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ON STATISTICAL CONVERGENCE OF SET SEQUENCES IN FUZZY ANTI–NORMED LINEAR SPACE

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Abstract. The current study aims to investigate the notion of set sequences in the context of fuzzy anti-norm linear space. We define the fuzzy α-anti Wijsman convergence, fuzzy α-anti Wijsman statistical convergence, and fuzzy α-anti Hausdorff statistical convergence, for $0<\alpha<1$, and presented some novel examples to support the definitions. Additionally, some original theorems which show how the notions are connected, are established.

1. Introduction

The notion of set convergence has an extensive background in mathematics; nevertheless, mathematicians have now realised that the concept is important in dealing with the system of equations, approximation in optimization, and other correlated topics. Wijsman [20] analysed the sequences of convex sets, cones, and functions in 1964 and presented some important conclusions on the subject. Since then, numerous authors worked on the topic and developed some notable conclusions. Recently, Nuray and Rhoades [11] provided the notion of Wijsman statistical convergence and Hausdorff statistical convergence of set sequences and studied summability methods for set sequences in 2012, which emerged as a helpful tool in further investigation of the topics. In addition, Ulusu and Nuray [19] introduced the concept of asymptotically lacunary statistical equivalent sequences of sets in the same year. Following that, Hazarika and Esi [5] presented the definitions of λ-asymptotically equivalent sequences of sets and introduced certain properties of the theory.

The ongoing study draw the idea of set sequences and statistical convergence in the fuzzy anti norm linear space. The concept of statistical convergence was first identified by Fast [4] in 1951, and the same is investigated by Schoenberg [18] in 1959, with some further basic properties of statistical convergence and, proposed the idea of summability theory. As Fast [4], the idea is based on the theory of natural density, where the natural density of a set $B\subseteq\mathbb{N}$, is denoted by $\delta(B)$ and defined as

$$\lim_{q\to\infty}\frac{1}{q}\left|\{r\leqslant q\ :\ r\in B\}\right|. \tag{1}$$

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A sequence $x = \langle x_r \rangle$ is called statistical convergent to λ if for every $\varepsilon > 0$ natural density of the set $\{r \leqslant q \,:\, |x_r - \lambda| \geqslant \varepsilon\}$ is 0. In notation, we denote it by $\mathrm{st}-\lim x = \lambda$.

The notion of fuzzy set was first examined by L. Zadeh [21] in 1965. Following that, other authors also explored the concept from their own perspectives and used it widely in various fields of sciences and engineering. Some scholars have recently published various types of work based on the idea of fuzzy theory and its extensions, notably on intuitionistic fuzzy theory (see [7, 10, 12, 13, 14, 16]). Bag and Samanta [1] proposed the concept of fuzzy norm in 2008. Afterwards, Jebril and Samanta [6] investigated the idea of fuzzy anti norm in slightly different way and made a remarkable comparison with the earlier notion of fuzzy anti-norm studied in [1] in 2010. Further, Dinda et al. [3] made a necessary revision in the literature studied in [6] and offered a new way to define the fuzzy anti-norm linear space by dropping redundant conditions from definition given in [6]. Recently, some authors analysed the concept of fuzzy anti-normed linear spaces and studied a relationship to various types of sequence spaces in their own approach [9, 8, 15] and drawn some key conclusions to the related topic.

The content are divided into four sections. Section 1, comprises introduction with some historical context; Section 2 contains the background, methodology, and some useful results with an original example that are employed in the research; Sections 3 and 4 are motivated to the primary results that are explored in the proposed work; and Section 5 is a conclusion.

Throughout the article $\mathbb{N}$ denotes set of natural numbers, $\mathbb{R}$ stands for set of real numbers and M is taken to deal with a normed linear space, further E^2 deals the cross product of set E to itself.

2. Background

DEFINITION 1. [17] Let $D = [0,1]$ then a binary operation $\diamond : D^2 \to D$ such that $\diamond$ is continuous, $\diamond$ is commutative, $\diamond$ is associative, $\diamond$ is monotonically increasing, i.e. for all $x,y,w,z \in [0,1]$; $x \diamond w \leqslant y \diamond z$, whenever $x \leqslant y$ and $w \leqslant z$, and for all $w \in [0,1]$, $w \diamond 0 = w$ is defined as a continuous t-conorm.

REMARK 1. [9]

(i) For every pair $0 < a_1 < a_2 < 1$ we can find $0 < b < 1$ such that $b \geqslant a_1 \diamond a_2$.

