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ON A FAMILY OF DIRICHLET SERIES GENERATED BY HARMONIC NUMBERS AND THEIR LAURENT EXPANSION

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Abstract. This paper explores a family of Dirichlet series with two variables, generated by the generalized harmonic numbers and the generalized skew-harmonic numbers. We investigate their analytic continuation to negative arguments and derive argument interchange formulas. Additionally, we determine the general coefficients of the Laurent series for $\mathscr{H}(s,z) = \sum_{n=1}^{\infty} \frac{H_n^{(z)}}{n^s}$ and $\overline{\mathscr{H}}(s,z) = \sum_{n=1}^{\infty} \frac{\overline{H}_n^{(z)}}{n^s}$ in terms of a newly defined mathematical object

$$\mathscr{D}\{f\}(a,b) = \int_a^b (f(\lfloor x \rfloor) - f(x))\, dx.$$

1. Introduction

The analytic function $\mathscr{H}(s,z) = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{k=1}^{n} \frac{1}{k^z}$ has been investigated in [2], with a specific focus on the case where $z = 1$, as explored in [9], and by utilizing techniques of Euler-Macluarin summation formula in [7]. Different versions of harmonic zeta functions are studied in [4]. In the first part of this paper, we will study the generalized version of those three functions in [4], the formulas were described but not stated explicitly. We derive the Euler-Maclaurin formula for each function and consider an analytic extension to the entire complex plane. We will also state the negative values, the residues and derive argument interchange formulas similar to the formula $\mathscr{H}(s,z) + \mathscr{H}(z,s) = \zeta(s+z) + \zeta(s)\zeta(z)$ in [2]. In the second part of this paper, we will use the Euler-Maclaurin formula and a newly defined mathematical object to find the general term of the Laurent series centered at $s = 1$ for these functions. The Laurent expansion of the harmonic zeta function $\mathscr{H}(s)$ (see Definition 7) and the generalized harmonic zeta function $\mathscr{H}(s,z)$ centered at $z = 1$ are expanded in the paper [5] using Ramanujan summation. We offer a different proof using the Euler-Macluarin summation formula. Furthermore, we will write the general term of their Laurent expansion in terms of the difference transform (see Definition 6). $\tilde{\gamma}_n$ is defined as

$$\mathscr{H}(s) = \frac{1}{(s-1)^2} + \frac{\gamma}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \tilde{\gamma}_k (s-1)^k$$

which was proposed to be called the harmonic Stieltjes constants in [5]. In this paper, I will call it as such.

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In this study, we will consider using the notation B_n denoting the n-th Bernoulli number, $B_n(x)$ denoting the n-th Bernoulli polynomial (see [8] and [1, pp. 804–806]) and $\widetilde{B}_n(x) = B_n(\{x\})$ denoting the n-th periodic Bernoulli polynomial (see [10]) where $\{x\}$ denotes the fractional part of x. As for the n-th Euler numbers, Euler polynomials, and the periodic Euler polynomials, we will denote them using E_n, $E_n(x)$ and $\widetilde{E}_n(x)$ respectively.

We will use the notation of the harmonic numbers, skew harmonic numbers, and the generalized version of them as the following

$$H_n = \sum_{k=1}^{n} \frac{1}{k}, \qquad \overline{H}_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}, \qquad H_n^{(z)} = \sum_{k=1}^{n} \frac{1}{k^z}, \qquad \overline{H}_n^{(z)} = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k^z}.$$

We also make use of the rising factorial and falling factorials respectively:

$$x^{\overline{n}} := \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)\dots(x+n-1), \quad x^{\underline{n}} := \frac{\Gamma(x+1)}{\Gamma(x-n+1)} = x(x-1)\dots(x-n+1).$$

Here $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$ denotes the classical gamma function (see [1, pp. 255–258 and 260–263] and [6, 5]).

Lastly, we recall the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad (\Re(s) > 1)$$

and the Dirichlet eta function

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}, \qquad (\Re(s) > 0).$$

2. Results of the generalized harmonic zeta functions using Euler-Maclaurin and Euler-Boole summation formula

Apostol-Vu [2] studied the generalized harmonic zeta function (see Definition 7)

$$\mathscr{H}(s,z) = \sum_{n=1}^{\infty} \frac{H_n^{(z)}}{n^s}.$$

They showed that the function is meromorphic on the entire complex plane in both variables using the Euler-Macluarin summation formula [7].

Boyadzhiev [4] studied the functions

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}H_n}{n^s}, \qquad \sum_{n=1}^{\infty} \frac{\overline{H}_n}{n^s}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\overline{H}_n}{n^s}.$$

They gave an analytic extension to the negative values using Euler-Macluarin summation and Euler-Boole summation formulas [3]. In this section, we study the generalized version of the other three functions, by replacing the harmonic numbers and skew harmonic numbers using the generalized versions of them.

