Journal of
Classical
Analysis
Volume 23, Number 2 (2024), 151–163
doi:10.7153/jca-2024-23-12

D'ALEMBERT'S OTHER FUNCTIONAL EQUATION WITH AN AUTOMORPHISM

YOUSSEF ASERRAR* AND ELHOUCIEN ELQORACHI

Abstract. Let S be a semigroup, and $\sigma : S \to S$ an automorphism that need not be involutive. We determine the complex-valued solutions of the following functional equation

$$f(xy) - \mu(y)f(\sigma(y)x) = 2f(x)g(y), \;\; x,y \in S,$$

where $\mu : S \to \mathbb{C}$ is a multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in S$. This enables us to solve the functional equation

$$f(x\varphi(y)) - f(\psi(y)x) = 2f(x)g(y), \;\; x,y \in S,$$

where $\varphi, \psi : S \to S$ are automorphisms such that φ is involutive and ψ is not necessarily involutive. Some consequences of these results are presented.

1. Introduction

In [5, 6], d'Alembert studied among others, the functional equation

$$f(x+y) - f(x-y) = g(x)h(y), \quad x,y \in \mathbb{R}.$$

This functional equation has been extended from $\mathbb{R}$ to abelian groups and has been solved in that setting. See for example [11, Equation (VS) in section 3.4.9]. Over the last century, many generalizations of the above equation have been treated. In 1997, Stetkær [12, Corollary III.5] obtained the solutions of the functional equation

$$f(x+y) - f(x+\sigma(y)) = 2g(x)h(y), \quad x,y \in G,$$

on an abelian group G, where $\sigma : G \to G$ is an involution. Ebanks and Stetkær [7] solved the functional equation

$$f(xy) - f(\sigma(y)x) = g(x)h(y), \quad x,y \in M, \tag{1}$$

for $f,g,h : M \to \mathbb{C}$, where M is a group or a monoid generated by its squares and $\sigma : M \to M$ is an involutive automorphism. That is $\sigma(xy) = \sigma(x)\sigma(y)$ and $\sigma(\sigma(x)) = x$ for all $x,y \in M$. In [4], Bouikhalene and Elqorachi determined the complex-valued solutions of the functional equation

$$f(xy) - \mu(y)f(\sigma(y)x) = g(x)h(y), \quad x,y \in M, \tag{2}$$

Mathematics subject classification (2020): 39B52, 39B32.
Keywords and phrases: Semigroup, d'Alembert's equation, automorphism, multiplicative function.
* Corresponding author.

where M is a group or a monoid generated by its squares, and $\mu : M \to \mathbb{C}$ is a multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in M$ and $\sigma : M \to M$ is an involutive automorphism. Moreover, the solutions of (2) on a semigroup generated by its squares can be found in [1]. Recently, Ebanks [9] described the solutions of Eq. (1) on compatible monoids which is a class of monoids that contains strictly the class of monoids generated by their squares, and regular monoids.

Inspired by these results, the present paper studies the functional equation

$$f(xy) - \mu(y)f(\sigma(y)x) = 2f(x)g(y), \quad x,y \in S, \tag{3}$$

on a semigroup S, where $\mu : S \to \mathbb{C}$ is a multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in S$ and $\sigma : S \to S$ is an automorphism not necessarily involutive. The condition $\sigma \circ \sigma = id$ played an important role in the previous results on the functional equation (2). Here, we solve (3) on any semigroup without using that condition, and we impose no further condition on the automorphism σ. The well known sine addition law

$$g(xy) = g(x)f(y) + g(y)f(x), \quad x,y \in S,$$

plays an important role in our investigation. It should be mentioned that its solutions with $g \neq 0$ are all abelian even in the setting of semigroups. See for example [3], [10, Therem 3.1] and [13, Theorem 4.1]. So, by using the previous results on the sine addition law, we prove that the solutions of Eq. (3) can be expressed in terms of multiplicative functions. This allows us to solve the functional equation

$$f(x\varphi(y)) - f(\psi(y)x) = 2f(x)g(y), \quad x,y \in S, \tag{4}$$

where $\varphi, \psi : S \to S$ are automorphisms such that φ is involutive and ψ is not necessarily involutive.