DEFINITION 2. [3] Let M be a linear space over the field $\mathbb{F}$ and $\diamond$ be t-conorm then the triplet $(M, \psi, \diamond)$ where $\psi : M \times \mathbb{R} \to [0,1]$ is a fuzzy anti-norm on M with respect to t-conorm $\diamond$, is said to be a fuzzy anti-normed linear space (FANLS), if for all $w,\ z \in M$ and for $\tau,\ t \in \mathbb{R}$ the following holds:

(i) for $\tau \leqslant 0$; $\psi(w, \tau) = 1$,

(ii) for $\tau > 0$; $\psi(w, \tau) = 0$ iff $w = 0$,

(iii) for $\tau > 0$; $\psi(\lambda w, \tau) = \psi\left(w, \dfrac{\tau}{|\lambda|}\right)$ for $\lambda \neq 0$,

(iv) for t and $\tau \in \mathbb{R}$; $\psi(w+z,\tau+t) \leqslant \psi(w,\tau) \diamond \psi(z,t)$,

(v) $\lim_{\tau \to 0} \psi(w,\tau) = 0$.

NOTE. The function ψ is called fuzzy anti-norm (FAN) on M and, further $\psi(w,\tau)$ is a non-increasing function of τ, that is, $\psi(w,\tau_1) \leqslant \psi(w,\tau_2)$ whenever $\tau_1 > \tau_2$.

REMARK 2. Throughout the article we deal with FANLS $(M,\psi,\diamond)$ and assume some further conditions which are follows as:
(P_1) if $\psi(w,\tau) < 1$ for all $\tau > 0$ then $w = 0$.
(P_2) $\psi(w,\tau) : \mathbb{R} \to [0,1]$ is a continuous function of τ and strictly decreasing on the set $\{\tau : 0 < \psi(w,\tau) < 1\} \subset \mathbb{R}$.
(P_3) t-conorm $\diamond$ is idempotent, i.e. $w \diamond w = w \ \forall \ w \in [0,1]$.

We now present some examples of fuzzy anti-norm and fuzzy anti-norm linear space with respect to t-conorm $\diamond$.

EXAMPLE 1. Let $(X, \|.\|)$ be a normed linear space with norm $\|.\|$ and if we define $x \diamond y = x + y - xy$ and $\psi : X \times \mathbb{R} \to [0,1]$ given by

$$\psi(w,\tau) = \begin{cases} 0; & \text{if } \tau > \|w\| \\ 1; & \text{if } \tau \leqslant \|w\| \end{cases}$$

then $(X,\psi,\diamond)$ is a fuzzy anti-norm linear space.

EXAMPLE 2. Let $(X, \|.\|)$ be a normed linear space with norm $\|.\|$ and if we define $x \diamond y = \min\{x+y,1\}$ and $\psi : X \times \mathbb{R} \to [0,1]$ given by

$$\psi(w,\tau) = \begin{cases} 0; & \text{if } \tau > \|w\| \\ \dfrac{\|w\|}{\tau + \|w\|}; & \text{if } \tau \leqslant \|w\| \\ 1; & \text{if } \tau \leqslant 0 \end{cases}$$

then $(X,\psi,\diamond)$ is a fuzzy anti-norm linear space with respect to t-conorm $\diamond$.

DEFINITION 3. [3] Let $(M,\psi,\diamond)$ FANLS with respect to idempotent t-conorm $\diamond$ and satisfies (P_1) then the function $\|u\|_\alpha^* : M \to [0,\infty)$, where $0 < \alpha < 1$ defines a norm and

$$\|u\|_\alpha^* = \min\{\tau > 0 : \psi(u,\tau) \leqslant 1-\alpha\}. \tag{2}$$

Note if $0 < \alpha_1 \leqslant \alpha_2 < 1 \Rightarrow \|u\|_{\alpha_1}^* \leqslant \|u\|_{\alpha_2}^*$ on $(M,\psi,\diamond)$.

LEMMA 1. *Let $(M,\psi,\diamond)$ be an FANLS with respect to idempotent t-conorm $\diamond$ and satisfies (P_1) then for non-zero $u \in M$ and for $0 < \alpha < 1$ and $\tau > 0$ we have*

$$\|u\|_\alpha^* = \tau \iff \psi(u,\tau) = 1-\alpha \tag{3}$$

where $\|u\|_\alpha$ is given by equation (2).

DEFINITION 4. [11] Let (M,p) be a norm linear space with norm p and let $E \subseteq M$ then for any $x \in M$, we define $d(x,E)$ as

$$d(x.E) = \inf_{y \in E} p(x-y) \tag{4}$$

where $d(x,E)$ gives distance of E from x in M.

DEFINITION 5. Let (M,p) be a normed linear space with norm p, then for a sequence $\langle E_r \rangle$, $E_r \neq \emptyset$ and closed for each r in M we define

$$d_r(x) = d(x,E_r) = \inf_{y \in E_r} p(x-y).$$

Further if $E \subseteq M$ where $E \neq \emptyset$ and closed in M then we define

$$d(x) = d(x,E) = \inf_{y \in E} p(x-y).$$

The set sequence $\langle E_r \rangle$ of non-empty and closed subsets in M is said to be Wijsman convergent to a closed and non-empty set E if for each $x \in M$ and for a given $\varepsilon > 0$ there exists a $r_0 \in \mathbb{N}$ such that

$$|d_r(x) - d(x)| < \varepsilon \quad \forall \ r \geqslant r_0 \tag{5}$$

or

$$\lim_{r \to \infty} d_r(x) = d(x). \tag{6}$$

In notation, we denote the case as $W - \lim E_r = E$ (see [11]).