2.1. Skew harmonic zeta function

DEFINITION 1. We define the skew harmonic zeta function as

$$\overline{\mathscr{H}}(s,z) := \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k^z}.$$

The special case where $z = 1$ is written as

$$\overline{\mathscr{H}}(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}.$$

LEMMA 1. *The following formula holds: for* $\Re(s) > 1-m$*,*

$$\overline{H}_n^{(s)} = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k^s} = \frac{(-1)^{n+1}}{n^s} + \eta(s) + \frac{1}{2}\sum_{k=0}^{m-1} \frac{s^{\overline{k}} E_k(0)(-1)^{n+k}}{k!\, n^{s+k}} + \frac{(-s)^{\underline{m}}}{2(m-1)!} \int_n^{\infty} \frac{\widetilde{E}_{m-1}(-x)}{x^{s+m}}\, dx.$$

Here $E_n(x)$ *and* $\widetilde{E}_n(x)$ *denote the Euler polynomials and periodic Euler polynomials respectively.*

Proof. Start with the Euler-Boole summation formula [3]

$$\sum_{j=a}^{n-1} (-1)^j f(j+h) = \frac{1}{2}\sum_{k=0}^{m-1} \frac{E_k(h)}{k!} \left((-1)^{n-1} f^{(k)}(n) + (-1)^a f^{(k)}(a) \right) + \frac{1}{2(m-1)!} \int_a^n f^{(m)}(x) \widetilde{E}_{m-1}(h-x)\, dx$$

for positive integers a, n and $h \in [0,1)$. For our purpose, we let $a = 1$, $h = 0$ and $f(x) = \frac{1}{x^s}$ implies $f^{(k)}(x) = (-s)^{\underline{k}} x^{-k-s}$. By such, we have

$$\sum_{j=1}^{n-1} \frac{(-1)^j}{j^s} = \frac{1}{2}\sum_{k=0}^{m-1} \frac{E_k(0)}{k!} \left((-1)^{n-1} \frac{(-s)^{\underline{k}}}{n^{s+k}} - (-s)^{\underline{k}} \right) + \frac{(-s)^{\underline{m}}}{2(m-1)!} \int_1^n \frac{\widetilde{E}_{m-1}(-x)}{x^{s+m}}\, dx.$$

Take the case where $n \longrightarrow \infty$ and subtract with the above case

$$\sum_{k \geqslant n} \frac{(-1)^k}{k^s} = \frac{1}{2}\sum_{k=0}^{m-1} \frac{E_k(0)}{k!} (-1)^{n-k} \frac{s^{\overline{k}}}{n^{s+k}} + \frac{(-s)^{\underline{m}}}{2(m-1)!} \int_n^{\infty} \frac{\widetilde{E}_{m-1}(-x)}{x^{s+m}}\, dx.$$

Writing the sum $\sum_{k \geqslant n} \frac{(-1)^k}{k^s}$ as $-\eta(s) + \overline{H}_n^{(s)} - \frac{(-1)^{n+1}}{n^s}$ finishes the proof. □

THEOREM 1. (Euler-Boole summation formula of skew harmonic zeta function) *For $z \neq 1$, $\Re(s) > 1-m$ and $\Re(z)+\Re(s)+m > 2$, the following formula holds:*

$$\overline{\mathscr{H}}(z,s) = \eta(z+s)+\eta(s)\zeta(z)+\frac{1}{2}\sum_{k=0}^{m-1}\frac{E_k(0)s^{\overline{k}}}{k!}(-1)^{k+1}\eta(s+z+k)$$
$$+\frac{(-s)^{\underline{m}}}{2(m-1)!}\sum_{n\geqslant 1}\frac{1}{n^z}\int_n^\infty \frac{\widetilde{E}_{m-1}(-x)}{x^{s+m}}\,dx.$$

Proof. Multiplying both sides of Lemma 1 by n^{-z} and summing both sides gives the formula. □

To figure out its domain, we consider the latter infinite sum. Like the periodic Bernoulli polynomials, it is evident that there exists a bound for periodic Euler polynomials, say $|\widetilde{E}_n(x)| \leqslant M_n$ for all $x \in \mathbb{R}$ for some $M_n \geqslant 0$.

$$\left|\sum_{n\geqslant 1}\frac{1}{n^z}\int_n^\infty \frac{\widetilde{E}_{m-1}(-x)}{x^{s+m}}\,dx\right| \leqslant \sum_{n\geqslant 1}\left|\frac{1}{n^z}\right|\int_n^\infty \frac{M_{m-1}}{|x^{s+m}|}\,dx \leqslant \sum_{n\geqslant 1}\frac{1}{n^{\Re(z)}}\int_n^\infty \frac{M_{m-1}}{x^{\Re(s+m)}}\,dx$$
$$\Longrightarrow \left|\sum_{n\geqslant 1}\frac{1}{n^z}\int_n^\infty \frac{\widetilde{E}_{m-1}(-x)}{x^{s+m}}\,dx\right| \leqslant \frac{M_{m-1}}{\Re(s+m-1)}\sum_{n\geqslant 1}\left|\frac{1}{n^{\Re(s+z+m-1)}}\right|.$$

The latter sum converges for $\Re(s+z+m-1) > 1$, therefore the domain would be such. We can do analytic continuation by increasing the value of m.