As consequences of our main result, we determine the complex-valued solutions of the d'Alembert functional equations

$$f(xy) - \mu(y)f(\sigma(y)x) = 2f(x)f(y), \quad x,y \in S, \tag{5}$$

$$f(x\varphi(y)) - f(\psi(y)x) = 2f(x)f(y), \quad x,y \in S, \tag{6}$$

and the Jensen-d'Alembert functional equations

$$f(xy) - \mu(y)f(\sigma(y)x) = 2f(x), \quad x,y \in S, \tag{7}$$

$$f(x\varphi(y)) - f(\psi(y)x) = 2f(x), \quad x,y \in S, \tag{8}$$

on semigroups. We also give the general solution of the functional equations

$$f(x\varphi(y)) - f(\varphi(y)x) = 2f(x)g(y), \quad x,y \in S, \tag{9}$$

$$f(x\varphi(y)) - f(yx) = 2f(x)g(y), \quad x,y \in S. \tag{10}$$

The outline of the paper is as follows. Our notation and notions are described in the following section. Section 3 consists of three subsections. Subsection 3.1 contains some preliminaries. The general solution of the functional equations (3) and (4) is given in subsection 3.2. Some consequences are presented in subsection 3.3. The last section contain some examples.

2. Set up, and notations

We impose as blanket assumption that $(S,\cdot)$ is a semigroup. We define the set $S^2 := \{xy \mid x,y \in S\}$. If S is a topological semigroup, let $C(S)$ denote the set of continuous functions mapping S into $\mathbb{C}$. We also need the following definitions.

DEFINITION 1. Let $f : S \to \mathbb{C}$.
f is multiplicative, if $f(xy) = f(x)f(y)$ for all $x,y \in S$.
f is central, if $f(xy) = f(yx)$ for all $x,y \in S$.
f is abelian, if f is central and $f(xyz) = f(xzy)$ for all $x,y,z \in S$.

Throughout this paper, $\sigma : S \to S$ denotes an automorphism, and $\mu : S \to \mathbb{C}$ a multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in S$. For any function $f : S \to \mathbb{C}$ we define the function $f^*(x) := \mu(x)f(\sigma(x))$ for all $x \in S$, and the functions $f^e := \frac{f+f^*}{2}$, $f^\circ := \frac{f-f^*}{2}$.

Let $\chi : S \to \mathbb{C}$ be a multiplicative function. The function $\phi : S \to \mathbb{C}$ denotes a non-zero solution of the special sine addition law

$$\phi(xy) = \phi(x)\chi(y) + \phi(y)\chi(x), \quad x,y \in S. \tag{11}$$

The following lemma will be helpful later.

LEMMA 1. *Let $\phi : S \to \mathbb{C}$ be a non-zero solution of Eq. (11) such that $\chi \neq 0$. Then ϕ and χ are linearly independent.*

Proof. Suppose that $\phi = \alpha\chi$ for some constant $\alpha \in \mathbb{C}$. Then Eq. (11) implies that

$$\alpha\chi(xy) = \alpha\chi(x)\chi(y) + \alpha\chi(x)\chi(y) = 2\alpha\chi(xy).$$

That is $\alpha = 0$, since $\chi \neq 0$. This contradicts the fact $\phi \neq 0$. This completes the proof. $\square$

3. Main results

3.1. Preliminaries

In the following lemmas we give some key properties of solutions of Eq. (3).

LEMMA 2. *Let $f,g : S \to \mathbb{C}$ be a solution of Eq. (3) such that $f \neq 0$ and $g \neq 0$. Let $a \in S$ such that $f(a) \neq 0$. Define the function $f_a : S \to \mathbb{C}$ by $f_a(x) := \dfrac{f(ax) - f(a)g(x)}{f(a)}$ for $x \in S$. The following statements hold:*

(1) $g(xy) = f_a(x)g(y) + f_a(y)g(x)$ for all $x,y \in S$. Hence, g and f_a are abelian, in particular central.

(2) $f^\circ(xy) = f(x)g(y) + f^*(y)g(x)$ for all $x,y \in S$.

(3) If f and g are linearly independent, then g and f^* are linearly independent.

(4) If f and g are linearly independent, then there exists two functions $h_1, h_2 : S \to \mathbb{C}$ such that

$$g(xy) = f(x)h_1(y) + g(x)h_2(y), \textit{ for all } \quad x, y \in S. \tag{12}$$

(5) $g \neq 0$ on S^2.

Proof. (1) Let x, y, z be arbitrary. Replacing (x, y) by (x, yz) in Eq. (3) and multiplying the result by -1, we obtain

$$\mu(yz)f(\sigma(yz)x) - f(xyz) = -2f(x)g(yz). \tag{13}$$

In addition, replacing (x, y) by $(\sigma(z)x, y)$ in Eq. (3) and multiplying the identity obtained by $\mu(z)$, we get

$$\mu(z)f(\sigma(z)xy) - \mu(yz)f(\sigma(yz)x) = 2\mu(z)f(\sigma(z)x)g(y). \tag{14}$$