NOTE. Throughout the study, the notion requires that each member of the set sequence $\langle E_r \rangle$ and the limit set E are to be non-empty and closed subset of M otherwise in Wijsman convergence, the limit set may not be well defined (see [2, 4]).

DEFINITION 6. Let (M,p) be a normed linear space with norm p, then a set sequence $\langle E_r \rangle$ is said to be Wijsman statistical convergence to the limit set $E \subseteq M$ if for each $x \in M$ the sequence $\langle d_r(x) \rangle$ is statistically convergent to $d(x)$ i.e. for every $\varepsilon > 0$

$$\lim_{q \to \infty} \frac{1}{q} |\{r \leqslant q : |d_r(x) - d(x)| \geqslant \varepsilon\}| = 0. \tag{7}$$

The case is also stated as $|d_r(x) - d(x)| < \varepsilon$ for a. a. r (read as for almost all r) and in notation, we denote it by $st - \lim_W E_r = E$.

DEFINITION 7. Let $(M, \psi, \diamond)$ be an FANLS. Then a set sequence $\langle E_r \rangle$ in M is said to be Wijsman convergent to the limit set $E \subseteq M$ with respect to FAN ψ and ψ satisfies property (P_1), if for every $0 < \varepsilon < 1$ and $\tau > 0$ there exists a $r_0 \in \mathbb{N}$ such that

$$\psi(d_r(x) - d(x), \tau) < \varepsilon \quad \text{for every} \quad r \geqslant r_0 \tag{8}$$

for each $x \in M$.

DEFINITION 8. [3] Let $(M,\psi,\diamond)$ be an FANLS. Then a set sequence $\langle E_r\rangle$ in M for $0<\alpha<1$ is said to be fuzzy α-anti Wijsman convergent to the limit set $E\subseteq M$ with respect to FAN ψ and ψ satisfies property (P_1), if for each $x\in M$ and $\tau>0$ there exists a $r_0\in\mathbb{N}$ such that

$$\lim_{r\to\infty}\psi\left(d_r(x)-d(x),\tau\right)<1-\alpha. \tag{9}$$

In notation, we state it as α-anti $\lim_W E_r \xrightarrow{\psi} E$ and E is called fuzzy α-anti Wijsman limit of set sequence $\langle E_r\rangle$.

EXAMPLE 3. Let $\left(X=\mathbb{R}^2,\|.\|\right)$ is a normed linear space and if $x\diamond y=\max\{x,y\}$ and $\psi: X\times\mathbb{R}\to[0,1]$ given by

$$\psi(w,\tau)=\begin{cases}\dfrac{\|w\|}{\tau+\|w\|}: & \text{if } \tau>0\\ 1: & \text{if } \tau\leqslant 0.\end{cases}$$

For the sequence $\langle E_r\rangle$ of subsets of $\mathbb{R}^2$ over FANLS $\left(\mathbb{R}^2,\psi,\diamond\right)$ defined by

$$E_r=\left\{(u,v): u^2+4v^2=\frac{1}{r^2}\right\} \tag{10}$$

then fuzzy α-anti $\lim_W E_r \xrightarrow{\psi} \{(0,0)\}$.

EXPLANATION 1. We define

$$d_r(x)=\inf_{z\in E_r}\|x-z\| \text{ where } x=(a,b)\in X \text{ and } z=(u,v)\in E_r \tag{11}$$

then for $0\leqslant\theta\leqslant 2\pi$

$$\|x-z\|=\sqrt{(u-a)^2+(v-b)^2}=\sqrt{\left(\frac{1}{r}\cos\theta-a\right)^2+\left(\frac{1}{2r}\sin\theta-b\right)^2} \tag{12}$$

and thus for every $x\in M$

$$\lim_{r\to\infty} d_r(x)=\inf_{z\in E_r}\|x-z\|=\sqrt{a^2+b^2}=d(x) \tag{13}$$

and then

$$\lim_{r\to\infty}\left(d_r(x)-d(x),\tau\right)<1-\alpha \text{ for every } x\in M,\ 0<\alpha<1 \text{ and } \tau>0. \tag{14}$$

THEOREM 1. *Let $(M,\psi,\diamond)$ be an FANLS with a idempotent t-conorm $\diamond$ which satisfies the property (P_1) and if for $0<\alpha<1$, the set sequence $\langle E_r\rangle$ in M fuzzy α-anti Wijsman convergent to E with regard to FAN ψ, then fuzzy α-anti Wijsman limit E is unique.*