Let's let $s=-n$ and consider some $m > n$ where n is some positive integer, so that the formula is valid for $\Re(z) > 2-m+n$:

$$\overline{\mathscr{H}}(z,-n) = \eta(z-n)+\eta(-n)\zeta(z)+\frac{1}{2}\sum_{k=0}^{m-1}\frac{E_k(0)(-n)^{\overline{k}}}{k!}(-1)^{k+1}\eta(-n+z+k)$$
$$= \frac{\eta(z-n)}{2}+\eta(-n)\zeta(z)-\frac{1}{2}\sum_{k=1}^{m-1}\binom{n}{k}E_k(0)\eta(-n+z+k)$$

note that $\binom{n}{k}=0$ for $k > n$

$$= \frac{\eta(z-n)}{2}+\eta(-n)\zeta(z)-\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}E_k(0)\eta(-n+z+k).$$

The formula is valid for $\Re(z) > 2-m+n$, $z \neq 1$. However, it is also independent of m, hence we can take $m \to \infty$. Reindexing the sum to get rid of the zeros of $E_{2n}(0)$ gives

COROLLARY 1. (Negative integers at the second argument) *For all $z \in \mathbb{C}\backslash\{1\}$ and $n \in \mathbb{N}$, the following formula holds:*

$$\overline{\mathscr{H}}(z,-n) = \frac{\eta(z-n)}{2}+\eta(-n)\zeta(z)-\frac{1}{2}\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2k+1}E_{2k+1}(0)\eta(z+2k+1-n).$$

LEMMA 2. (see [2, pp. 88]) *For $\Re(s) > -2q$ and positive integers q, the following formula holds:*

$$\sum_{k\geqslant n}\frac{1}{k^s}=\frac{n^{1-s}}{s-1}+\frac{1}{2n^s}+\sum_{m=1}^{q}\frac{s^{\overline{2m-1}}B_{2m}}{(2m)!n^{s+2m-1}}-\frac{s^{\overline{2q+1}}}{(2q+1)!}\int_n^\infty\frac{\widetilde{B}_{2q+1}(t)}{t^{s+2q+1}}\,dt.$$

Proof. Euler-Macluarin of $f(x)=\frac{1}{x^s}$ from n to ∞ (see formula 1) gives this formula. □

Multiplying both sides by $\frac{(-1)^{n+1}}{n^z}$ and summing both sides gives

$$\overline{\mathscr{H}}(s,z)=\frac{\eta(z+s-1)}{s-1}+\frac{\eta(s+z)}{2}+\sum_{m=1}^{q}\frac{s^{\overline{2m-1}}B_{2m}}{(2m)!}\eta(s+z+2m-1)$$
$$-\frac{s^{\overline{2q+1}}}{(2q+1)!}\sum_{n\geqslant1}\frac{(-1)^{n+1}}{n^z}\int_n^\infty\frac{\widetilde{B}_{2q+1}(t)}{t^{s+2q+1}}\,dt.$$

Again, we can bound the integral term,

$$\left|\int_n^\infty\frac{\widetilde{B}_{2q+1}(t)}{t^{s+2q+1}}\,dt\right|\leqslant\int_n^\infty\frac{|\widetilde{B}_{2q+1}(t)|}{t^{\Re(s)+2q+1}}\,dt\leqslant\frac{C_{2q+1}}{2q+\Re(s)}\frac{1}{n^{2q+\Re(s)}}.$$

THEOREM 2. *The following formula holds: for $\Re(s) > -2q$ and $\Re(z)+\Re(s)+2q > 1$ for any positive integers q,*

$$\overline{\mathscr{H}}(s,z)=\frac{\eta(z+s-1)}{s-1}+\frac{\eta(s+z)}{2}+\sum_{m=1}^{q}\frac{s^{\overline{2m-1}}B_{2m}}{(2m)!}\eta(s+z+2m-1)$$
$$-\frac{s^{\overline{2q+1}}}{(2q+1)!}\sum_{n\geqslant1}\frac{(-1)^{n+1}}{n^z}\int_n^\infty\frac{\widetilde{B}_{2q+1}(t)}{t^{s+2q+1}}\,dt.$$

Let $s=-n$ and choose a q such that $2q+1>n$, we get the following

COROLLARY 2. (Negative values at the first argument) *For all $z\in\mathbb{C}$ and $n\in\mathbb{N}$, the following formula holds:*

$$\overline{\mathscr{H}}(-n,z)=-\frac{\eta(z-n-1)}{n+1}+\frac{\eta(z-n)}{2}-\sum_{m=1}^{\lfloor\frac{n+1}{2}\rfloor}\binom{n+1}{2m}\frac{B_{2m}}{n+1}\eta(z-n+2m-1).$$

Note that the domain is for all $z\in\mathbb{C}$ because we can always increase the value of q if we want a larger domain. This is similar to Corollary 1.

