)}$ class in which α,β,p and q satisfies the conditions $0 \leqslant \alpha q + \beta p \leqslant q$, $1-\alpha p \geqslant 0$, and $1-\beta q \geqslant 0$, and the variation for any f in this class by $V_{pm^{\alpha},qn^{\beta}}(f,[a,b],[c,d])$. When $\Lambda = \Lambda' = \{1\}$ and $p = q = 1$, the class is referred to as the class of functions of bounded variation in the sense of Hardy and Krause ($f \in \mathrm{BV}_H(R)$) and we denote the variation of any $f \in \mathrm{BV}_H$ by $V(f,[a,b],[c,d])$. When $\Lambda = \{m^{\alpha}\}$ and $\Lambda' = \{n^{\beta}\}$ for $\alpha,\beta \geqslant 0$ with $\alpha+\beta \leqslant 1$ and $p = q = 1$, we refer this class by $(m^{\alpha},n^{\beta})\mathrm{BV}$ class and we denote the variation for any f in this class by $V_{m^{\alpha},n^{\beta}}(f,[a,b],[c,d])$. When $\Lambda = \Lambda' = \{1\}$ and $p = q$, the class is referred to as the class of functions of p-bounded variation in the sense of Hardy and Krause ($f \in \mathrm{BV}_H^p(R)$) and we denote the variation of any $f \in \mathrm{BV}_H^p$ by $V_p(f,[a,b],[c,d])$. When $\Lambda = \Lambda' = \{1\}$, the class is referred to as the class of functions of (p,q)-bounded variation in the sense of Hardy and Krause ($f \in \mathrm{BV}_H^{(p,q)}(R)$) and we denote the variation of any $f \in \mathrm{BV}_H^{(p,q)}$ by $V_{p,q}(f,[a,b],[c,d])$.

REMARK 1. If $f \in (\Lambda,\Lambda')\mathrm{BV}^{(p,q)}(R)$ then f is bounded over R. In fact, let $f \in (\Lambda,\Lambda')\mathrm{BV}^{(p,q)}(R)$. Then for any $(x,y) \in R$, we have

$$\begin{aligned}
|f(x,y)| &\leqslant |f(x,y) - f(a,y) - f(x,c) + f(a,c)| + |f(x,c) - f(a,c)| \\
&\quad + |f(a,y) - f(a,c)| + |f(a,c)| \\
&= |f(a,c)| + (\lambda'_1)^{1/pq} \left(\frac{|f(x,y) - f(a,y) - f(x,c) + f(a,c)|^q}{{\lambda'_1}^{1/p}} \right)^{1/q} \\
&\quad + (\lambda_1)^{1/p} \left(\frac{|f(x,c) - f(a,c)|^p}{\lambda_1} \right)^{1/p} + (\lambda'_1)^{1/q} \left(\frac{|f(a,y) - f(a,c)|^q}{\lambda'_1} \right)^{1/q}
\end{aligned}$$

$$= |f(a,c)| + (\lambda_1)^{1/p}(\lambda_1')^{1/q}\left(\frac{1}{\lambda_1}\left(\frac{|f(x,y)-f(a,y)-f(x,c)+f(a,c)|^q}{\lambda_1'}\right)^{p/q}\right)^{1/p}$$
$$+ (\lambda_1)^{1/p}\left(\frac{|f(x,c)-f(a,c)|^p}{\lambda_1}\right)^{1/p} + (\lambda_1')^{1/q}\left(\frac{|f(a,y)-f(a,c)|^q}{\lambda_1'}\right)^{1/q}$$
$$\leqslant (\lambda_1)^{1/p}(\lambda_1')^{1/q}V_{p\Lambda,q\Lambda'}(f,[a,b],[c,d]) + (\lambda_1)^{1/p}V_{p\Lambda}(f(\cdot,c),[a,b])$$
$$+ (\lambda_1')^{1/q}V_{q\Lambda'}(f(a,\cdot),[c,d]) + |f(a,c)|.$$

Thus f is bounded on R.

DEFINITION 4. The rectangular oscillation of a function $f:[a,b]\times[c,d]\to\mathbb{C}$ over a subrectangle $J\times K$ of $[a,b]\times[c,d]$ is defined as

$$\operatorname{osc}_2(f,J\times K) = \sup_{u,u'\in J;\, v,v'\in K} |f(u,v)-f(u',v)-f(u,v')+f(u',v')|.$$

Here we shall consider the class $(\Lambda,\Lambda')\mathrm{BV}^{(p,q)}$, where $\Lambda=\{m^\alpha\}$ and $\Lambda'=\{n^\beta\}$, for $\alpha,\beta\geqslant 0$. Also we will prove (in Lemma 2) that if $f(x,y)\in\left(m^\alpha,n^\beta\right)\mathrm{BV}^{(p,q)}$ for $\alpha,\beta\geqslant 0$, $\alpha q+\beta p\leqslant q$ then all the four limits $f(x\pm 0,y\pm 0)$ exist at every point (x,y). We denote

$$s(f,x,y) = \frac{1}{4}[f(x+0,y+0)+f(x-0,y+0)+f(x+0,y-0)+f(x-0,y-0)],$$

$$\phi(u,v) = \begin{cases} f(u,v)-f(x+0,y+0), & \text{if } u>x, v>y,\\ f(u,v)-f(x-0,y+0), & \text{if } u<x, v>y,\\ f(u,v)-f(x+0,y-0), & \text{if } u>x, v<y,\\ f(u,v)-f(x-0,y-0), & \text{if } u<x, v<y,\\ f(u,y)-f(x+0,y), & \text{if } u>x, v=y,\\ f(u,y)-f(x-0,y), & \text{if } u<x, v=y,\\ f(x,v)-f(x,y+0), & \text{if } u=x, v>y,\\ f(x,v)-f(x,y-0), & \text{if } u=x, v<y,\\ 0, & \text{if } (u,v)=(x,y), \end{cases}$$

and

$$g(u,v) = \phi(u,v)-\phi(x,v)-\phi(u,y). \tag{3}$$

We will also denote $x+\frac{j(1-x)}{n}$ and $x-\frac{j(1+x)}{n}$ by $s_{j,x}$ and $t_{j,x}$ respectively for $x\in(-1,1)$, $j=0,1,\dots,n$, also denote intervals $I_{j,x}:=[t_{j+1,x},t_{j,x}]$ and $J_{j,x}:=[s_{j,x},s_{j+1,x}]$, for $j=1,2,\dots,n-1$.

Throughout this paper, we write $u\ll v$ if there exists a positive constant K, such that $u\leqslant Kv$ and K need not be the same at each occurrence.

2. Main Theorem

Our main theorem is as follows.

THEOREM 3. *Let* $f \in \left(m^{\alpha}, n^{\beta}\right)\mathrm{BV}^{(p,q)}(I^2)$, $0 \leqslant \alpha q + \beta p \leqslant q$, $1 - \alpha p \geqslant 0$, *and* $1 - \beta q \geqslant 0$. *Then, for* $(x,y) \in (-1,1)\times(-1,1)$, *we have* $S_{m,n}(f,x,y) \to s(f,x,y)$ *in Pringsheim sense as* $m,n \to \infty$.

By putting $\alpha = \beta = 0$ in Theorem 3, we get the following corollary for the class of functions of (p,q)-bounded variation in the sense of Hardy and Krause.

COROLLARY 1. *Let* $f \in \mathrm{BV}_H^{(p,q)}(I^2)$. *Then, for* $(x,y) \in (-1,1)\times(-1,1)$, *we have* $S_{m,n}(f,x,y) \to s(f,x,y)$ *in Pringsheim sense as* $m,n \to \infty$.

Also, putting $q = p$ and $\alpha = \beta = 0$ in Theorem 3, we get the following corollary for the class of functions of p-bounded variation in the sense of Hardy and Krause.

COROLLARY 2. *Let* $f \in \mathrm{BV}_H^{p}(I^2)$. *Then, for* $(x,y) \in (-1,1)\times(-1,1)$, *we have* $S_{m,n}(f,x,y) \to s(f,x,y)$ *in Pringsheim sense as* $m,n \to \infty$.