Proof. If possible, we suppose the sequence $\langle E_r \rangle$ of sets in $(M, \psi, \diamond)$, fuzzy α-anti Wijsman convergent to two limit sets E and F where $0 < \alpha < 1$, the sets E and F are non-empty and closed in M. We define $d' = d(x, F)$ and then for every $\tau > 0$ and every $x \in M$ we have

$$\lim_{r \to \infty} \psi\left(d_r(x) - d(x), \frac{\tau}{2}\right) < 1 - \alpha \text{ and } \lim_{r \to \infty} \psi\left(d_r(x) - d'(x), \frac{\tau}{2}\right) < 1 - \alpha. \tag{15}$$

Now

$$\psi\left(d(x) - d'(x), \tau\right) \leqslant \psi\left(d_r(x) - d(x), \frac{\tau}{2}\right) \diamond \psi\left(d_r(x) - d'(x), \frac{\tau}{2}\right). \tag{16}$$

If $r \to \infty$ then

$$\lim_{r \to \infty} \psi\left(d(x) - d'(x), \frac{\tau}{2}\right) < (1 - \alpha) \diamond (1 - \alpha) < 1 - \alpha < 1 \quad \text{for every } x \text{ and } \tau. \tag{17}$$

By property (P_1) we obtain $d(x) = d'(x)$ that is E coincides with F. □

DEFINITION 9. Let $(M, \psi, \diamond)$ be an FANLS with a t-conorm $\diamond$ which satisfies the property (P_1). Then for $0 < \alpha < 1$, the set sequence $\langle E_r \rangle$ in M is said to be fuzzy α-anti Wijsman Cauchy sequence with regard to FAN ψ if for every $\tau > 0$ we can find a $r_0 \in \mathbb{N}$ such that $\psi(d_r(x) - d_q(x)) < 1 - \alpha \ \forall \ r, q \geqslant r_0$.

THEOREM 2. *Let $(M, \psi, \diamond)$ be an FANLS with a idempotent t-conorm $\diamond$ which satisfies the property (P_1). Then for $0 < \alpha < 1$ every fuzzy α-anti Wijsman cgt set sequence $\langle E_r \rangle$ in M is fuzzy α-anti Wijsman Cauchy sequence with regard to FAN ψ.*

Proof. Let $\langle E_r \rangle$ be a fuzzy α-anti Wijsman cgt sequence of sets which converges to E with respect to FAN ψ, then for $0 < \alpha < 1$ and $\tau > 0$ there is a $r_0 \in \mathbb{N}$ such that

$$\psi\left(d_r(x) - d(x), \frac{\tau}{2}\right) < 1 - \alpha \quad \text{for each } r \geqslant r_0 \text{ and } x \in M. \tag{18}$$

Take r, $q > r_0$ then for each $x \in M$ and $\tau > 0$ we have

$$\begin{aligned} \psi\left(d_r(x) - d_q(x), \tau\right) &\leqslant \psi\left(d_r(x) - d(x), \frac{\tau}{2}\right) \diamond \psi\left(d_q(x) - d(x), \frac{\tau}{2}\right) \\ &< (1 - \alpha) \diamond (1 - \alpha) < 1 - \alpha. \quad \square \end{aligned} \tag{19}$$

3. Statistical convergence of set sequences in FANLS

In this section, we investigate the statistical convergence of set sequences in FANLS, and establish some key conclusion with some topological aspect of the concepts with respect to a fuzzy anti norm.

DEFINITION 10. Let $(M, \psi, \diamond)$ be an FANLS with a idempotent t-conorm $\diamond$ which satisfies the property (P_1). Then for $0 < \alpha < 1$, a set sequence $\langle E_r \rangle$ in M is said to be fuzzy α-anti Wijsman statistical convergent to the set $E \subseteq M$ with regard to FAN ψ if for $\tau > 0$ and for each $x \in M$

$$\lim_{q\to\infty} \frac{1}{q} \left|\{r \leqslant q \ : \ \psi\left(d_r(x) - d(x), \tau\right) \geqslant 1-\alpha\}\right| = 0 \tag{20}$$

in other words, the same is stated as $d\left(x, E_r\right) = d\left(x, E\right)$ for a.a.r (for almost all r) and, in notation we write it as $st - \lim E_r \xrightarrow{\psi} E(WS)$. The set E is known as fuzzy α-anti Wijsman statistical limit of the sequence.