With the help of Corollary 2, by setting $z=1$ we easily get

$$\overline{\mathscr{H}}(-n)=-\frac{\eta(-n)}{n+1}+\frac{\eta(1-n)}{2}-\sum_{m=1}^{\lfloor\frac{n+1}{2}\rfloor}\binom{n+1}{2m}\frac{B_{2m}}{n+1}\eta(2m-n).$$

Now we consider the following type of sum

$$S=\sum_{n\geqslant 1} n^m\Big(\overline{H}_n-\log(2)+\frac{(-1)^n}{2n}-\frac{(-1)^n}{2}\sum_{k=1}^{m-1}\frac{(-1)^k}{n^{k+1}}E_k(0)\Big).$$

The sum is convergent because the summand is alternating and in the magnitude of $O(\frac{1}{n})$.

$$\begin{aligned}S=&-\frac{\eta(-m)}{m+1}-\sum_{k=1}^{\lfloor\frac{m+1}{2}\rfloor}\binom{m+1}{2k}\frac{B_{2k}}{m+1}\eta(2k-m)-\zeta(-m)\log(2)\\&+\frac{1}{2}\sum_{k=1}^{m-1}E_k(0)(-1)^k\eta(k+1-m).\end{aligned}$$

Note $E_{2n}(0)=0$ for all $n\in\mathbb{N}$

$$\begin{aligned}&=-\frac{\eta(-m)}{m+1}-\zeta(-m)\log(2)-\frac{1}{2}\sum_{k=1}^{\lfloor\frac{m}{2}\rfloor}E_{2k-1}(0)\eta(2k-m)\\&-\sum_{k=1}^{\lfloor\frac{m+1}{2}\rfloor}\binom{m+1}{2k}\frac{B_{2k}}{m+1}\eta(2k-m).\end{aligned}$$

THEOREM 3. *For all positive integers m:*

$$\begin{aligned}&\sum_{n\geqslant 1} n^m\Big(\overline{H}_n-\log(2)+\frac{(-1)^n}{2n}-\frac{(-1)^n}{2}\sum_{k=1}^{m-1}\frac{(-1)^k}{n^{k+1}}E_k(0)\Big)\\&=-\frac{\eta(-m)}{m+1}-\zeta(-m)\log(2)-\frac{1}{2}\sum_{k=1}^{\lfloor\frac{m}{2}\rfloor}E_{2k-1}(0)\eta(2k-m)\\&-\sum_{k=1}^{\lfloor\frac{m+1}{2}\rfloor}\binom{m+1}{2k}\frac{B_{2k}}{m+1}\eta(2k-m)\end{aligned}$$

where $E_n(x)$ is the n-th Euler polynomial and B_n is the n-th Bernoulli number.

If m is positive even, we have $\eta(2k-m)$ being 0 for all $2k<m$, thus obtaining the following corollary:

COROLLARY 3. *For all $m\in\mathbb{N}$, the following formula holds:*

$$\sum_{n\geqslant 1} n^{2m}\Big(\overline{H}_n-\log(2)+\frac{(-1)^n}{2n}+\frac{(-1)^n}{2}\sum_{k=1}^{m}\frac{E_{2j-1}(0)}{n^{2j}}\Big)=\Big(\frac{4^{m-1}}{m}-\frac{1}{2}-\frac{1}{4m}\Big)B_{2m}.$$

Note that $E_n(0)=\frac{2}{n+1}(1-2^{n+1})B_{n+1}$ (see [4, pp. 27]).

EXAMPLES.

$$\sum_{n\geqslant 1} n^2\left(\overline{H}_n-\log 2+\frac{(-1)^n}{2n}-\frac{(-1)^n}{4n^2}\right)=\frac{1}{24}$$

$$\sum_{n\geqslant 1}(-1)^n n^4\left((\overline{H}_n-\log 2)(-1)^n+\frac{1}{2n}-\frac{1}{4n^2}+\frac{1}{8n^4}\right)=-\frac{11}{240}.$$

2.2. Harmonic eta function

In the below section we study the harmonic eta function $\mathscr{J}(s,z)$.

DEFINITION 2. We define the harmonic eta function as

$$\mathscr{J}(s,z):=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s}\sum_{k=1}^{n}\frac{1}{k^z}.$$

The special case where $z=1$ will be written as

$$\mathscr{J}(s):=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s}\sum_{k=1}^{n}\frac{1}{k}.$$

THEOREM 4. (Euler-Boole and EM of harmonic eta function) *The following formula holds: for* $\Re(s)+\Re(z)>2-m$,

$$\mathscr{J}(s,z)=\frac{1}{2}\sum_{k=0}^{m-1}\frac{s^{\overline{k}}E_k(0)(-1)^k}{k!}\eta(s+z+k)-\frac{(-s)_{\underline{m}}}{2(m-1)!}\sum_{n\geqslant 1}\frac{1}{n^z}\int_n^{\infty}\frac{\widetilde{E}_{m-1}(-x)}{x^{s+m}}dx$$

$$\mathscr{J}(z,s)=\eta(z)\zeta(s)-\frac{\eta(z+s-1)}{s-1}+\frac{\eta(z+s)}{2}-\sum_{m=1}^{q}\frac{s^{\overline{2m-1}}B_{2m}}{(2m)!}\eta(s+z+2m-1)$$
$$+\frac{s^{\overline{2q+1}}}{(2q+1)!}\sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n^z}\int_n^{\infty}\frac{\widetilde{B}_{2q+1}(t)}{t^{s+2q+1}}dt.$$