At last, putting $p = q = 1$ in Theorem 3, we get the following corollary for the class of functions of generalized bounded variation, which is a two-dimensional analogue of Theorem 2.

COROLLARY 3. *Let* $f \in \left(m^{\alpha}, n^{\beta}\right)\mathrm{BV}(I^2)$, $0 \leqslant \alpha + \beta \leqslant 1$. *Then, for* $(x,y) \in (-1,1)\times(-1,1)$, *we have* $S_{m,n}(f,x,y) \to s(f,x,y)$ *in Pringsheim sense as* $m,n \to \infty$.

3. Lemmas

To prove main theorem, we require following lemmas.

LEMMA 2. *If* $f \in (m^{\alpha}, n^{\beta})\mathrm{BV}^{(p,q)}(R)$ *for* $\alpha, \beta \geqslant 0$, $0 \leqslant \alpha q + \beta p \leqslant q$, *and every* $(x_0, y_0) \in I^2$, *the four limits* $f(x_0 \pm 0, y_0 \pm 0)$ *of* $f(x,y)$ *as* $(x,y) \to (x_0,y_0)$ *and* (x,y) *is in the corresponding open coordinate quadrant, exist.*

Proof of Lemma 2. Our proof is similar to that of Theorem 7 of [2]. Suppose $f(x,y) \in (m^{\alpha}, n^{\beta})\mathrm{BV}^{(p,q)}(R)$, $\alpha, \beta \geqslant 0$, $\alpha q + \beta p \leqslant q$. Suppose also that there is a point $(x_0,y_0) \in R$ such that $f(x,y)$ does not have a limit as $(x,y) \to (x_0,y_0)$ within an open coordinate quadrant with vertex (x_0,y_0). Without loss of generality, we may assume that the quadrant is $\{(x,y) : x_0 < x < b, y_0 < y < d\} = S$, say.

Then, by Cauchy criterion (see, e.g., [9, Proposition 2.54]), there is an $\varepsilon > 0$ such that for every $\delta > 0$, there are points (x_1,y_1), (x_1',y_1') in $S \cap S_{\delta}(x_0,y_0) \setminus \{(x_0,y_0)\}$ such that

$$|f(x_1,y_1) - f(x_1',y_1')| > 4\varepsilon. \tag{4}$$

Choose $(s,t) \in (x_0,b)\times(y_0,d)$. Then, since $f(x,t)$ and $f(s,y)$ are in $(m^{\alpha})\mathrm{BV}^{(p)}([a,b])$ and $(n^{\beta})\mathrm{BV}^{(q)}([c,d])$ respectively in each variable separately, $\lim_{x\to x_0+} f(x,t)$ and

$\lim_{y\to y_{0+}} f(s,y)$ exist (see, e.g., [13, Theorem 2]). Therefore, as $\varepsilon > 0$, by Cauchy criterion, there are $\delta_1, \delta_2 > 0$ such that $\delta_1 < b - x_0$, $\delta_2 < d - y_0$, and

$$x_0 < x_1, x_1' < x_0 + \delta_1 \implies |f(x_1,t) - f(x_1',t)| < \varepsilon;$$
$$y_0 < y_1, y_1' < y_0 + \delta_2 \implies |f(s,y_1) - f(s,y_1')| < \varepsilon.$$

Put $\delta = \min\{\delta_1, \delta_2\}$. Then, for this δ, as above, there are points (x_1,y_1), (x_1',y_1') in $S \cap S_\delta(x_0,y_0) \setminus \{(x_0,y_0)\}$ such that (4) holds. Observe that

$$\begin{aligned}&(x_1,y_1),\,(x_1',y_1') \in S \cap S_\delta(x_0,y_0) \setminus \{(x_0,y_0)\}\\ &\qquad \implies |f(x_1,t) - f(x_1',t)| < \varepsilon \text{ and } |f(s,y_1) - f(s,y_1')| < \varepsilon. \end{aligned} \tag{5}$$

Now, letting

$$P = f(s,t) - f(s,y_1) - f(x_1,t) + f(x_1,y_1)$$

and

$$Q = f(s,t) - f(s,y_1') - f(x_1',t) + f(x_1',y_1'),$$

in view of (4) and (5), we have

$$\begin{aligned}|P - Q| &\geqslant |f(x_1,y_1) - f(x_1',y_1')| - |f(x_1,t) - f(x_1',t)| - |f(s,y_1) - f(s,y_1')|\\ &\geqslant 4\varepsilon - \varepsilon - \varepsilon = 2\varepsilon.\end{aligned}$$

So, at least one $|P|$ or $|Q|$, must exceed ε. Hence renaming endpoints, we obtained a rectangle, say, $I_1 \times J_1 := [x_1, x_1'] \times [y_1, y_1']$ with $x_0 < x_1 < x_1' < b$ and $y_0 < y_1 < y_1' < d$, for which $|f(I_1 \times J_1)| > \varepsilon$.

Now, let $(s,t) \in (x_0,x_1) \times (y_0,y_1)$. Then arguing as above, and now choosing $\delta_1 < x_1 - x_0$ and $\delta_2 < y_1 - y_0$, we can obtain a rectangle, say, $I_2 \times J_2 := [x_2, x_2'] \times [y_2, y_2']$ with $x_0 < x_2 < x_2' < x_1$ and $y_0 < y_2 < y_2' < y_1$, for which $|f(I_2 \times J_2)| > \varepsilon$. Note that this construction gives $I_1 \cap I_2 = \emptyset$ and $J_1 \cap J_2 = \emptyset$. Continuing in this way, we can form a sequence $\{I_n \times J_n\}$ of rectangles in $(x_0,b) \times (y_0,d)$ such that the intervals I_n's are disjoint, the intervals J_n's are disjoint, and that $|f(I_n \times J_n)| > \varepsilon$ for each $n \in \mathbb{N}$. But then, as $0 \leqslant q\alpha + p\beta \leqslant q$, we have

$$\begin{aligned}\left(\sum_{j=1}^{N} \frac{1}{j^\alpha}\left(\sum_{k=1}^{N} \frac{|f(I_j \times J_k)|^q}{k^\beta}\right)^{p/q}\right)^{1/p} &> \left(\sum_{j=1}^{N} \frac{1}{j^\alpha}\left(\sum_{k=1}^{j} \frac{|f(I_j \times J_k)|^q}{k^\beta}\right)^{p/q}\right)^{1/p}\\ &\geqslant \left(\sum_{j=1}^{N} \frac{1}{j^\alpha}\left(\frac{|f(I_j \times J_j)|^q}{j^\beta}\right)^{p/q}\right)^{1/p}\\ &> \varepsilon\left(\sum_{j=1}^{N} \frac{1}{j^{(q\alpha+p\beta)/q}}\right)^{1/p} \to \infty\end{aligned}$$

as $N \to \infty$, which is a contradiction as $f(x,y) \in (m^\alpha, n^\beta)\mathrm{BV}^{(p,q)}(R)$. This completes the proof of Lemma 2. □

LEMMA 3. *If f is of (p,q)-(Λ,Λ')-bounded variation on $[a,b]\times[c,d]$ then*

(a) $\lim_{\delta\to 0^+} V_{p\Lambda,q\Lambda'}(f,(a,a+\delta],[c,d])=0$.

(b) $\lim_{\delta\to 0^+} V_{p\Lambda,q\Lambda'}(f,[a,a+\delta],[c,d])=0$, *if $f(x,\cdot)$ is continuous at a point a.*

(c) $\lim_{\delta\to 0^+} V_{p\Lambda,q\Lambda'}(f,[a,b],(c,c+\delta])=0$.

(d) $\lim_{\delta\to 0^+} V_{p\Lambda,q\Lambda'}(f,[a,b],[c,c+\delta])=0$, *if $f(\cdot,y)$ is continuous at a point c.*

(e) $\lim_{\delta\to 0^+} V_{p\Lambda,q\Lambda'}(f,(a,a+\delta],(c,c+\delta])=0$.