Now, replacing (x, y) by (xy, z) in Eq. (3), we find that

$$f(xyz) - \mu(z)f(\sigma(z)xy) = 2f(xy)g(z). \tag{15}$$

Thus, by adding Eq. (13), Eq. (14) and Eq. (15), we obtain

$$f(xy)g(z) + \mu(z)f(\sigma(z)x)g(y) - f(x)g(yz) = 0.$$

Since $\mu(z)f(\sigma(z)x) = f(xz) - 2f(x)g(z)$ for all $x, y, z \in S$, the identity above can be written as

$$f(xy)g(z) + g(y)\,[f(xz) - 2f(x)g(z)] = f(x)g(yz).$$

That is

$$g(z)\,[f(xy) - f(x)g(y)] + g(y)\,[f(xz) - f(x)g(z)] = f(x)g(yz). \tag{16}$$

Now, the result follows easily from Eq. (16) by putting $x = a$. Thus g and f_a are abelian, in particular central according to [13, Theorem 4.1 (e)].

(2) Replacing (x, y) by $(\sigma(y), x)$ in Eq. (3) and multiplying the identity obtained by $\mu(y)$ we get

$$\mu(y)f(\sigma(y)x) - f^*(xy) = 2f^*(y)g(x). \tag{17}$$

Thus, by adding (17) to (3), we get

$$f(xy) - f^*(xy) = 2f(x)g(y) + 2f^*(y)g(x).$$

That is

$$f^\circ(xy) = f(x)g(y) + f^*(y)g(x). \tag{18}$$

This is the result (2) of Lemma 2.

(3) Assume that f and g are linearly independent. We use a proof by contradiction. Suppose g and f^* are linearly dependent. That is $f^* = \lambda g$ for some constant

$\lambda \in \mathbb{C}$. So f^* is central, since g is central. Then f is central, since σ is an automorphism. Thus f° is central. Therefore Eq. (18) implies that

$$f(x)g(y)+f^*(y)g(x) = f(y)g(x)+f^*(x)g(y),$$

for all $x,y \in S$. That is $f(x)g(y)+\lambda g(y)g(x) = f(y)g(x)+\lambda g(x)g(y)$. This implies that f and g are linearly dependent which is a contradiction. This proves (3).

(4) Applying (18) to $f^\circ((xy)z) = f^\circ(x(yz))$, we see that

$$f(xy)g(z)+f^*(z)g(xy) = f(x)g(yz)+f^*(yz)g(x). \tag{19}$$

Since $f = f^e + f^\circ$ and $f^* = f^e - f^\circ$, we get in view of Eq. (18)

$$f(xy) = f^e(xy)+f^\circ(xy) = f^e(xy)+f(x)g(y)+f^*(y)g(x),$$

and

$$f^*(yz) = f^e(yz)-f^\circ(yz) = f^e(yz)-f(y)g(z)-f^*(z)g(y).$$

Substituting the last two identities in (19), we get after some rearrangements

$$\begin{aligned} &g(z)\left[f^e(xy)+g(x)f(y)\right]+f^*(z)\left[g(xy)+g(x)g(y)\right] \\ &= f(x)\left[g(yz)-g(y)g(z)\right]+g(x)\left[f^e(yz)-f^*(y)g(z)\right]. \end{aligned}$$

Now, suppose that f and g are linearly independent, then g and f^* are also linearly independent, according to Lemma 2 (3). Thus, by fixing $z = z_1$ and $z = z_2$ such that $g(z_1)f^*(z_2)-g(z_2)f^*(z_1) \neq 0$ in the identity above we obtain two equations from which we get Eq. (12). This is case (4).

(5) Suppose $g = 0$ on S^2. Then Eq. (19) becomes $f(xy)g(z) = f^*(yz)g(x)$. This implies that $f(xy) = g(x)l(y)$ for some function $l : S \to \mathbb{C}$, since $g \neq 0$. So $f(xyz) = g(xy)l(z) = 0$ for all $x,y,z \in S$. Thus, replacing (x,y) by (xy,z) in the functional equation (3), we get $0 = 2f(xy)g(z)$ which implies $f = 0$ on S^2 since $g \neq 0$. Therefore, Eq. (3) yields $0 = 2f(x)g(y)$ for all $x,y \in S$. That is $f = 0$ or $g = 0$. This contradicts the fact $f \neq 0$ and $g \neq 0$. This completes the proof of Lemma 2. $\square$

LEMMA 3. *Let $f,g : S \to \mathbb{C}$ be a solution of Eq. (3) such that $f \neq 0$ and $g \neq 0$. Then g is not proportional to a multiplicative function.*