EXAMPLE 4. Let's consider an FANLS $\left(\mathbb{R}^2, \psi, \diamond\right)$, where $x \diamond y = \max\{x,y\}$ and

$$\psi(w,\tau) = \begin{cases} \dfrac{\|w\|}{\tau + \|w\|} : \text{if } \ \tau > 0 \\ 1 : \qquad\qquad \text{if } \ \tau \leqslant 0 \end{cases}$$

consider the sequence $\langle E_r \rangle$ of subsets of $\mathbb{R}^2$ over FANLS $\left(\mathbb{R}^2, \psi, \diamond\right)$ where

$$E_r = \begin{cases} (x, e^{-rx}) : & 0 \leqslant x \leqslant 1 \ \text{ and } \ r \neq 5^q; \quad q \in \mathbb{N} \\ (0,y) : & 0 \leqslant y \leqslant 1 \ \text{ and } \ r = 5^q; \quad q \in \mathbb{N} \end{cases} \tag{21}$$

then fuzzy α-anti $\lim E_r \xrightarrow{\psi} \{(x,0) : 0 \leqslant x \leqslant 1\}\ (WS)$.

EXPLANATION 2. Let

$$E = \{(x,0) : 0 \leqslant x \leqslant 1\}, \ \ P = \{r : \ \ r \neq 5^q\} \ \ \text{ and } \ \ Q = \{r : \ \ r = 5^q\}. \tag{22}$$

Now

$$\delta(Q) = \lim_{q\to\infty} \frac{1}{q} |\{k \leqslant q : k = 5^n\}| \leqslant \lim_{q\to\infty} \frac{\log_5 q}{q} = \lim_{q\to\infty} \left(\frac{\ln q}{q}\right) \log_5 e = 0 \tag{23}$$

thus, for every $0 < \alpha < 1$ and $\tau > 0$ we obtain

$$T = \{k : \psi\left(d_r(x) - d(x), \tau\right) \geqslant 1-\alpha\} = F \cup Q \ \ \text{(say)} \tag{24}$$

where $F \subseteq P$ such that $|F|$ is a finite number. Then

$$\delta(T) \leqslant \delta(F) + \delta(Q) = 0 \ \text{ for every } x \tag{25}$$

which yields, fuzzy α-anti $\lim E_r \xrightarrow{\psi} E(WS)$.

DEFINITION 11. Let $(M, \psi, \diamond)$ be an FANLS with a idempotent t-conorm $\diamond$ which satisfies the property (P_1). Then for $0 < \alpha < 1$, a set sequence $\langle E_r \rangle$ in M is said to be fuzzy α-anti Wijsman statistical Cauchy sequence with regard to FAN ψ, if for $\tau > 0$ and for each $x \in M$, there exists a $q \in \mathbb{N}$ such that

$$\lim_{q\to\infty} \frac{1}{q} \left|\left\{r \ : \ \psi\left(d_q(x) - d_r(x), \tau\right) \geqslant 1-\alpha\right\}\right| = 0. \tag{26}$$

Equivalently, we state it as $\psi\left(d_q(x) - d_r(x), \tau\right) < 1-\alpha$ for $a.a.r$

THEOREM 3. *Let $(M,\psi,\diamond)$ be an FANLS with a idempotent t-conorm $\diamond$ which satisfies the property (P_1). Then for $0<\alpha<1$, every fuzzy α-anti Wijsman statistical convergent set sequence $\langle E_r\rangle$ in M is a fuzzy α-anti Wijsman statistical Cauchy sequence with regard to FAN ψ.*

Proof. Let $\langle E_r\rangle$ be a set sequence in $(M,\psi,\diamond)$ such that for $0<\alpha<1$ and for each x, fuzzy α-anti $\lim E_r \xrightarrow{\psi} E(WS)$ this implies that for each $\tau>0$ we have $\psi(d_r(x)-d(x),\tau)<1-\alpha$ for *a.a.r*.

Fix $q\in\mathbb{N}$ such that $\psi(d_q(x)-d(x),\tau)<1-\alpha$ for *a.a.r* and this implies

$$\begin{aligned}\psi(d_q(x)-d_r(x),\tau) &= \psi(d_q(x)-d(x)+d(x)-d_r(x),\tau)\\ &\leqslant \psi\left(d_q(x)-d(x),\frac{\tau}{2}\right)\diamond\psi\left(d_r(x)-d(x),\frac{\tau}{2}\right)\\ &< (1-\alpha)\diamond(1-\alpha)<1-\alpha \quad \text{for } a.a.r. \end{aligned}\tag{27}$$

Hence, there exists a $q\in\mathbb{N}$ such that $\psi(d_q(x)-d_r(x),\tau)<1-\alpha$ for *a.a.r*. □

THEOREM 4. *Let $(M,\psi,\diamond)$ be an FANLS with a idempotent t-conorm $\diamond$ which satisfies the property (P_1). Then for every fuzzy α-anti Wijsman statistically Cauchy set sequence $\langle E_r\rangle$ in M where $0<\alpha<1$, there is a fuzzy α-anti Wijsman statistically convergent sequence $\langle F_r\rangle$ in M with regard to FAN ψ such that $E_r=F_r$ for a.a.r.*

Proof. Let $\langle E_r\rangle$ be a fuzzy α-anti Wijsman statistically Cauchy set sequence in M then for every $0<\alpha<1$ and $\tau>0$ we can choose a $p_0\in\mathbb{N}$ such that

$$\psi\left(d_r(x)-d_{p_0}(x),\tau\right)<1-\alpha \quad \text{for } a.a.r \text{ and for } \ r\geqslant p_0. \tag{28}$$