(f) $\lim_{\delta\to 0^+} V_{p\Lambda,q\Lambda'}(f,[a,a+\delta],[c,c+\delta])=0$, *if $f(x,y)$ is continuous at a point (a,c).*

Proof of Lemma 3. Clearly, $V_{p\Lambda,q\Lambda'}(f,(a,a+\delta],[c,d])\geqslant 0$. Suppose that there is $\varepsilon>0$ such that

$$V_{p\Lambda,q\Lambda'}(f,(a,a+\delta],[c,d])>\varepsilon,\ \ \forall\delta\geqslant 0. \tag{6}$$

Now, as f is of (p,q)-(Λ,Λ')-bounded variation on $[a,b]\times[c,d]$, f is of p-Λ-bounded variation on $[a,b]$ for a fixed variable y then the right-hand limit $f(x+0,\cdot)$ exists. Therefore, for $m_0',n_0'\in\mathbb{N}$, and given ε there exist $\delta_0\leqslant\delta$ satisfies

$$|f(I^0,\cdot)|\leqslant\left(\frac{\lambda_1}{m_0'}\right)^{1/p}\left(\frac{\lambda_1'}{n_0'}\right)^{1/q}\frac{\varepsilon}{2^{2+1/p}},$$

for all subintervals $I^0\subset(a,a+\delta_0]$.

Now, for this δ_0 and from (6), existence of disjoint rectangles $I_i^0\times J_j^0\subset(a,a+\delta_0]\times[c,d]$: $i=0,1,\ldots,m_0$, $j=0,1,2,\ldots,n_0$, $m_0,n_0\in\mathbb{N}$ such that

$$\sum_{i=1}^{m_0}\frac{1}{\lambda_i}\left(\sum_{j=1}^{n_0}\frac{|f(I_i^0\times J_j^0)|^q}{\lambda_j'}\right)^{p/q}\geqslant\varepsilon^p.$$

Now, as

$$|f(I_i^0\times J_j^0)|\leqslant 2|f(I^0,\cdot)|\leqslant\left(\frac{\lambda_1}{m_0'}\right)^{1/p}\left(\frac{\lambda_1'}{n_0'}\right)^{1/q}\frac{\varepsilon}{2^{1+1/p}},$$

and as numerical inequalities (see [1, p. 16]) for any positive numbers a,b, and p, we have

$$(a+b)^p\leqslant\begin{cases}2^p(a^p+b^p) & \text{for } p\geqslant 1,\\ a^p+b^p & \text{for } 0\leqslant p<1,\end{cases}\implies (a+b)^p\leqslant 2^p(a^p+b^p)\ \text{for } p\geqslant 0,$$

we have

$$\sum_{i=1}^{m_0}\frac{1}{\lambda_i}\left(\sum_{j=1}^{n_0}\frac{|f(I_i^0\times J_j^0)|^q}{\lambda_j'}\right)^{p/q}\geqslant \varepsilon^p$$

$$\Longrightarrow \sum_{i=1}^{m_0}\frac{1}{\lambda_i}\left(\sum_{j=1}^{n_0'}\frac{|f(I_i^0\times J_j^0)|^q}{\lambda_j'}\right)^{p/q}+\sum_{i=1}^{m_0}\frac{1}{\lambda_i}\left(\sum_{j=n_0'+1}^{n_0}\frac{|f(I_i^0\times J_j^0)|^q}{\lambda_j'}\right)^{p/q}\geqslant\frac{\varepsilon^p}{2^{p/q}}$$

$$\Longrightarrow \left[\sum_{i=1}^{m_0'}+\sum_{i=m_0'+1}^{m_0}\right]\frac{1}{\lambda_i}\left(\sum_{j=1}^{n_0'}\frac{|f(I_i^0\times J_j^0)|^q}{\lambda_j'}\right)^{p/q}+\sum_{i=1}^{m_0}\frac{1}{\lambda_i}\left(\sum_{j=n_0'+1}^{n_0}\frac{|f(I_i^0\times J_j^0)|^q}{\lambda_j'}\right)^{p/q}$$
$$\geqslant\frac{\varepsilon^p}{2^{p/q}}$$

$$\Longrightarrow \sum_{i=m_0'+1}^{m_0}\frac{1}{\lambda_i}\left(\sum_{j=1}^{n_0'}\frac{|f(I_i^0\times J_j^0)|^q}{\lambda_j'}\right)^{p/q}+\sum_{i=1}^{m_0}\frac{1}{\lambda_i}\left(\sum_{j=n_0'+1}^{n_0}\frac{|f(I_i^0\times J_j^0)|^q}{\lambda_j'}\right)^{p/q}$$
$$\geqslant\frac{\varepsilon^p}{2^{p/q}}-\sum_{i=1}^{m_0'}\frac{1}{\lambda_i}\left(\sum_{j=1}^{n_0'}\frac{|f(I_i^0\times J_j^0)|^q}{\lambda_j'}\right)^{p/q}$$

$$\Longrightarrow \sum_{i=m_0'+1}^{m_0}\frac{1}{\lambda_i}\left(\sum_{j=1}^{n_0'}\frac{|f(I_i^0\times J_j^0)|^q}{\lambda_j'}\right)^{p/q}+\sum_{i=1}^{m_0}\frac{1}{\lambda_i}\left(\sum_{j=n_0'+1}^{n_0}\frac{|f(I_i^0\times J_j^0)|^q}{\lambda_j'}\right)^{p/q}$$
$$\geqslant\frac{\varepsilon^p}{2^{p/q}}-\frac{\varepsilon^p}{2^{p+1}}=\frac{\varepsilon^p}{2^{p/q}}-\frac{\varepsilon^p}{2^{2+p/q}}=\frac{3\varepsilon^p}{2^{p+1}}.$$

Now again, suppose all subintervals $I^0\subset(a_0,a+\delta_0]$ then removing the part $(a_0,a+\delta]$, we proceed similarly for $m_1',n_1'\in\mathbb{N}$, and given ε there exist $\delta_1\leqslant\delta$ satisfies

$$|f(I^1,\cdot)|\leqslant\left(\frac{\lambda_1}{m_1'}\right)^{1/p}\left(\frac{\lambda_1'}{n_1'}\right)^{1/q}\frac{\varepsilon}{2^{2+1/p}},$$

for all subintervals $I^1\subset(a,a+\delta_1]$, $a+\delta_1\leqslant a_0$. Now again, for this δ_1 and from (6), existence of disjoint rectangles $I_i^1\times J_j^1\subset(a,a+\delta_1]\times[c,d]$: $i=0,1,\dots,m_1$, $j=0,1,2,\dots,n_1$, $m_1,n_1\in\mathbb{N}$ there is a collection of rectangles $(I_i\times J_j : i=m_1',\dots,m_1;\ j=0,1,\dots,n)$ satisfy

$$\sum_{i=m_1'+1}^{m_1}\frac{1}{\lambda_i}\left(\sum_{j=1}^{n_1'}\frac{|f(I_i^1\times J_j^1)|^q}{\lambda_j'}\right)^{p/q}+\sum_{i=1}^{m_1}\frac{1}{\lambda_i}\left(\sum_{j=n_1'+1}^{n_1}\frac{|f(I_i^1\times J_j^1)|^q}{\lambda_j'}\right)^{p/q}\geqslant\frac{\varepsilon^p}{2^{p/q}}-\frac{\varepsilon^p}{2^{1+p}}$$
$$=\frac{3\varepsilon^p}{2^{p+1}}.$$

Similarly, we can constuct an finite families of disjoint rectangles inductively on remainig rectangle, so that

$$\sum_{k=0}^{n}\left[\sum_{i=m_k'+1}^{m_k}\frac{1}{\lambda_i}\left(\sum_{j=1}^{n_k'}\frac{|f(I_i^k\times J_j^k)|^q}{\lambda_j'}\right)^{p/q}+\sum_{i=1}^{m_k}\frac{1}{\lambda_i}\left(\sum_{j=n_k'+1}^{n_k}\frac{|f(I_i^k\times J_j^k)|^q}{\lambda_j'}\right)^{p/q}\right]\geqslant\frac{3n\varepsilon^p}{2^{p+1}},$$

for all n. It follows that $f\notin(\Lambda,\Lambda')BV^{(p,q)}([a,b]\times[c,d])$, a contradiction. Thus, it must be $V(f,(a,a+\delta],[c,d])\to 0$ as $\delta\to 0$. This completes the proof of (a). Similarly, one can prove (c) and (e).