Proof. We use a proof by contradiction. Suppose $g = \lambda m$, where $m : S \to \mathbb{C}$ is a non-zero multiplicative function and $\lambda \in \mathbb{C}\backslash\{0\}$ is a constant. Then, Eq. (16) can be written as

$$m(z)\left[f(xy)-\lambda f(x)m(y)\right]+m(y)\left[f(xz)-\lambda f(x)m(z)\right] = f(x)m(y)m(z). \tag{20}$$

By putting $z = z_0$ in Eq. (20) such that $m(z_0) \neq 0$, we get

$$f(xy) = \left[(2\lambda+1)f(x)+k(x)\right]m(y), \tag{21}$$

for all $x,y \in S$, where $k(x) := \dfrac{-f(xz_0)}{m(z_0)}$. Now, replacing (x,y) by $(\sigma(y),x)$ in Eq. (21) and multiplying the identity obtained by $\mu(y)$, we get

$$\mu(y)f(\sigma(y)x) = [(2\lambda+1)f^*(y)+k^*(y)]\,m(x). \tag{22}$$

Thus, by subtracting Eq. (22) from Eq. (21) and taking into account Eq. (3), we obtain

$$[f(x)+k(x)]\,m(y) = [(2\lambda+1)f^*(y)+k^*(y)]\,m(x). \tag{23}$$

This implies, since $m \neq 0$

$$f+k = \delta m,$$

where $\delta \in \mathbb{C}$. So

$$f^*+k^* = \delta m^*, \tag{24}$$

and then from Eq. (23), we obtain that

$$2\lambda f^* = \delta m - \delta m^*. \tag{25}$$

Theorefore, since $m \neq 0$, Eq. (23) can be written as

$$\delta m(y) = (2\lambda+1)f^*(y)+k^*(y). \tag{26}$$

On the other hand, computing $f(xyz)$ in two different ways using Eq. (21) and comparing the results, we find that

$$2\lambda f(xy)+k(xy) = 0, \text{ for all } x,y \in S.$$

Thus, applying Eq. (26) to yz and taking into account the last identity, we get

$$f^*(yz) = \delta m(y)m(z), \text{ for all } y,z \in S.$$

Then, applying Eq. (21) to $(\sigma(y),\sigma(z))$ and multiplying by $\mu(yz)$, we obtain

$$\delta m(y)m(z) = \delta m(y)m^*(z).$$

This implies $\delta m = \delta m^*$. Thus Eq. (25) yields $\lambda f^* = 0$ which implies $f = 0$, since $\lambda \neq 0$ and σ is an automorphism. This contradicts $f \neq 0$. This completes the proof. □

The final preliminary is the following

LEMMA 4. *Let $f,g : S \to \mathbb{C}$ be a solution of Eq. (3) such that $f \neq 0$ and $g \neq 0$. Then*

(1) $g = c(\chi_1 - \chi_2)$ or $g = \phi$, where $\chi_1,\chi_2 : S \to \mathbb{C}$ are two different non-zero multiplicative functions, $\phi : S \to \mathbb{C}$ is a non-zero solution of Eq. (11), namely

$$\phi(xy) = \phi(x)\chi(y)+\phi(y)\chi(x), \quad x,y \in S,$$

where $\chi : S \to \mathbb{C}$ is a non-zero multiplicative function and $c \in \mathbb{C}\backslash\{0\}$ is a constant.

(2) Assume that f and g are linearly independent. Then

(a) If $g = c(\chi_1 - \chi_2)$ where $\chi_1, \chi_2 : S \to \mathbb{C}$ are two different non-zero multiplicative functions and $c \in \mathbb{C}\setminus\{0\}$ is a constant, then $f = \alpha\chi_1 + \beta\chi_2$ for some constants $\alpha, \beta \in \mathbb{C}$.

(b) If $g = \phi$ where $\phi : S \to \mathbb{C}$ is a non-zero solution of Eq. (11), then $f = \gamma\chi + \delta\phi$ for some constants $\gamma, \delta \in \mathbb{C}$.

Proof. (1) Follows from Lemma 2 (1) and (5), [13, Theorem 4.1] and Lemma 3.

(2) Suppose that f and g are linearly independent. According to Lemma 2 (4), Eq. (12) is satisfied for some functions $h_1, h_2 : S \to \mathbb{C}$. Now, we discuss two cases: $h_1 \neq 0$ and $h_1 = 0$.

Case 1: $h_1 \neq 0$. There exist $y_0 \in S$ such that $h_1(y_0) \neq 0$. Thus, by putting $y = y_0$ in Eq. (12), we can see that

$$f(x) = \alpha_1 g(xy_0) + \alpha_2 g(x), \text{ for all } x \in S, \tag{27}$$

where $\alpha_1, \alpha_2 \in \mathbb{C}$ are constants. Now, the formulas for f described in (2) (a) and (2) (b) follows easily from Eq. (27) and Lemma 4 (1).