We form an increasing sequence $\langle\alpha_n\rangle$ in $0<\alpha_n\leqslant 1$ such that $\alpha_n\to 1$ as $n\to\infty$ and we again choose a $p_1\in\mathbb{N}$ and we then for all $\tau>0$ obtain that

$$A_1=\left\{r:\psi(d_r(x)-d_{p_1}(x),\tau)<1-\alpha_1\right\} \quad \text{contains } \ E_r \text{ for } a.a.r \tag{29}$$

we now choose $p_2\in\mathbb{N}$ and for all $\tau>0$ we obtain that

$$A_2=\left\{r:\psi(d_r(x)-d_{p_2}(x),\tau)<1-\alpha_2\right\} \quad \text{contains } \ E_r \text{ for } a.a.r \tag{30}$$

we further assert that $A_1\cap A_2$ contains E_r for *a.a.r*, since

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\left|\{k\leqslant n:k\notin A_1\cap A_2\}\right| &\leqslant \lim_{n\to\infty}\frac{1}{n}\left|\{k\leqslant n:k\notin A_1\}\right|\\ &\quad+\lim_{n\to\infty}\frac{1}{n}\left|\{k\leqslant n:k\notin A_2\}\right|=0.\end{aligned}\tag{31}$$

If we continue the process to choose p_n and α_n as above then we arrive that $A=\bigcap_{n\geqslant 1}A_n$ contains E_r for *a.a.r*.

Now, we define a set sequence $\langle F_r \rangle$ by $F_r = E_r$ for $r \in A$ and $F_r \neq E_r$ for $r \notin A$ and then for set $\{r : F_r \neq E_r\} = A^c$ we conclude that

$$\delta(A^c) = \delta\left(\bigcup_{n\geqslant 1} A_n^c\right) \leqslant \sum_{n\geqslant 1} \delta(A_n^c) = 0. \tag{32}$$

Hence, $F_r = E_r$ for *a.a.r.* □

THEOREM 5. *Let $(M,\psi,\diamond)$ be an FANLS with a idempotent t-conorm $\diamond$ which satisfies the property (P_1). If the set sequence $\langle E_r \rangle$ in M such that for $0<\alpha<1$, fuzzy α-anti $\lim E_r \xrightarrow{\psi} E(WS)$ and for each $\tau > 0$, $\psi(d_{r+1}(x) - d_r(x), \tau) < 1-\alpha$ as $r \to \infty$ then fuzzy α-anti $\lim_W E_r \xrightarrow{\psi} E$.*

Proof. Let for $0<\alpha<1$, fuzzy α-anti $\lim E_r \xrightarrow{\psi} E(WS)$ holds true. We can find a set sequence $\langle F_r \rangle$ such that fuzzy α-anti $\lim_W F_r \xrightarrow{\psi} E$ and $E_r = F_r$ for $a.a.r$ further, for every $r \in \mathbb{N}$ we find two numbers $p(r)$ and $q(r)$ such that $p(r)+q(r) = r$ where $p(r) = \max\{j \leqslant r : E_j = F_j\}$. If $\{j \leqslant r : E_j = F_j\} = \emptyset$ then we put $p(r) = -1$ but this happens for at most finite number of r and hence we can write

$$\lim_{r\to\infty} \frac{q(r)}{p(r)} = 0 \tag{33}$$

if possible, we suppose

$$\frac{q(r)}{p(r)} = \varepsilon > 0 \text{ then } \frac{1}{r}\left|\{j \leqslant r : E_j \neq F_j\}\right| \leqslant \frac{q(r)}{r} \leqslant \frac{p(r)}{p(r)(1+\varepsilon)} \leqslant \frac{\varepsilon}{1+\varepsilon} \tag{34}$$

which leads to contradiction that $E_r = F_r$ for $a.a.r$. Hence item in equation (33) must holds. Further for $0<\alpha<1$, we are given that $\psi(d_{r+1}(x) - d_r(x), \tau) < 1-\alpha$ as $r \to \infty$. Now for every $\tau > 0$ and for every $x \in M$

$$\begin{aligned}
&\psi\left(d\left(x, F_{p(r)}\right) - d\left(x, E_r\right), \tau\right) \\
&= \psi\left(d\left(x, E_{p(r)}\right) - (x, E_r), \tau\right) \\
&\leqslant \psi\left(d\left(x, E_{p(r)}\right) - d\left(x, E_{p(r)+1}\right), \tau_1\right) \diamond \psi\left(d\left(x, E_{p(r)+1}\right) - d\left(x, E_{p(r)+2}\right), \tau_2\right) \\
&\quad \diamond \ldots, \psi\left(d\left(x, E_{r-1}\right) - d\left(x, E_r\right), \tau_r\right) \\
&< (1-\alpha) \diamond (1-\alpha) \ldots, \diamond (1-\alpha) < (1-\alpha)
\end{aligned} \tag{35}$$