Now, suppose $V_{p\Lambda,q\Lambda'}(f,(a,a+\delta],[c,d])\to 0$ as $\delta\to 0$. Then for any $\varepsilon>0$ there is $\delta>0$ such that

$$V_{p\Lambda,q\Lambda'}(f,(a,a+\delta_0],[c,d])\leqslant\varepsilon,\ \forall\,\delta_0<\delta. \tag{7}$$

We will prove that condition (b) holds. Given any family $\{I_i\times J_j\}_{i=1,j=1}^{m,n}$ of $[a,a+\delta_0]\times[c,d]$, also an interval I_i which contain a, denoting it by $I_{i'}$, we have

$$\sum_{i=1}^{m}\frac{1}{\lambda_i}\left(\sum_{j=1}^{n}\frac{1}{\lambda_j'}|f(I_i\times J_j)|^q\right)^{p/q}=\frac{1}{\lambda_{i'}}\left(\sum_{j=1}^{n}\frac{1}{\lambda_j'}|f(I_{i'}\times J_j)|^q\right)^{p/q}+\sum_{i=1,i\neq i'}^{m}\frac{1}{\lambda_i}\left(\sum_{j=1}^{n}\frac{1}{\lambda_j'}|f(I_i\times J_j)|^q\right)^{p/q} \tag{8}$$

By our assumption $f(x,\cdot)$ is continuous at point a, then for given $\varepsilon>0$ there is $0<\delta_1<\delta_0$ satisfying

$$|f(a,\cdot)-f(a+\delta_1,\cdot)|\leqslant\left((\lambda_{i'})^{1/p}(\lambda_1')^{1/q}\right)\varepsilon/2^{2+1/q}n^{1/q}.$$

Now, decomposing $I_{i'}=[a,a+\delta_0]=[a,a+\delta_1]\cup[a+\delta_1,a+\delta_0]:=I_{i'}^1\cup I_{i'}^2$, we have

$$\frac{1}{\lambda_{i'}}\left(\sum_{j=1}^{n}\frac{1}{\lambda_j'}|f(I_{i'}\times J_j)|^q\right)^{p/q}\leqslant\frac{2^{p(1+1/q)}}{\lambda_{i'}}\left[\left(\sum_{j=1}^{n}\frac{1}{\lambda_j'}|f(I_{i'}^1\times J_j)|^q\right)^{p/q}+\left(\sum_{j=1}^{n}\frac{1}{\lambda_j'}|f(I_{i'}^2\times J_j)|^q\right)^{p/q}\right]. \tag{9}$$

Now, as $|f(I_{i'}^1\times J_j)|\leqslant 2\sup_{y\in[c,d]}|f(a,\cdot)-f(a+\delta_1,\cdot)|\leqslant\left((\lambda_{i'})^{1/p}(\lambda_1')^{1/q}\right)\varepsilon/2^{1+1/q}n^{1/q}$, we have

$$\frac{2^{p(1+1/q)}}{\lambda_{i'}}\left(\sum_{j=1}^{n}\frac{1}{\lambda_j'}|f(I_{i'}^1\times J_j)|^q\right)^{p/q}\leqslant\varepsilon^p.$$

Also from (7), the second terms of right hand side of an inequality (8) and (9) becomes

$$\frac{2^{p(1+1/q)}}{\lambda_{i'}}\left(\sum_{j=1}^{n}\frac{1}{\lambda'_j}|f(I^2_{i'}\times J_j)|^q\right)^{p/q}+\sum_{i=1,i\neq i'}^{m}\frac{1}{\lambda_i}\left(\sum_{j=1}^{n}\frac{1}{\lambda'_j}|f(I_i\times J_j)|^q\right)^{p/q}$$
$$\leqslant KV^p_{p\Lambda,q\Lambda'}(f,(a,a+\delta_0],[c,d])\leqslant K\varepsilon^p,$$

where K is some positive constant. Since ε is arbitrary, condition (b) holds true. Similarly, one can prove (d) and (f). □

We also recall the partial summation formulas for single and double sequences, which are as follows.

LEMMA 4. *Consider $n\in\mathbb{N}$. For $j=0,1,\dots,n$, let a_j and b_j be real numbers. Let $B_j=\sum_{k=j}^{n}b_k$ for $j=0,1,2,\dots,n$, and $B_{n+1}=0$. Then*

$$\sum_{j=1}^{n}a_jb_j=\sum_{j=1}^{n}\left(a_j-a_{j-1}\right)B_j+a_0B_1. \tag{10}$$

Also, for $B_j=\sum_{k=0}^{j}b_k$, we have

$$\sum_{j=0}^{n}a_jb_j=\sum_{j=0}^{n-1}\left(a_j-a_{j+1}\right)B_j+a_nB_n. \tag{11}$$

LEMMA 5. *Consider $(m,n)\in\mathbb{N}^2$.Let $a_{j,k}$ and $b_{j,k}$ be real numbers, and let $B_{j,k}=\sum_{j'=j}^{m}\sum_{k'=k}^{n}b_{j',k'}$ and $B_{m+1,n+1}=B_{j,n+1}=B_{m+1,k}=0$, for $j=0,1,\dots,m$, $k=0,1,\dots,n$, then*

$$\sum_{j=1}^{m}\sum_{k=1}^{n}a_{j,k}b_{j,k}=\sum_{j=1}^{m}\sum_{k=1}^{n}\left(a_{j,k}-a_{j,k-1}-a_{j-1,k}+a_{j-1,k-1}\right)B_{j,k}$$
$$+\sum_{j=1}^{m}(a_{j,0}-a_{j-1,0})B_{j,1}+\sum_{k=1}^{n}(a_{0,k}-a_{0,k-1})B_{1,k}+a_{0,0}B_{1,1}.$$

4. Proof of the main Theorem

Proof of Theorem 3. For any $m,n\in\mathbb{N}$, $x,y\in(-1,1)$, and using the facts (see [4, Lemma 1])

$$\int_x^1 K_n(x,t)dt=\frac{1}{2}-\frac{1}{2}P_n(x)P_{n+1}(x)\ \text{ and }\ \int_{-1}^{x}K_n(x,t)dt=\frac{1}{2}+\frac{1}{2}P_n(x)P_{n+1}(x), \tag{12}$$

we have the representation (see [4, Proof of Theorem 1])

$$
\begin{aligned}
&S_{m,n}(f,x,y)-s(f,x,y)\\
&=\int_{-1}^{1}\int_{-1}^{1} f(u,v)K_m(x,u)K_n(y,v)\,dudv-s(f,x,y)\\
&=\int_{-1}^{1}\int_{-1}^{1} \phi(u,v)K_m(x,u)K_n(y,v)\,dudv+\frac{1}{4}[f(x+0,y+0)-f(x+0,y-0)\\
&\quad -f(x-0,y+0)+f(x-0,y-0)]P_m(x)P_{m+1}(x)P_n(y)P_{n+1}(y)-\frac{1}{4}(f(x+0,y+0)\\
&\quad +f(x+0,y-0)-f(x-0,y+0)-f(x-0,y-0))P_m(x)P_{m+1}(x)-\frac{1}{4}(f(x+0,y+0)\\
&\quad -f(x+0,y-0)+f(x-0,y+0)-f(x-0,y-0))P_n(y)P_{n+1}(y). \qquad (13)
\end{aligned}
$$