Proof. Using Lemma 1 and Lemma 2, we multiply both sides by $\frac{1}{n^z}$ and $\frac{(-1)^{n+1}}{n^z}$ respectively and sum both sides to obtain the Theorem. □

This is good news, because if we add the above formula with Theorem 1 (or Theorem 2) we get the following formula

$$\mathscr{J}(s,z)+\overline{\mathscr{H}}(z,s)=\eta(z+s)+\eta(s)\zeta(z)$$

for $\Re(s)+\Re(z)>2-m$. However it is also obvious that we can take $m\longrightarrow\infty$ and the formula will still hold, hence we get the following corollary.

COROLLARY 4. *For all* $(s,z)\in\mathbb{C}^2$ *and* $z\neq 1$*, the following formula holds:*

$$\mathscr{J}(s,z)+\overline{\mathscr{H}}(z,s)=\eta(z+s)+\eta(s)\zeta(z).$$

Letting $s=-n$ in Theorem 4 gives us

COROLLARY 5. (Negative values at one argument) *For all* $z\in\mathbb{C}$ *and* $n\in\mathbb{N}$*, the following two formulas holds:*

$$\mathscr{J}(-n,z)=\frac{\eta(z-n)}{2}+\frac{1}{2}\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2k+1}E_{2k+1}(0)\eta(z+2k+1-n)$$

$$\mathscr{J}(z,-n)=\frac{\eta(z-n)}{2}+\eta(z)\zeta(-n)+\frac{\eta(z-n-1)}{n+1}+\sum_{m=1}^{\lfloor\frac{n+1}{2}\rfloor}\binom{n+1}{2m}\frac{B_{2m}}{n+1}\eta(z-n+2m-1).$$

2.3. Skew harmonic eta function

In the below section we study the function $\overline{\mathscr{J}}(s,z)$.

DEFINITION 3. The skew harmonic eta function $\overline{\mathscr{J}}$ is defined as follows

$$\overline{\mathscr{J}}(s,z)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s}\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k^z}$$

where the special case $z=1$ will be written as

$$\overline{\mathscr{J}}(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s}\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}.$$

THEOREM 5. *For* $\Re(z)+\Re(s)+2m>2$ *and* $\Re(s)>1-2m$*, the following formula holds for* $m=1,2,3,\ldots$*:*

$$\overline{\mathscr{J}}(z,s)=\frac{\zeta(z+s)}{2}+\eta(s)\eta(z)+\frac{1}{2}\sum_{k=0}^{m-1}\frac{s^{\overline{2k+1}}E_{2k+1}(0)}{(2k+1)!}\zeta(z+s+2k+1)+\frac{(-s)^{\underline{2m}}}{2(2m-1)!}\sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n^z}\int_n^{\infty}\frac{\widetilde{E}_{2m-1}(-x)}{x^{s+2m}}\,dx$$

$$\overline{\mathscr{J}}(s,z)=\frac{\zeta(z+s)}{2}-\frac{1}{2}\sum_{k=0}^{m-1}\frac{s^{\overline{2k+1}}E_{2k+1}(0)}{(2k+1)!}\zeta(z+s+2k+1)-\frac{(-s)^{\underline{2m}}}{2(2m-1)!}\sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n^z}\int_n^{\infty}\frac{\widetilde{E}_{2m-1}(-x)}{x^{s+2m}}\,dx.$$

Proof. Consider Lemma 1, multiplying by $(-1)^{n+1}n^{-z}$ on both sides

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^z}\overline{H}_n^{(s)} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^z}\left[\frac{(-1)^{n+1}}{n^s}+\eta(s)+\frac{1}{2}\sum_{k=0}^{m-1}\frac{s^{\overline{k}}E_k(0)(-1)^{n+k}}{k!n^{s+k}}\right.$$
$$\left.+\frac{(-s)^{\underline{m}}}{2(m-1)!}\int_n^{\infty}\frac{\widetilde{E}_{m-1}(-x)}{x^{s+m}}\,dx\right]$$

$$\overline{\mathscr{J}}(z,s)=\zeta(z+s)+\eta(s)\eta(z)-\frac{1}{2}\sum_{k=0}^{m-1}\frac{s^{\overline{k}}E_k(0)(-1)^k}{k!}\zeta(z+s+k)$$
$$+\frac{(-s)^{\underline{m}}}{2(m-1)!}\sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n^z}\int_n^{\infty}\frac{\widetilde{E}_{m-1}(-x)}{x^{s+m}}\,dx.$$

Since $E_{2n}(0)=0$ for any natural numbers n, the summand only adds something every two terms. As such, let's change m to $2m$ for simplicity:

$$\overline{\mathscr{J}}(z,s)=\zeta(z+s)+\eta(s)\eta(z)-\frac{1}{2}\sum_{k=0}^{2m-1}\frac{s^{\overline{k}}E_k(0)(-1)^k}{k!}\zeta(z+s+k)$$
$$+\frac{(-s)^{\underline{2m}}}{2(2m-1)!}\sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n^z}\int_n^{\infty}\frac{\widetilde{E}_{2m-1}(-x)}{x^{s+2m}}\,dx.$$