where $\tau_1 + \tau_2 + \tau_3 \ldots, + \tau_r = \tau$ and each $\tau_i > 0$, since $\tau > 0$ was arbitrary, hence we therefore conclude that $\psi\left(d\left(x, F_{p(r)}\right) - d\left(x, E_r\right), \tau\right) < 1-\alpha < 1$ which yields that $d\left(x, F_{p(r)}\right) = d\left(x, E_r\right)$ for $r \to \infty$. Since fuzzy α-anti $\lim_W F_r = E$ therefore fuzzy α-anti $\lim_W E_r = E$. □

4. Hausdorff statistical convergence of set sequences in FANLS

This section aims to introduce and define the Hausdorff convergence and Hausdorff statistical convergence of set sequences in FANLS. The concept of Hausdorff convergence is obtained when point-wise convergence of sequences $\langle d_r(x)\rangle$ is replaced by uniform convergence in the study, *i.e.* we investigate the convergence of sequence $\langle d_r(x)\rangle$ to $d(x)$ for all $x \in M$ instead for each x.

DEFINITION 12. Let $(M,\psi,\diamond)$ be an FANLS with a idempotent t-conorm $\diamond$ which satisfies the property (P_1). Then for $0<\alpha<1$, a set sequence $\langle E_r\rangle$ in M is said to be fuzzy α-anti Hausdorff convergent to the set $E \subseteq M$ with regard to FAN ψ, if for $\tau > 0$

$$\sup_{x\in M}\psi\left(d_r(x)-d(x),\tau\right) < 1-\alpha \quad \text{as} \quad r\to\infty. \tag{36}$$

In notation, we denote the case as, fuzzy α-H-anti lim $E_r \xrightarrow{\psi} E$.

DEFINITION 13. Let $(M,\psi,\diamond)$ be an FANLS with a idempotent t-conorm $\diamond$ which satisfies the property (P_1). Then for $0<\alpha<1$, a set sequence $\langle E_r\rangle$ in M is said to be fuzzy α-anti Hausdorff statistically convergent to the set $E \subseteq M$ with regard to FAN ψ, if for $\tau > 0$

$$\delta\left(\{r\in\mathbb{N} : \sup_{x\in M}\psi\left(d_r(x)-d(x),\tau\right) \geqslant 1-\alpha\}\right) = 0. \tag{37}$$

In notation, we denote the case as, fuzzy α-anti lim $E_r \xrightarrow{\psi} E(HS)$.

THEOREM 6. *Let $(M,\psi,\diamond)$ be an FANLS with a idempotent t-conorm $\diamond$ which satisfies the property (P_1). Then for $0<\alpha<1$, fuzzy α-H-anti* $\lim E_r \xrightarrow{\psi} E$ *implies fuzzy α-anti* $\lim_W E_r \xrightarrow{\psi} E$.

Proof. Let's consider a set sequence $\langle E_r\rangle$ of the subsets of M is such that, for $0<\alpha<1$, fuzzy α-H-anti $\lim E_r \xrightarrow{\psi} E$, where $E \subseteq M$ therefore there exists a positive integer r_0 such that

$$\sup_{x\in M}\psi\left(d_r(x)-d(x),\tau\right) < 1-\alpha \quad \text{for all} \quad r \geqslant r_0. \tag{38}$$

We now work with the same r_0 and thus obtain

$$\psi\left(d_r(x)-d(x),\tau\right) \leqslant \sup_{x\in M}\psi\left(d_r(x)-d(x),\tau\right) < 1-\alpha \quad \text{for all} \quad r \geqslant r_0. \tag{39}$$

□

THEOREM 7. *Let $(M,\psi,\diamond)$ be an FANLS with a idempotent t-conorm $\diamond$ which satisfies the property (P_1). Then for $0<\alpha<1$, the fuzzy α-anti* $\lim E_r \xrightarrow{\psi} E(HS)$ *implies, fuzzy α-anti* $\lim E_r \xrightarrow{\psi} E(WS)$.

That is in FANLS, the Hausdorff statistical convergence implies the Wijsman statistical convergence.