Case 2: $h_1 = 0$. In this case, Eq. (12) becomes

$$g(xy) = g(x)h_2(y), \quad x, y \in S. \tag{28}$$

According to Lemma 4 (1), we have two possibilities:

Subcase 2.1: $g = \phi$ where $\phi : S \to \mathbb{C}$ is a non-zero solution of Eq. (11). In this case, the functional equation (28) yields

$$\phi(x)\left[\chi(y) - h_2(y)\right] + \phi(y)\chi(x) = 0, \quad x, y \in S.$$

If $\chi = 0$, the identity above implies $h_2 = 0$, since $\phi \neq 0$. Thus $g = 0$ on S^2 but $g \neq 0$ on S^2 according to Lemma 2 (5). So $\chi \neq 0$, then according to Lemma 1, ϕ and χ are linearly independent. Thus, the identity above implies that $\chi = h_2$ and $\phi = 0$. This is a contradiction, since $\phi \neq 0$. This case does not occur.

Subcase 2.2: $g = c(\chi_1 - \chi_2)$ where $\chi_1, \chi_2 : S \to \mathbb{C}$ are two different non-zero multiplicative functions and $c \in \mathbb{C}\setminus\{0\}$ is a constant. The functional equation (28) can be written as

$$\chi_1(x)\left[\chi_1(y) - h_2(y)\right] + \chi_2(x)\left[h_2(y) - \chi_2(y)\right] = 0, \quad x, y \in S.$$

Now, by the help of [13, Theorem 3.18], we deduce from the last identity that $\chi_1 = h_2$ and $\chi_2 = h_2$. That is $\chi_1 = \chi_2$. This contradicts the fact $\chi_1 \neq \chi_2$. So, this case does not occur. This completes the proof. $\square$

3.2. Solutions of Eq. (3) and Eq. (4)

In this section, we solve the functional equations (3) and (4) on semigroups. The principal result of the paper is the following

THEOREM 1. *The solutions $f,g:S\to\mathbb{C}$ of the functional equation (3) with $g\neq 0$ are the following pairs:*

(1) $f=0$ and $g\neq 0$ arbitrary.

(2) $f=\alpha\chi$ and $g=\dfrac{\chi-\chi^*}{2}$ where $\chi:S\to\mathbb{C}$ is a multiplicative function and $\alpha\in\mathbb{C}\backslash\{0\}$ is a constant such that $\chi^*\neq\chi$.

Note that, f and g are abelian in case (2).

Furthermore, in case (2), if S is a topological semigroup, $f,\mu\in C(S)$ and $\sigma: S\to S$ is a continuous automorphism of S then $g,\chi,\chi^*\in C(S)$.

Proof. Let $f,g:S\to\mathbb{C}$ be a solution of Eq. (3) such that $g\neq 0$. If $f=0$ then g is arbitrary, so we are in family (1). Henceforth, we suppose $f\neq 0$. According to Lemma 4 (1), we have two possible forms for g

$$g=c(\chi_1-\chi_2)\quad\text{or}\quad g=\phi,$$

where $\chi_1,\chi_2:S\to\mathbb{C}$ are two different non-zero multiplicative functions, $\phi:S\to\mathbb{C}$ is a non-zero solution of Eq. (11) and $c\in\mathbb{C}\backslash\{0\}$ is a constant.

We split the discussion into two cases according to whether f and g are linearly dependent or not.

First case: f and g are linearly dependent. That is $f=\alpha_1 g$ for some $\alpha_1\in\mathbb{C}\backslash\{0\}$.

Subcase A: $g=\phi$ where $\phi:S\to\mathbb{C}$ is a non-zero solution of Eq. (11). This implies $f=\alpha_1\phi$. Inserting these forms in Eq. (3), we get after some rearrangements

$$\phi(x)\left[\chi(y)-\chi^*(y)-2\phi(y)\right]+\chi(x)\left[\phi(y)-\phi^*(y)\right]=0.$$

Since $\chi\neq 0$ (because $g\neq 0$ on S^2 according to Lemma 2 (5)), then according to Lemma 1, ϕ and χ are linearly independent. So, the identity above implies

$$\chi-\chi^*=2\phi\quad\text{and}\quad\phi-\phi^*=0.$$

Since $\phi\neq 0$, we deduce from the formula in the left hand side that $\chi\neq\chi^*$. On the other hand, by the help of [8, Lemma 4], we see that $\chi=\chi^*$. So, this case does not occur.