To justify the latter infinite sum, we consider

$$\left|\sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n^z}\int_n^{\infty}\frac{\widetilde{E}_{2m-1}(-x)}{x^{s+2m}}\,dx\right|\leqslant\sum_{n\geqslant 1}\frac{1}{n^{\Re(z)}}\int_n^{\infty}\frac{|\widetilde{E}_{2m-1}(-x)|}{x^{\Re(s)+2m}}\,dx$$
$$\leqslant\frac{M_{2m-1}}{\Re(s)+2m-1}\sum_{n\geqslant 1}\frac{1}{n^{\Re(s)+\Re(z)+2m-1}}.$$

Which converges for $\Re(z)+\Re(s)+2m>2$, reindexing the finite sum and taking out the $k=0$ case gives the first formula.

Now we go back to Lemma 1 which states that

$$\sum_{k\geqslant n}\frac{(-1)^k}{k^s}=\frac{1}{2}\sum_{k=0}^{m-1}\frac{s^{\overline{k}}E_k(0)(-1)^{n+k}}{k!n^{s+k}}+\frac{(-s)^{\underline{m}}}{2(m-1)!}\int_n^{\infty}\frac{\widetilde{E}_{m-1}(-x)}{x^{s+m}}\,dx$$

for $\Re(s)>1-m$. We multiply both sides by $(-1)^n n^{-z}$, sum both sides and reindexing the finite sum proves the second formula. □

COROLLARY 6. (Negative integers at the both arguments) *The following formula holds for $z\neq n+1$ and $z\neq n-2k$ for $0\leqslant k\leqslant\lfloor\frac{n-1}{2}\rfloor$:*

$$\overline{\mathscr{J}}(-n,z)=\frac{\zeta(z-n)}{2}+\frac{1}{2}\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2k+1}E_{2k+1}(0)\zeta(z-n+2k+1)$$

$$\overline{\mathscr{J}}(z,-n) = \frac{\zeta(z-n)}{2} + \eta(-n)\eta(z) - \frac{1}{2}\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} E_{2k+1}(0)\zeta(z-n+2k+1).$$

Proof. Letting $s=-n$ in Theorem 5 gives the formula. The steps are similar to Corollary 1, 2, 5. □

In Theorem 5, the domain is $\Re(z)+\Re(s)+2m>2$. If we add them up, we get

$$\overline{\mathscr{J}}(s,z)+\overline{\mathscr{J}}(s,z) = \zeta(z+s)+\eta(s)\eta(z)$$

for $\Re(z)+\Re(s)+m>2$. Taking m to infinity gives

COROLLARY 7. *For all $(s,z)\in\mathbb{C}^2$ and $z+s\neq 1$, the following formula holds:*

$$\overline{\mathscr{J}}(z,s)+\overline{\mathscr{J}}(s,z) = \zeta(z+s)+\eta(s)\eta(z).$$

2.4. Residues

Using Theorem 2, we see that $\overline{\mathscr{H}}(s,z)$ is entire in z and is meromorphic in s with a pole of $s=1$. It is evident that

$$\operatorname*{Res}_{s=1}\left[\overline{\mathscr{H}}(s,z)\right] = \eta(z).$$

Similarly, using Theorem 4, it is also obvious that $\mathscr{J}(s,z)$ is both entire in s and in z.

Now we study the residues of $\overline{\mathscr{J}}(z,s)$, Using Theorem 5, we can see that its poles all come from the zeta functions:

$$\overline{\mathscr{J}}(z,s) = \frac{\zeta(z+s)}{2} + \eta(s)\eta(z) + \frac{1}{2}\sum_{k=0}^{m-1}\frac{s^{\overline{2k+1}}E_{2k+1}(0)}{(2k+1)!}\zeta(z+s+2k+1)$$
$$+\frac{(-s)^{\underline{2m}}}{2(2m-1)!}\sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n^z}\int_n^\infty \frac{\widetilde{E}_{2m-1}(-x)}{x^{s+2m}}\,dx$$

for $\Re(s)+\Re(z)>2-2m$. As we increase the value of m, we have a representation for $\overline{\mathscr{J}}(z,s)$ on the entire complex plane. Using the limit $\lim_{s\to 1}(s-1)\zeta(s)=1$, we get

$$\operatorname*{Res}_{z+s=1}\left[\overline{\mathscr{J}}(z,s)\right] = \frac{1}{2}$$
$$\operatorname*{Res}_{z=-2n-s}\left[\overline{\mathscr{J}}(z,s)\right] = \frac{s^{\overline{2n+1}}}{2(2n+1)!}E_{2n+1}(0)$$
$$\operatorname*{Res}_{s=-2n-z}\left[\overline{\mathscr{J}}(z,s)\right] = -\frac{E_{2n+1}(0)}{2}\binom{2n+z}{2n+1}$$

for any non-negative integers n.