For fixed x and y, for simplicity denote $K_m(x,u)$ and $K_n(y,v)$ by $X_m(u)$ and $Y_n(v)$ respectively, and using notation as in (3), we decompose the double integral as

$$
\begin{aligned}
&\int_{-1}^{1}\int_{-1}^{1} \phi(u,v)X_m(u)Y_n(v)\,dudv\\
&\quad=\left(\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\int_{I_{j,x}}\int_{I_{k,y}}+\int_{x}^{1}\int_{-1}^{y}+\int_{-1}^{x}\int_{y}^{1}+\int_{x}^{1}\int_{y}^{1}\right)g(u,v)X_m(u)Y_n(v)\,dudv\\
&\quad\quad+\int_{-1}^{1}\int_{-1}^{1}\phi(u,y)X_m(u)Y_n(v)\,dudv+\int_{-1}^{1}\int_{-1}^{1}\phi(x,v)X_m(u)Y_n(v)\,dudv\\
&\quad=\mathrm{A}_1+\mathrm{A}_2+\ldots+\mathrm{A}_5+\mathrm{A}_6,\ \text{say}. \qquad (14)
\end{aligned}
$$

First decomposing A_1, we have

$$
\begin{aligned}
\mathrm{A}_1&=\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\left[\int_{I_{j,x}}\int_{I_{k,y}}(g(u,v)-g(t_{j,x},v)-g(u,t_{k,y})+g(t_{j,x},t_{k,y})\right.\\
&\quad+\int_{I_{j,x}}\int_{I_{k,y}}\big(g(u,t_{k,y})-g(t_{j,x},t_{k,y})\big)+\int_{I_{j,x}}\int_{I_{k,y}}\big(g(t_{j,x},v)-g(t_{j,x},t_{k,y})\big)\\
&\quad\left.+\int_{I_{j,x}}\int_{I_{k,y}}g(t_{j,x},t_{k,y})\right]X_m(u)Y_n(v)\,dudv\\
&=\mathrm{A}_{11}+\mathrm{A}_{12}+\mathrm{A}_{13}+\mathrm{A}_{14},\ \text{say}. \qquad (15)
\end{aligned}
$$

Now for $m\geqslant 2,\ j=1,2,\ldots,m-1$, and for fixed $x\in(-1,1)$, using an inequality (see [3, (3.12)])

$$
\int_{I_{j,x}}|X_m(t)|\,dt\leqslant\frac{4\sqrt{2m}}{\pi j(1-x^2)(m-j)^{1/2}}\ll\frac{1}{(j+1)}\sqrt{\frac{m}{m-j}}, \qquad (16)
$$

and an inequality (see [4, Lemma 2])

$$
\int_{t_{1,x}}^{s_{1,x}}|X_m(t)|\,dt\leqslant\frac{4}{1-x^2}\ll 1, \qquad (17)
$$

we proceed as follows

$$
\begin{aligned}
|\mathrm{A}_{11}| &= \left|\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\int_{I_{j,x}}\int_{I_{k,y}}(g(u,v)-g(t_{j,x},v)-g(u,t_{k,y})+g(t_{j,x},t_{k,y}))X_m(u)Y_n(v)dudv\right| \\
&\leqslant \sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\int_{I_{j,x}}\int_{I_{k,y}}|\phi(u,v)-\phi(t_{j,x},v)-\phi(u,t_{k,y})+\phi(t_{j,x},t_{k,y})||X_m(u)||Y_n(v)|dudv \\
&\ll \sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\mathrm{osc}_2(\phi,I_{j,x},I_{k,y})\frac{\sqrt{m}}{(j+1)(m-j)^{1/2}}\frac{\sqrt{n}}{(k+1)(n-k)^{1/2}}.
\end{aligned}
\tag{18}
$$

Also, defining

$$
R'_{k,n}=\int_{-1}^{t_{k,y}}Y_n(v)dv \ \text{ for } \ k=1,2,\dots,n-1 \ \text{ and } \ R'_{n,n}=0,
\tag{19}
$$

and using partial summation formula (see (10) of Lemma 4) with $a_k=g(u,t_{k,y})-g(t_{j,x},t_{k,y})$ and $b_k=(R'_{k,n}-R'_{k+1,n})$, and as $a_0=0$, we have

$$
\begin{aligned}
A_{12} &= \sum_{j=0}^{m-1}\int_{I_{j,x}}\sum_{k=1}^{n-1}(g(u,t_{k,y})-g(t_{j,x},t_{k,y}))(R'_{k,n}-R'_{k+1,n})X_m(u)du \\
&= \sum_{j=0}^{m-1}\int_{I_{j,x}}\Bigg(\sum_{k=1}^{n-1}\left(g(u,t_{k,y})-g(t_{j,x},t_{k,y})-g(u,t_{k-1,y})+g(t_{j,x},t_{k-1,y})\right)(R'_{k,n}-R'_{n,n}) \\
&\qquad\qquad +(g(u,y)-g(t_{j,x},y))(R'_{1,n}-R'_{n,n})\Bigg)X_m(u)du. \\
&= \sum_{j=1}^{m-1}\int_{I_{j,x}}\sum_{k=1}^{n-1}\left(g(u,t_{k,y})-g(t_{j,x},t_{k,y})-g(u,t_{k-1,y})+g(t_{j,x},t_{k-1,y})\right)R'_{k,n}X_m(u)du,
\end{aligned}
$$

as $g(u,y)-g(t_{j,x},y)=0$ by (3), and as $R'_{n,n}=0$ by (19). Since

$$
\begin{aligned}
g(u,t_{k,y})-g(t_{j,x},t_{k,y})&-g(u,t_{k-1,y})+g(t_{j,x},t_{k-1,y}) \\
&= \phi(u,t_{k,y})-\phi(t_{j,x},t_{k,y})-\phi(u,t_{k-1,y})+\phi(t_{j,x},t_{k-1,y})
\end{aligned}
\tag{20}
$$

and for fixed $n\geqslant 2$ and $-1\leqslant t<y<1$, we have an inequality (see [4, Lemma 3])

$$
\begin{aligned}
&\left|\int_{-1}^{t}Y_n(v)dv\right|\leqslant\frac{6}{n(y-t)}(1-y^2)^{-1/2}\ll\frac{1}{n(y-t)} \implies \\
&|R'_{k,n}|=\left|\int_{-1}^{t_{k,y}}Y_n(v)dv\right|\ll\frac{1}{n(y-t_{k,y})}=\frac{1}{n\left(y-\left(y-\frac{k(1+y)}{n}\right)\right)}=\frac{1}{k(1+y)}\ll\frac{1}{k}.
\end{aligned}
\tag{21}
$$

Now, using (16), (20), and (21), we have

$$
\begin{aligned}
|\mathrm{A}_{12}| &\leqslant \sum_{j=0}^{m-1}\sum_{k=1}^{n-1}\int_{I_{j,x}}|\phi(u,t_{k,y})-\phi(t_{j,x},t_{k,y})-\phi(u,t_{k-1,y})+\phi(t_{j,x},t_{k-1,y})||R'_{k,n}|\,|X_m(u)|\,du \\
&\ll \sum_{j=0}^{m-1}\sum_{k=1}^{n-1}\frac{\sqrt{m}}{(j+1)k\sqrt{m-j}}\mathrm{osc}_2(\phi,I_{j,x},I_{k-1,y}).
\end{aligned}
\tag{22}
$$

Since A_{13} is symmetric to A_{12}, defining

$$R_{j,m} = \int_{-1}^{t_{j,x}} X_m(u)du, \text{ for } j = 1, 2, \ldots, m-1 \text{ and } R_{m,m} = 0, \tag{23}$$

we can prove

$$|\mathrm{A}_{13}| \ll \sum_{j=1}^{m-1}\sum_{k=0}^{n-1} \frac{\sqrt{n}}{j(k+1)\sqrt{(n-k)}} \operatorname{osc}_2(\phi, I_{j-1,x}, I_{k,y}). \tag{24}$$