Subcase B: $g=c(\chi_1-\chi_2)$ where $\chi_1,\chi_2:S\to\mathbb{C}$ are two different non-zero multiplicative functions and $c\in\mathbb{C}\backslash\{0\}$. Then $f=\alpha_1 c(\chi_1-\chi_2)$. Substituting these forms in Eq. (3), we obtain

$$\begin{aligned}&\chi_1(x)\left[\chi_1(y)-\chi_1^*(y)\right]+\chi_2(x)\left[\chi_2^*(y)-\chi_2(y)\right]\\&=2c\chi_1(x)\left[\chi_1(y)-\chi_2(y)\right]+2c\chi_2(x)\left[\chi_2(y)-\chi_1(y)\right].\end{aligned}$$

By the help of [13, Theorem 3.18], the identity above reduces to

$$\frac{\chi_1-\chi_1^*}{2c}=\chi_1-\chi_2 \quad\text{and}\quad \frac{\chi_2^*-\chi_2}{2c}=\chi_2-\chi_1.$$

This implies $\chi_1-\chi_1^*=\chi_2-\chi_2^*$. That is $\chi_1+\chi_2^*=\chi_1^*+\chi_2$. Now, according to [13, Corollary 3.19], since $\chi_1\neq\chi_2$, we deduce that $\chi_1=\chi_1^*$ and $\chi_2=\chi_2^*$. Thus $\chi_1-\chi_2=0$. This is a contradiction, since $\chi_1\neq\chi_2$. This case does not occur.

Second case: f and g are linearly independent. According to Lemma 4 (2), we have two cases to consider:

Subcase A: $f=\alpha\chi_1+\beta\chi_2$ and $g=c(\chi_1-\chi_2)$ where $\chi_1,\chi_2:S\to\mathbb{C}$ are two different non-zero multiplicative functions, $c\in\mathbb{C}\backslash\{0\}$, and $\alpha,\beta\in\mathbb{C}$. Therefore, by proceeding similarly to the first case (Subcase B), we show that

$$\frac{\alpha\,(\chi_1-\chi_1^*)}{2c}=\alpha\,(\chi_1-\chi_2) \quad\text{and}\quad \frac{\beta\,(\chi_2-\chi_2^*)}{2c}=\beta\,(\chi_1-\chi_2). \tag{29}$$

Now, since $f\neq 0$, we have three cases to discuss:

(i) $\alpha\neq 0$ and $\beta\neq 0$. We get from Eq. (29) that $\chi_1-\chi_1^*=\chi_2-\chi_2^*$. Thus, by [13, Corollary 3.19], we see that $\chi_1=\chi_1^*$ and $\chi_2=\chi_2^*$. Therefore $\chi_1=\chi_2$. This case does not occur.

(ii) $\alpha=0$ and $\beta\neq 0$. Eq. (29) yields $\dfrac{\chi_2-\chi_2^*}{2c}=\chi_1-\chi_2$. This implies $(1+2c)\chi_2=\chi_2^*+2c\chi_1$. Since $\chi_1\neq\chi_2$, we can see that $\chi_2\neq\chi_2^*$. Thus, if $\chi_1\neq\chi_2^*$, we obtain by the help of [13, Theorem 3.18] that $c=0$ which is a contradiction. So $\chi_1=\chi_2^*$, then $c=\dfrac{-1}{2}$. Let $\chi:=\chi_2$. So $\chi^*=\chi_1$. This occurs in case (2).

(iii) $\alpha\neq 0$ and $\beta=0$. Proceeding similarly to the above case (Subcase A (ii)), we show that $c=\dfrac{1}{2}$. Therefore $\chi_1^*=\chi_2$. This is case (2) with $\chi:=\chi_1$.

Subcase B: $f=\gamma\chi+\delta\phi$ and $g=\phi$ where $\phi:S\to\mathbb{C}$ is a non-zero solution of Eq. (11) and $\gamma,\delta\in\mathbb{C}$ are constants. Thus, inserting the forms of f and g in the functional equation (3) and taking into account that ϕ and χ are linearly independent according to Lemma 1, we find that

$$\delta(\chi-\chi^*)=2\delta\phi \;\text{ and }\; \gamma(\chi-\chi^*)+\delta(\phi-\phi^*)=2\gamma\phi.$$

If $\delta=0$ then $\gamma\neq 0$, since $f\neq 0$. Then $\chi-\chi^*=2\phi$. So $\chi\neq\chi^*$, since $\phi\neq 0$. This a contradiction, since $\chi=\chi^*$ according to [8, Lemma 4]. We get the same contradiction if $\delta\neq 0$. This case does not occur.

Conversely, we check by elementary computations that the forms described in Theorem 1 satisfy the functional equation (3).

The topological statements follow from the functional equation and [13, Theorem 3.18]. This completes the proof. $\square$

Now, we turn our attention to the functional equation (4). The next theorem gives the general solution of Eq. (4) on semigroups.