The residues of the generalized harmonic zeta function are evaluated in [2].

2.5. Other possible harmonic zeta functions

In this section, we consider slightly different forms of harmonic zeta functions, for example the below function.

DEFINITION 4. The function $\mathscr{I}(s,z)$ is defined as the following

$$\mathscr{I}(s,z) := \sum_{n\geqslant 0} \frac{(-1)^n}{(2n+1)^s} \sum_{k=0}^{n} \frac{1}{(2k+1)^z}.$$

To study the analytic continuation of this function, we apply a similar technique in Lemma 1.

Starting with the Euler-Boole summation formula

$$\sum_{j=a}^{n-1}(-1)^j f(j+h) = \frac{1}{2}\sum_{k=0}^{m-1}\frac{E_k(h)}{k!}\Big((-1)^{n-1}f^{(k)}(n)+(-1)^a f^{(k)}(a)\Big) + \frac{1}{2(m-1)!}\int_a^n f^{(m)}(x)\widetilde{E}_{m-1}(h-x)\,dx$$

for $a,n\in\mathbb{N}$, $h\in[0,1)$. Let $f(x)=\frac{1}{(x+\frac{1}{2})^s} \implies f^{(k)}(x)=(-s)^{\underline{k}}(x+\frac{1}{2})^{-s-k}$, $h=0$ and let a be anything ($a=1$ for example):

$$\sum_{j=1}^{n-1}\frac{(-1)^j}{(j+\frac{1}{2})^s} = \frac{1}{2}\sum_{k=0}^{m-1}\frac{E_k(0)}{k!}\left((-1)^{n-1}\frac{(-s)^{\underline{k}}}{(n+\frac{1}{2})^{s+k}} - f^{(k)}(1)\right) + \frac{(-s)^{\underline{m}}}{2(m-1)!}\int_a^n \frac{\widetilde{E}_{m-1}(-x)}{(x+\frac{1}{2})^{s+m}}\,dx.$$

Take $n \longrightarrow \infty$ and subtract it from the above case

$$\sum_{j\geqslant n}\frac{(-1)^j}{(j+\frac{1}{2})^s} = \frac{1}{2}\sum_{k=0}^{m-1}\frac{E_k(0)}{k!}(-1)^n\frac{(-s)^{\underline{k}}}{(n+\frac{1}{2})^{s+k}} + \frac{(-s)^{\underline{m}}}{2(m-1)!}\int_n^\infty \frac{\widetilde{E}_{m-1}(-x)}{(x+\frac{1}{2})^{s+m}}\,dx$$

$$\sum_{j\geqslant n}\frac{(-1)^j}{(2j+1)^s} = \sum_{k=0}^{m-1}\frac{E_k(0)}{k!}(-1)^n\frac{(-s)^{\underline{k}}2^{k-1}}{(2n+1)^{s+k}} + \frac{(-s)^{\underline{m}}}{2^{s+1}(m-1)!}\int_n^\infty \frac{\widetilde{E}_{m-1}(-x)}{(x+\frac{1}{2})^{s+m}}\,dx.$$

The integral converges for $\Re(s) > 1-m$. Multiplying both sides by $\frac{1}{(2n+1)^z}$ and summing both sides gives

$$\sum_{n\geqslant 0}^{\infty}\frac{1}{(2n+1)^z}\sum_{j\geqslant n}\frac{(-1)^j}{(2j+1)^s} = \sum_{k=0}^{m-1}\frac{E_k(0)}{k!}(-s)^{\underline{k}}2^{k-1}\sum_{n\geqslant 0}^{\infty}\frac{(-1)^n}{(2n+1)^{s+k+z}} + \frac{1}{2^{s+1}(m-1)!}\sum_{n\geqslant 0}^{\infty}\frac{(-s)^{\underline{m}}}{(2n+1)^z}\int_n^\infty \frac{\widetilde{E}_{m-1}(-x)}{(x+\frac{1}{2})^{s+m}}\,dx.$$

Interchanging the summation gives us the following:

$$\mathscr{I}(s,z)=\sum_{k=0}^{m-1}\frac{E_k(0)}{k!}(-s)^{\underline{k}}2^{k-1}\beta(s+z+k)$$
$$+\frac{1}{2^{s+1}(m-1)!}\sum_{n\geqslant 0}^{\infty}\frac{(-s)^{\underline{m}}}{(2n+1)^z}\int_n^\infty\frac{\widetilde{E}_{m-1}(-x)}{(x+\frac{1}{2})^{s+m}}\,dx.$$

To find the convergence of the infinite sum, we use a similar technique as before

$$\left|\sum_{n\geqslant 0}^{\infty}\frac{1}{(2n+1)^z}\int_n^\infty\frac{\widetilde{E}_{m-1}(-x)}{(x+\frac{1}{2})^{s+m}}\,dx\right|\leqslant\sum_{n\geqslant 0}^{\infty}\frac{1}{(2n+1)^{\Re(z)}}\int_n^\infty\frac{M_{m-1}}{(x+\frac{1}{2})^{\Re(s)+m}}\,dx$$
$$\leqslant\frac{M_{m-1}}{\Re(s)+m-1}\sum_{n\geqslant 0}\frac{2^{\Re(s)+m-1}}{(2n+1)^{\Re(s)+\Re(z)+m-1}}$$

which converges for $\Re(z)+\Re(s)>2-m$. Since $E_{2n}(0)=0$ for $n\geqslant 1$, we can only consider m being *even*: $m\mapsto 2m$.