Now, from (19), (23), using double summation formula (see Lemma 5) with $a_{j,k} = g(t_{j,x}, t_{k,y})$ and $b_{j,k} = (R_{j,m} - R_{j+1,m})(R'_{k,n} - R'_{k+1,n})$, and as $a_{0,k} = a_{j,0} = a_{0,0} = 0$, we have

$$\begin{aligned}
\mathrm{A}_{14} &= \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} \int_{t_{j+1,x}}^{t_{j,x}} \int_{t_{k+1,y}}^{t_{k,y}} g(t_{j,x}, t_{k,y}) X_m(u) Y_n(v) du dv \\
&= \sum_{j=1}^{m-1}\sum_{k=1}^{n-1} g(t_{j,x}, t_{k,y})(R_{j,m} - R_{j+1,m})(R'_{k,n} - R'_{k+1,n}) \\
&= \sum_{j=1}^{m-1}\sum_{k=1}^{n-1} \big(g(t_{j,x}, t_{k,y}) - g(t_{j,x}, t_{k-1,y}) - g(t_{j-1,x}, t_{k,y}) + g(t_{j-1,x}, t_{k-1,y})\big) R_{j,m} R'_{k,n}.
\end{aligned}$$

Therefore, from (21) and putting $u = t_{j-1,x}$ in (20), we have

$$\begin{aligned}
|\mathrm{A}_{14}| &\leqslant \sum_{j=1}^{m-1}\sum_{k=1}^{n-1} |\phi(t_{j,x}, t_{k,y}) - \phi(t_{j-1,x}, t_{k,y}) - \phi(t_{j,x}, t_{k-1,y}) + \phi(t_{j-1,x}, t_{k-1,y})||R_{j,m}||R'_{k,n}| \\
&\ll \sum_{j=1}^{m-1}\sum_{k=1}^{n-1} \frac{1}{jk} \operatorname{osc}_2(\phi, I_{j-1,x}, I_{k-1,y}).
\end{aligned} \tag{25}$$

Therefore, from (15), (18), (22), (24), and (25), we have

$$\begin{aligned}
\mathrm{A}_1 &\ll \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} \frac{\sqrt{mn}}{(j+1)(k+1)(m-j)^{1/2}(n-k)^{1/2}} \operatorname{osc}_2(\phi, I_{j,x}, I_{k,y}) \\
&\ll \left[\sum_{j=0}^{[m/2]} \frac{1}{j+1} + \sum_{j=[m/2]+1}^{m-1} \frac{\sqrt{m}}{(j+1)(m-j)^{1/2}}\right] \\
&\quad \times \left[\sum_{k=0}^{[n/2]} \frac{1}{k+1} + \sum_{k=[n/2]+1}^{n-1} \frac{\sqrt{n}}{(k+1)(n-k)^{1/2}}\right] \operatorname{osc}_2(\phi, I_{j,x}, I_{k,y}) \\
&= \left[\sum_{j=0}^{[m/2]}\sum_{k=0}^{[n/2]} \frac{1}{(j+1)(k+1)} + \sum_{j=[m/2]+1}^{m-1}\sum_{k=0}^{[n/2]} \frac{\sqrt{m}}{(j+1)(k+1)(m-j)^{1/2}}\right. \\
&\quad + \sum_{j=0}^{[m/2]}\sum_{k=[n/2]+1}^{n-1} \frac{\sqrt{n}}{(j+1)(k+1)(n-k)^{1/2}}
\end{aligned}$$

$$+\sum_{j=[m/2]+1}^{m-1}\sum_{k=[n/2]+1}^{n-1}\frac{\sqrt{mn}}{(j+1)(k+1)(m-j)^{1/2}(n-k)^{1/2}}\Bigg]\operatorname{osc}_2(\phi,I_{j,x},I_{k,y})$$
$$=\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D},\ \text{say.} \tag{26}$$

Now, applying Hölder's inequality with given p, q and taking r, s such that $1/p+1/r=1$, $1/q+1/s=1$, we have

$$\begin{aligned}
\mathrm{A}&=\sum_{j=0}^{[m/2]}\frac{1}{(j+1)}\left(\sum_{k=0}^{[n/2]}\frac{1}{(k+1)^{1-1/q^2+1/q^2}}\operatorname{osc}_2(\phi,I_{j,x},I_{k,y})\right)\\
&\leqslant\sum_{j=0}^{[m/2]}\frac{1}{(j+1)}\left(\sum_{k=0}^{[n/2]}\frac{1}{(k+1)^{1/q}}(\operatorname{osc}_2(\phi,I_{j,x},I_{k,y}))^q\right)^{1/q}\left(\sum_{k=0}^{[n/2]}\frac{1}{(k+1)^{s-s/q^2}}\right)^{1/s}\\
&=\sum_{j=0}^{[m/2]}\frac{1}{(j+1)}\left(\sum_{k=0}^{[n/2]}\frac{1}{(k+1)^{1/q}}(\operatorname{osc}_2(\phi,I_{j,x},I_{k,y}))^q\right)^{1/q}\left(\sum_{k=0}^{[n/2]}\frac{1}{(k+1)^{1+1/q}}\right)^{1/s}\\
&\ll\sum_{j=0}^{m}\frac{1}{(j+1)^{1-1/p^2+1/p^2}}\left(\sum_{k=0}^{n}\frac{1}{(k+1)^{1/q}}(\operatorname{osc}_2(\phi,I_{j,x},I_{k,y}))^q\right)^{1/q}\\
&\ll\left(\sum_{j=0}^{m}\frac{1}{(j+1)^{1/p}}\left(\sum_{k=0}^{n}\frac{1}{(k+1)^{1/q}}(\operatorname{osc}_2(\phi,I_{j,x},I_{k,y}))^q\right)^{p/q}\right)^{1/p}\\
&\quad\times\left(\sum_{j=0}^{m}\frac{1}{(j+1)^{r-r/p^2}}\right)^{1/r}\\
&\ll\left(\sum_{j=0}^{m}\frac{1}{(j+1)^{1/p}}\left(\sum_{k=0}^{n}\frac{1}{(k+1)^{1/q}}(\operatorname{osc}_2(\phi,I_{j,x},I_{k,y}))^q\right)^{p/q}\right)^{1/p}.
\end{aligned}\tag{27}$$

Now, first fixing j, we define

$$a_{j,0}=0,\text{for } t\in[0,1) \text{ and } a_{j,t}=\sum_{l=0}^{[t]-1}\frac{1}{(l+1)^{\beta}}(\operatorname{osc}_2(\phi,I_{j,x},I_{l,y}))^q,\text{ for } t\in[1,n+1]. \tag{28}$$

Now, as $1-\beta q\geqslant 0\implies\frac{1}{q}\geqslant\beta$, and using partial summation formula (see (11) of Lemma 4), we have

$$\begin{aligned}
\sum_{k=0}^{n}\frac{1}{(k+1)^{1/q-\beta}}\frac{(\operatorname{osc}_2(\phi,I_{j,x},I_{k,y}))^q}{(k+1)^{\beta}}&=\sum_{k=0}^{n}\frac{1}{(k+1)^{1/q-\beta}}(a_{j,k+1}-a_{j,k})\\
&=\sum_{k=0}^{n-1}\left(\frac{1}{(k+1)^{1/q-\beta}}-\frac{1}{(k+2)^{1/q-\beta}}\right)a_{j,k+1}\\
&\quad+\frac{1}{(n+1)^{1/q-\beta}}a_{j,n+1}.
\end{aligned}\tag{29}$$

Since $a_{j,t}$ is non-decreasing on $t \in [k+1,k+2]$ for fixed j and $k \in [0,n-1]$, we have

$$\frac{1}{(1/q-\beta)}\sum_{k=0}^{n-1}\left(\frac{1}{(k+1)^{1/q-\beta}}-\frac{1}{(k+2)^{1/q-\beta}}\right)a_{j,k+1}=\sum_{k=0}^{n-1}a_{j,k+1}\int_{k+1}^{k+2}\frac{1}{t^{1+1/q-\beta}}dt \leqslant \int_{1}^{n+1}\frac{a_{j,t}}{t^{1+1/q-\beta}}dt. \tag{30}$$