Proof. Let's consider a set sequence $\langle E_r \rangle$ of the subsets of M is such that, for $0 < \alpha < 1$, fuzzy α-anti $\lim E_r \xrightarrow{\psi} E(HS)$ and, $E \subseteq M$. Further for $0 < \alpha < 1$ and $\tau > 0$ we define two sets as

$$A = \{k \in \mathbb{N} : \sup_{x \in M} \psi(d_k(x) - d(x), \tau) \geqslant 1 - \alpha\} \tag{40}$$

where $\delta(A) = 0$ and for each $x \in M$

$$B = \{k \in \mathbb{N} : \psi(d_k(x) - d(x), \tau) \geqslant 1 - \alpha\}. \tag{41}$$

We now prove that $B \subseteq A$ which completes our requirements. Let $q \in B$ then

$$\psi(d_q(x) - d(x), \tau) \geqslant 1 - \alpha \tag{42}$$

this implies

$$\sup_{x \in M} \psi(d_q(x) - d(x), \tau) \geqslant \psi(d_q(x) - d(x), \tau) \geqslant 1 - \alpha \tag{43}$$

therefore $q \in A \Rightarrow B \subseteq A$. □

Conclusion

From the topological point of view, the obtained results in the current work are significant and reliable for the further study of the notion. The theorems and examples investigated in the literature generalise the topic of set convergence to statistical convergence in fuzzy anti-norm linear space. We also investigated Cauchy criterion for the statistical convergence of set sequences in fuzzy anti-norm linear space. In addition, Hausdorff convergence and Hausdorff statistical convergence in FANLS are examined, which are stronger than the ordinary statistical convergence.

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REFERENCES

[1] T. BAG AND S. K. SAMANTA, *A comparative study of fuzzy norms on a linear space*, Fuzzy Sets Syst., **159**, 6 (2008), 670–684.

[2] G. BEER, *On the compactness theorem for sequences of closed sets*, Math. Balkanica, **16**, (2002), 327–338.

[3] BIVAS DINDA, T. K. SAMANTA AND IQBAL H. JEBRIL *Fuzzy anti-norm and fuzzy α-anti convergence*, Demonstr. Math., **45**, 4 (2012), 739–754.

[4] H. FAST, *Sur la convergence statistique*, Colloq. Math., **2**, (1951), 241–244.

[5] B. HAZARIKA AND A. ESI, *On λ-asymptotically Wijsman generalized statistical convergence of sequences of sets*, Tatra Mt. Math. Publ., **56**, 1 (2013), 67–77.

[6] I. H. JEBRIL AND T. K. SAMANTA, *Fuzzy anti-normed linear space*, J. Math. Technol., **26**, (2010), 338–353.
[7] S. KARAKUS, K. DEMIRCI AND O. DUMAN, *Statistical convergence on intuitionistic fuzzy normed spaces*, Chaos Solitons Fractals, **35**, 4 (2008), 763–769.
[8] V. A. KHAN, H. FATIMA, A. AHMAD AND M. I. IDRISI, *Some fuzzy anti λ-ideal convergent double sequence spaces*, J. Intell. Fuzzy Systems, **38**, 2 (2020), 1617–1622.
[9] L. KOCINAC, *Some topological properties of fuzzy antinormed linear spaces*, Journal of Mathematics, Hindawi, **2018**, (2018), 1–6.
[10] M. MURSALEEN AND S. MOHIUDDINE, *Statistical convergence of double sequences in intuitionistic fuzzy normed spaces*, Chaos Solitons Fractals, **41**, 5 (2009), 2414–2421.
[11] F. NURAY AND B. RHOADES, *Statistical convergence of sequences of sets*, Fasc. Math., **49**, (2012), 87–99.
[12] S. PANDIT AND A. AHMAD, *A study on statistical convergence of triple sequences in intuitionistic fuzzy normed space*, Sahand Commun. Math. Anal., **19**, 3 (2022), 1–12.
[13] S. PANDIT AND A. AHMAD, *On asymptotically I-equivalent sequences in intuitionistic fuzzy normed spaces*, NeuroQuantology, **20**, 19 (2022), 502–512.
[14] S. PANDIT AND A. AHMAD, *On algebra of lacunary statistical limit of double sequences in intuitionistic fuzzy normed space*, J. Appl. Math. Informatics, **41**, 3 (2023), 541–552.
[15] S. PANDIT, A. AHMAD AND A. ESI, *A study of triple sequence spaces in fuzzy anti-normed linear spaces*, Filomat, **37**, 15 (2023), 4971–4980.
[16] S. PANDIT, A. AHMAD AND A. ESI, *On intuitionistic fuzzy metric space and ideal convergence of triple sequence space*, Sahand Commun. Math. Anal., **20**, 1 (2023), 35–44.
[17] J. H. PARK, *Intuitionistic fuzzy metric spaces*, Chaos Solitons Fractals, **22**, 5 (2004), 1039–1046.
[18] I. SCHOENBERG, *The integrability of certain functions and related summability methods*, Am. Math. Mon., **66**, (1959), 361–775.
[19] U. ULUSU AND F. NURAY, *Lacunary Statistical Convergence of Sequence of Sets*, Progr. Appl. Math., **4**, (2012), 99–109.
[20] R. A. WIJSMAN, *Convergence of sequences of convex sets, cones and functions*, Bull. Amer. Math. Soc., **70**, (1964), 186–188.
[21] L. ZADEH, *Fuzzy sets*, Inform. Control., **8**, 3 (1965), 338–353.

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