THEOREM 2. *The solutions* $f,g : S \to \mathbb{C}$ *of the functional equation* (4) *with* $g \neq 0$, *namely*

$$f(x\varphi(y)) - f(\psi(y)x) = 2f(x)g(y), \ \ x,y \in S,$$

are the following pairs:

(1) $f = 0$ *and* $g \neq 0$ *arbitrary.*

(2) $f = \alpha\chi$ *and* $g = \dfrac{\chi\circ\varphi - \chi\circ\psi}{2}$ *where* $\chi : S \to \mathbb{C}$ *is a multiplicative function and* $\alpha \in \mathbb{C}\backslash\{0\}$ *is a constant such that* $\chi\circ\varphi \neq \chi\circ\psi$.

Note that, f *and* g *are abelian in case (2).*

Moreover, in case (2), if S *is a topological semigroup,* $f \in C(S)$ *and* $\varphi,\psi : S \to S$ *are continuous automorphisms of* S *then* $g,\chi,\chi\circ\varphi,\chi\circ\psi \in C(S)$.

Proof. It is not difficult to verify that each pair described in Theorem 2 is a solution of the functional equation (4) with $g \neq 0$. Now, let $f,g : S \to \mathbb{C}$ be a solution of Eq. (4). Replacing y by $\varphi(y)$ in Eq. (4), we get

$$f(xy) - f(\psi\circ\varphi(y)x) = 2f(x)g\circ\varphi(y), \quad x,y \in S.$$

That is $(f,g\circ\varphi)$ satisfy the functional equation (3) with $\mu = 1$ and $\sigma := \psi\circ\varphi$. The rest of the proof follows easily from Theorem 1. This completes the proof of Theorem 2. □

3.3. Consequences

In this section, we give some interesting consequences of our main result. The first one is the following.

COROLLARY 1. *The functional equations* (5) *and* (6), *namely*

$$f(xy) - \mu(y)f(\sigma(y)x) = 2f(x)f(y), \ \ x,y \in S,$$

and

$$f(x\varphi(y)) - f(\psi(y)x) = 2f(x)f(y), \ \ x,y \in S,$$

do not have non-zero solutions.

Proof. Follows from Theorem 1 and Theorem 2 by taking $g \equiv f$, and using [13, Theorem 3.18]. □

The second consequence is the following corollary.

COROLLARY 2. *The only solution* $f : S \to \mathbb{C}$ *of the functional equations* (7) *and* (8), *namely*

$$f(xy) - \mu(y)f(\sigma(y)x) = 2f(x), \ \ x,y \in S,$$

and

$$f(x\varphi(y)) - f(\psi(y)x) = 2f(x), \ \ x,y \in S,$$

is $f \equiv 0$.

Proof. Follows from Theorem 1 and Theorem 2 by taking $g \equiv 1$, and using [13, Theorem 3.18]. $\square$

An other interesting consequence is the following result.

COROLLARY 3. *The only solution of the functional equation* (9) *with* $g \neq 0$, *namely*

$$f(x\varphi(y)) - f(\varphi(y)x) = 2f(x)g(y), \quad x,y \in S,$$

is the pair $f = 0$ *and* $g \neq 0$ *arbitrary.*

Proof. Follows easily from Theorem 2 by taking $\psi := \varphi$. $\square$

The last result of the paper reads as follows.

COROLLARY 4. *The solutions* $f,g : S \to \mathbb{C}$ *of the functional equation* (10) *with* $g \neq 0$, *namely*

$$f(x\varphi(y)) - f(yx) = 2f(x)g(y), \quad x,y \in S,$$

are the following

(1) $f = 0$ *and* $g \neq 0$ *arbitrary.*

(2) $f = \alpha\chi$ *and* $g = \dfrac{\chi \circ \varphi - \chi}{2}$ *where* $\chi : S \to \mathbb{C}$ *is a multiplicative function and* $\alpha \in \mathbb{C}\backslash\{0\}$ *is a constant such that* $\chi \circ \varphi \neq \chi$.

Note that, f *and* g *are abelian in case (2).*

Proof. Follows from Theorem 2 by taking $\psi := id$. $\square$

4. Examples

In this section, we apply our theory to two examples of groups. The first one is abelian, and the second one is not.