Now, changing variable t by $\frac{n+1}{s}$, we have $t\to 1 \iff s\to n+1$, $t\to n+1 \iff s\to 1$, and $\frac{dt}{ds}=(-1)\frac{n+1}{s^2}$. Therefore

$$\begin{aligned}\int_{1}^{n+1}\frac{a_{j,t}}{t^{1+1/q-\beta}}dt &= \int_{1}^{n+1}a_{j,\left[\frac{n+1}{s}\right]}\left(\frac{s}{n+1}\right)^{1+1/q-\beta}\left(\frac{n+1}{s^2}\right)ds\\ &= \frac{1}{(n+1)^{1/q-\beta}}\sum_{k=0}^{n-1}\int_{k+1}^{k+2}a_{j,\left[\frac{n+1}{s}\right]}\left(\frac{1}{s^{1-1/q+\beta}}\right)ds\\ &\leqslant \frac{1}{n^{1/q-\beta}}\sum_{k=0}^{n-1}\frac{a_{j,\left[\frac{n+1}{k+1}\right]}}{(k+1)^{1/s+\beta}} \ll a_{j,n+1}.\end{aligned} \tag{31}$$

Therefore, from (29), (30), and (31), we have

$$\sum_{k=0}^{n}\frac{1}{(k+1)^{1/q}}(\operatorname{osc}_2(\phi,I_{j,x},I_{k,y}))^q \ll a_{j,n+1}. \tag{32}$$

Similarly, for fixed n, defining

$$b_{0,n}=0 \ \text{ for } t\in[0,1) \ \text{ and } \ b_{t,n}=\sum_{j=0}^{[t]-1}\frac{1}{(j+1)^{\alpha}}\left(a_{j,n+1}\right)^{p/q} \ \text{ for } t\in[1,m+1],$$

we can prove the following inequality

$$\begin{aligned}&\sum_{j=0}^{m}\frac{1}{(j+1)^{1/p}}\left(\sum_{k=0}^{n}\frac{1}{(k+1)^{1/q}}(\operatorname{osc}_2(\phi,I_{j,x},I_{k,y}))^q\right)^{p/q}\\ &\quad\ll \sum_{j=0}^{m}\frac{1}{(j+1)^{1/p}}\left(a_{j,n+1}\right)^{p/q} \ll \frac{1}{m^{1/p-\alpha}}\sum_{j=1}^{m}\frac{1}{j^{1/r+\alpha}}b_{\left[\frac{m+1}{j}\right],n}.\end{aligned} \tag{33}$$

Now, by definition of $a_{j,n+1}$ and $b_{t,n}$, we have

$$\begin{aligned}b_{\left[\frac{m+1}{j}\right],n} &= \sum_{j'=0}^{\left[\frac{m+1}{j}\right]-1}\frac{1}{(j'+1)^{\alpha}}(a_{j',n+1})^{p/q}\\ &= \sum_{j'=0}^{\left[\frac{m+1}{j}\right]-1}\frac{1}{(j'+1)^{\alpha}}\left(\sum_{l=0}^{n}\frac{1}{(l+1)^{\beta}}(\operatorname{osc}_2(\phi,I_{j',x},I_{l,y}))^q\right)^{p/q}\\ &\leqslant \mathrm{V}^p_{pm^{\alpha},qn^{\beta}}\left(\phi,\left[x-\frac{(1+x)}{j},x\right],[-1,y]\right).\end{aligned} \tag{34}$$

Therefore, from (27), (33), and (34), we have

$$\mathrm{A} \ll \left(\frac{1}{m^{1/p-\alpha}} \sum_{j=1}^{m} \frac{1}{j^{1/r+\alpha}} \mathrm{V}^{p}_{pm^{\alpha},qn^{\beta}} \left(\phi, \left[x - \frac{(1+x)}{j}, x \right], [-1,y] \right) \right)^{1/p}$$
$$= o(1) \text{ as } m \to \infty. \tag{35}$$

Also, we have the following inequality

$$\begin{aligned}\sum_{j=[m/2]+1}^{m-1} \frac{\sqrt{m}}{(j+1)(m-j)^{1/2}} &\ll \frac{1}{m} \sum_{j=1}^{m-1} \left(\frac{m-j}{m} \right)^{-1/2} \\ &= \sum_{j=1}^{m-1} \left(\frac{m-j}{m} \right)^{-1/2} \left[\frac{m-j}{m} - \frac{m-j-1}{m} \right] \\ &\ll \sum_{j=1}^{m-1} \int_{\frac{m-j-1}{m}}^{\frac{m-j}{m}} x^{-1/2} dx \ll \int_0^1 x^{-1/2} dx = 2. \end{aligned} \tag{36}$$

Using (36) and Hölder's inequality with $1/q + 1/s = 1$, we have

$$\begin{aligned}\mathrm{B} &= \sum_{j=[m/2]+1}^{m-1} \frac{\sqrt{m}}{(j+1)(m-j)^{1/2}} \sum_{k=0}^{[n/2]} \frac{1}{(k+1)} \mathrm{osc}_2(\phi, I_{j,x}, I_{k,y}) \\ &\ll \sum_{k=0}^{n} \frac{1}{(k+1)^{1-1/q^2+1/q^2}} \mathrm{osc}_2(\phi, [-1,x], I_{k,y}) \\ &\ll \left(\sum_{k=0}^{n} \frac{1}{(k+1)^{1/q}} (\mathrm{osc}_2(\phi, [-1,x], I_{k,y}))^q \right)^{1/q} \left(\sum_{k=0}^{n} \frac{1}{(k+1)^{s-s/q^2}} \right)^{1/s} \\ &\ll \left(\sum_{k=0}^{n} \frac{1}{(k+1)^{1/q}} (\mathrm{osc}_2(\phi, [-1,x], I_{k,y}))^q \right)^{1/q} \\ &\ll \left(\frac{1}{n^{1/q-\beta}} \sum_{k=0}^{n-1} \frac{1}{(k+1)^{1/s+\beta}} \sum_{l=0}^{\left[\frac{n+1}{k+1}\right]-1} \frac{1}{(l+1)^{\beta}} (\mathrm{osc}_2(\phi, [-1,x], I_{l,y}))^q \right)^{1/q} \\ &\ll \left(\frac{1}{n^{1/q-\beta}} \sum_{k=0}^{n-1} \frac{1}{(k+1)^{1/s+\beta}} \mathrm{V}^{q}_{pm^{\alpha},qn^{\beta}} \left(\phi, [-1,x], \left[y - \frac{(1+y)}{k+1}, y \right] \right) \right)^{1/q} \\ &= o(1) \text{ as } n \to \infty. \end{aligned} \tag{37}$$

Since C is symmetric to B, we can prove

$$\mathrm{C} = o(1) \text{ as } m \to \infty. \tag{38}$$

Also, for $j \in \left[\left[\frac{m}{2}\right] + 1, m-1 \right]$ and $k \in \left[\left[\frac{n}{2}\right] + 1, n-1 \right]$, $\mathrm{osc}_2(\phi, I_{j,x}, I_{k,y}) \to 0$ as $m, n \to$

∞, we have

$$\begin{aligned}
\mathrm{D} &= \sum_{j=[m/2]+1}^{m-1} \sum_{k=[n/2]+1}^{n-1} \frac{\sqrt{mn}}{(j+1)(k+1)(m-j)^{1/2}(n-k)^{1/2}} \operatorname{osc}_2(\phi, I_{j,x}, I_{k,y}) \\
&\ll \sum_{j=[m/2]+1}^{m-1} \sum_{k=[n/2]+1}^{n-1} \frac{1}{m^{1/2}n^{1/2}(m-j)^{1/2}(n-k)^{1/2}} \operatorname{osc}_2(\phi, I_{j,x}, I_{k,y}) \\
&= o(1) \sum_{j=[m/2]+1}^{m-1} \sum_{k=[n/2]+1}^{n-1} \frac{1}{m^{1/2}n^{1/2}(m-j)^{1/2}(n-k)^{1/2}} = o(1) \text{ as } m,n \to \infty. \qquad (39)
\end{aligned}$$