EXAMPLE 1. Let $S = (\mathbb{R}, +)$, let $\gamma \in \mathbb{R}\backslash\{0,-1\}$ be a fixed element and let $\psi(x) = \gamma x$ and $\varphi(x) = -x$ for all $x \in \mathbb{R}$. The functional equation (4) can be written as follows:

$$f(x-y) - f(x+\gamma y) = 2f(x)g(y), \quad x,y \in \mathbb{R}. \tag{30}$$

According to [13, Example 3.7], the continuous characters on S are of the form

$$\chi_\lambda(x) = e^{\lambda x}, \quad x \in \mathbb{R},$$

where $\lambda \in \mathbb{C}$. On the other hand, if $\lambda \neq 0$ then $\chi_\lambda \circ \varphi \neq \chi_\lambda \circ \psi$ since $\gamma \in \mathbb{R}\backslash\{0,-1\}$. Therefore, the continuous non-zero solutions of Eq. (30) are the following

$$f(x) = \alpha e^{\lambda x} \quad \text{and} \quad g(x) = \frac{e^{-\lambda x} - e^{\gamma\lambda x}}{2},$$

where $\lambda \neq 0$ and $\alpha \neq 0$.

EXAMPLE 2. Let $S = H_3$ be the Heisenberg group defined by

$$H_3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x,y,z \in \mathbb{R} \right\}.$$

We consider the following automorphism

$$\sigma \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x & 2024z \\ 0 & 1 & 2024y \\ 0 & 0 & 1 \end{pmatrix}.$$

According to [13, Example 3.14], the continuous non-zero multiplicative functions on S have the form

$$\chi \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = e^{ax+by},$$

where $a,b \in \mathbb{C}$. In addition, $\chi \neq \chi \circ \sigma$ when $b \neq 0$. According to Theorem 1, the continuous non-zero solutions of Eq. (3) with $\mu = 1$ are the following

$$\begin{cases} f : \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto \alpha e^{ax+by} \\ g : \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto \dfrac{e^{ax+by} - e^{ax+2024by}}{2} \end{cases},$$

where $\alpha \in \mathbb{C} \setminus \{0\}$, $a \in \mathbb{C}$ and $b \in \mathbb{C} \setminus \{0\}$.

Acknowledgement. We would like to thank the referee for carefully reading our manuscript and for giving such constructive comments which substantially helped improving the quality of the paper.

REFERENCES

[1] O. AJEBBAR, E. ELQORACHI, *D'Alembert's other functional equation on semigroups*, Aequat. Math. vol. 94, no. 5 (2020), 913–930.

[2] Y. ASERRAR, E. ELQORACHI AND TH. M. RASSIAS, *A trigonometric functional equation with an automorphism*, Int. J. Nonlinear Anal. Appl (2023), https://doi.org/10.22075/IJNAA.2023.31988.4745.

[3] Y. ASERRAR, E. ELQORACHI, *Cosine and sine addition and subtraction law with an automorphism*, Annal. Math Silesianae (2023), https://doi.org/10.2478/amsil-2023-0021.

[4] B. BOUIKHALENE, E. ELQORACHI, *A class of functional equations of type d'Alembert on monoids*, Anastassiou, G., Rassias, J. M. (eds.) Frontiers in Functional Equations and Analytic Inequalities. Springer, Berlin (2019), 219–235.

[5] J. D'ALEMBERT, *Recherche sur la courbe que forme une corde tendue mise en vibration*, History of the Academy, vol. I, 1747, Berlin (1747), 214–219.

[6] J. D'ALEMBERT, *Recherche sur la courbe que forme une corde tendue mise en vibration*, History of the Academy, vol. II, 1747, Berlin (1747), 220–249.

[7] B. EBANKS, H. STETKÆR, *d'Alembert's other functional equation on monoids with an involution*, Aequat. Math. vol. 89, no. 1 (2015), 187–206.

[8] B. EBANKS, C. T. NG, *Levi-Civita functional equations and the status of spectral synthesis on semigroups*, Semigroup Forum. vol. 103, no. 2 (2021), 469–494.

[9] B. EBANKS, *D'Alembert's other functional equation on monoids revisited*, Publ. Math. Debrecen. vol. 101, no. 3–4 (2022), 497–508.

[10] B. EBANKS, *Around the Sine Addition Law and d'Alembert's Equation on Semigroups*, Results Math. vol. 77, no. 1, 11 (2022).

[11] P. KANNAPPAN, *Functional equations and inequalities with applications*, Springer monographs in Mathematics, Springer, New York (2009).

[12] H. STETKÆR, *Functional equations on abelian groups with involution*, Aequat. Math. vol. 54, no. 1–2 (1997), 144–172.

[13] H. STETKÆR, *Functional equations on groups*, World scientific Publishing CO. Pte. Ltd., Singapore (2013).

(Received June 9, 2024)

Youssef Aserrar
Ibn Zohr University, Faculty of sciences
Department of mathematics
Agadir, Morocco
e-mail: youssefaserrar05@gmail.com

Elhoucien Elqorachi
Ibn Zohr University, Faculty of sciences
Department of mathematics
Agadir, Morocco
e-mail: elqorachi@hotmail.com