Therefore, from (26), (35), (37), (38), and (39), we have

$$\mathrm{A}_1 \ll \mathrm{A} + \mathrm{B} + \mathrm{C} + \mathrm{D} = o(1) \text{ as } m,n \to \infty. \qquad (40)$$

Similar way, we can prove

$$\begin{aligned}
\mathrm{A}_2 &= \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \int_{J_{j,x}} \int_{I_{k,y}} g(u,v) X_m(u) Y_n(v)\, du\, dv = o(1) \text{ as } m,n \to \infty, \\
\mathrm{A}_3 &= \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \int_{I_{j,x}} \int_{J_{k,y}} g(u,v) X_m(u) Y_n(v)\, du\, dv = o(1) \text{ as } m,n \to \infty,
\end{aligned} \qquad (41)$$

and

$$\mathrm{A}_4 = \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \int_{J_{j,x}} \int_{J_{k,y}} g(u,v) X_m(u) Y_n(v)\, du\, dv = o(1) \text{ as } m,n \to \infty. \qquad (42)$$

Now, as $\int_{-1}^{1} Y_n(v)\,dv = 1$ by (12), and decomposing the integral on A_5, we have

$$\begin{aligned}
\mathrm{A}_5 &= \int_{-1}^{1} \int_{-1}^{1} \phi(u,y) X_m(u) Y_n(v)\, du\, dv \\
&= \int_{-1}^{1} \phi(u,y) X_m(u)\, du \\
&= \sum_{j=0}^{m-1} \int_{t_{j+1,x}}^{t_{j,x}} (\phi(u,y) - \phi(t_{j,x},y)) X_m(u)\, du + \sum_{j=1}^{m-1} \int_{t_{j+1,x}}^{t_{j,x}} \phi(t_{j,x},y) X_m(u)\, du \\
&\quad + \int_{x}^{1} \phi(u,y) X_m(u)\, du \\
&= \mathrm{A}_{51} + \mathrm{A}_{52} + \mathrm{A}_{53}, \text{ say.}
\end{aligned} \qquad (43)$$

Now, from (16), (17), (36) and using Hölder's inequality, we first have

$$\begin{aligned}
|\mathrm{A}_{51}| &\leqslant \sum_{j=0}^{m-1} \int_{t_{j+1,x}}^{t_{j,x}} |\phi(u,y) - \phi(t_{j,x},y)| |X_m(u)|\, du \\
&\leqslant \sum_{j=0}^{m-1} \frac{\sqrt{m}}{(j+1)\sqrt{m-j}} \operatorname{osc}_1(\phi(\cdot,y), [t_{j+1,x}, t_{j,x}])
\end{aligned}$$

$$\begin{aligned}
&\ll \sum_{j=0}^{[m/2]} \frac{1}{(j+1)^{1-1/p^2+1/p^2}} \operatorname{osc}_1(\phi(\cdot,y),[t_{j+1,x},t_{j,x}]) \\
&\quad + \sum_{j=[m/2]+1}^{m-1} \frac{1}{\sqrt{m(m-j)}} \operatorname{osc}_1(\phi(\cdot,y),[t_{j+1,x},t_{j,x}]) \\
&\ll \left(\sum_{j=0}^{[m/2]} \frac{1}{(j+1)^{1/p}} (\operatorname{osc}_1(\phi(\cdot,y),[t_{j+1,x},t_{j,x}]))^p \right)^{1/p} \\
&\quad + \sum_{j=[m/2]+1}^{m-1} \frac{1}{\sqrt{m(m-j)}} \operatorname{osc}_1(\phi(\cdot,y),[t_{j+1,x},t_{j,x}]). \qquad (44)
\end{aligned}$$

Now, defining

$$a'_t = 0 \text{ for } t \in [0,1) \text{ and } a'_t = \sum_{i=0}^{[t]-1} \frac{1}{(i+1)^{\alpha}} (\operatorname{osc}_1(\phi(\cdot,y),I_{i,x}))^p \text{ for } t \in [1,m+1].$$

Then, proceeding as in (28) to (31), we have

$$\begin{aligned}
&\sum_{j=0}^{m-1} \frac{1}{(j+1)^{1/p}} (\operatorname{osc}_1(\phi(\cdot,y),[t_{j+1,x},t_{j,x}]))^p \\
&\qquad \leqslant \frac{1+1/p-\alpha}{(m+1)^{1/p-\alpha}} \sum_{j=0}^{m} \frac{1}{(j+1)^{1-1/p+\alpha}} a'_{\left[\frac{m+1}{j+1}\right]} \\
&\qquad = o(1) \text{ as } m \to \infty. \qquad (45)
\end{aligned}$$

Also,

$$\sum_{j=[m/2]+1}^{m-1} \frac{1}{m^{1/2}(m-j)^{1/2}} \operatorname{osc}_1(\phi(\cdot,y),[t_{j+1,x},t_{j,x}]) = o(1) \text{ as } m \to \infty. \qquad (46)$$

Now, using partial summation formula (see (10) of Lemma 4) with $a_j = \phi(t_{j,x},y)$, $b_j = \int\limits_{t_{j+1,x}}^{t_{j,x}} X_m(u)du$ and as $a_0 = 0$, we have

$$\begin{aligned}
\mathrm{A}_{52} &= \sum_{j=1}^{m-1} \phi(t_{j,x},y) \int_{t_{j+1,x}}^{t_{j,x}} X_m(u)du \\
&= \sum_{j=1}^{m-1} \int_{-1}^{t_{j,x}} \left(\phi(t_{j,x},y) - \phi(t_{j-1,x},y)\right) X_m(u)du.
\end{aligned}$$

Now, using Hölder's inequality and in view of (21), we have

$$
\begin{aligned}
|\mathrm{A}_{52}| &\leqslant \sum_{j=1}^{m-1} |\phi(t_{j,x},y)-\phi(t_{j-1,x},y)| \left| \int_{-1}^{x-\frac{j(1+x)}{m}} X_m(u)du \right| \\
&\leqslant \sum_{j=1}^{m-1} \frac{1}{j} \operatorname{osc}_1(\phi(\cdot,y),[t_{j,x},t_{j-1,x}]) \\
&\leqslant \sum_{j=0}^{m-1} \frac{1}{(j+1)^{1-1/p^2+1/p^2}} \operatorname{osc}_1(\phi(\cdot,y),[t_{j+1,x},t_{j,x}]) \\
&\ll \left(\sum_{j=0}^{m-1} \frac{1}{(j+1)^{1/p}} (\operatorname{osc}_1(\phi(\cdot,y),[t_{j+1,x},t_{j,x}]))^p \right)^{1/p}. \qquad (47)
\end{aligned}
$$

Therefore, from (43)–(47), we have

$$\mathrm{A}_{51}+\mathrm{A}_{52}=o(1) \ \text{ as } \ m\to\infty, \qquad (48)$$

similarly, we can have

$$\mathrm{A}_{53}=\sum_{j=0}^{m-1}\int_{s_{j,x}}^{s_{j+1,x}} \phi(u,y)X_m(u)du=o(1) \ \text{ as } \ m\to\infty. \qquad (49)$$

Therefore, from (48) and (49), we have

$$\mathrm{A}_5=o(1) \ \text{ as } \ m\to\infty, \qquad (50)$$

and proceeding similarly as in A_5, we can have

$$\mathrm{A}_6=o(1) \ \text{ as } \ n\to\infty. \qquad (51)$$

Also, using an inequality (see [4, Lemma 1])

$$|P_m(x)| \leqslant \left(\frac{2}{\pi}\right)^{1/2} (1-x^2)^{-1/2} m^{-1/2}, \ x\in(-1,1),$$

last three terms on the right-hand side of the equation (13) tend to zero as $m,n\to\infty$. Therefore, from (13), (14), (40), (41), (50), and (51), we have

$$S_{m,n}(f,x,y)\to s(f,x,y) \ \text{ as } \ m,n\to\infty. \quad \square$$

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