THEOREM 6. *For $\Re(s)>1-m$ and $\Re(z)+\Re(s)>2-2m$, the following formula holds:*

$$\mathscr{I}(s,z)=\sum_{k=0}^{2m-1}\frac{E_k(0)}{k!}(-s)^{\underline{k}}2^{k-1}\beta(s+z+k)$$
$$+\frac{(-s)^{\underline{2m}}}{2^{s+1}(2m-1)!}\sum_{n\geqslant 0}^{\infty}\frac{1}{(2n+1)^z}\int_n^\infty\frac{\widetilde{E}_{2m-1}(-x)}{(x+\frac{1}{2})^{s+2m}}\,dx.$$

Letting $s=-n$ gives the following corollary.

COROLLARY 8. *For all $z\in\mathbb{C}$ and $n\in\mathbb{N}$, the following formula holds:*

$$\mathscr{I}(-n,z)=\frac{\beta(z-n)}{2}+\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2j+1}E_k(0)4^j\beta(z+2j+1-n).$$

3. Harmonic polylogarithm function

Seeing the above you may ask: "What about the function $\sum_{n=0}^{\infty}\frac{H_{2n+1}^{(s)}}{(2n+1)^z}(-1)^n$?"

DEFINITION 5. Define the function $\mathrm{Hl}_z^{(s)}(x)$ as the following:

$$\mathrm{Hl}_z^{(s)}(x)=\sum_{n=1}^{\infty}\frac{H_n^{(s)}}{n^z}x^n.$$

Let's consider the domain of convergence, since $\sum_{k=1}^{n} \frac{1}{k^s} = \zeta(s) + O\left(\frac{1}{n^{s-1}}\right)$ for $\Re(s) > 1$.

We see that the summand has the asymptotic behavior

$$\frac{H_n^{(s)}}{n^z} x^n = O\left(\frac{x^n}{n^{z+s-1}}\right).$$

Therefore the sum converges absolutely for $|x| < 1$, converges for $\Re(s) > 1$ and $\Re(z) > 1$ when $|x| = 1$, and diverges for $|x| > 1$.

Removing the $H_n^{(s)}$ in the summand turns the function into a polylogarithm. In this regard, we may expect that $\mathrm{Hl}_z^{(s)}(x)$ has similar properties.

Using Definition 5, it is easy to prove that

$$\mathrm{Hl}_z^{(s)}(x) = \int_0^x \frac{\mathrm{Hl}_{z-1}^{(s)}(t)}{t}\, dt$$

which is similar to the formula $\int_0^x \frac{\mathrm{Li}_s(t)}{t}\, dt = \mathrm{Li}_{s+1}(x)$.

LEMMA 3.
$$\sum_{n=1}^{\infty} H_n^{(s)} x^n = \frac{\mathrm{Li}_s(x)}{1-x}.$$

Proof. Multiplying both sides by $1-x$ and expanding the sum gives the formula. □

Using the above lemma, we let $x \mapsto xt$, multiply by $\frac{1}{x}(-\log x)^{z-1}$ and integrate both sides from 0 to 1

$$\implies \int_0^1 \frac{\mathrm{Li}_s(xt)}{x(1-xt)}(-\log x)^{z-1}\, dx = \int_0^1 \sum_{n=1}^{\infty} H_n^{(s)} t^n x^{n-1} (-\log x)^{z-1}\, dx.$$

To switch the order of integration, we can use Weierstrass-M test if we assume that $|t| < 1$. Using the fact that $\int_0^1 x^{a-1}(-\log x)^{b-1}\, dx = \frac{\Gamma(b)}{a^b}$, We get the integral representation for the harmonic polylogarithm:

$$\mathrm{Hl}_z^{(s)}(t) = \frac{1}{\Gamma(z)} \int_0^1 \frac{\mathrm{Li}_s(xt)}{x(1-xt)}(-\log x)^{z-1}\, dx.$$

We can consider using analytic continuation if we want t on a larger domain.

THEOREM 7. *The harmonic polylogarithm $\mathrm{Hl}_z^{(s)}(x)$ is holomorphic on $(x,s,z) \in \mathbb{C}^3$ for where it is defined.*

Proof. Using the Cauchy-Riemann equations easily finishes the proof. □

By Lemma 2, multiplying both sides by $\frac{x^n}{n^z}$ and summing both sides gives the following.

THEOREM 8. *The below formula holds for* $\Re(s) > -2q$ *and* $|x| < 1$, $\Re(s) + \Re(z) > 1 - 2q$ *for* $|x| = 1$*:*

$$\mathrm{Hl}_z^{(s)}(x) = \mathrm{Li}_z(x)\zeta(s) + \frac{1}{1-s}\mathrm{Li}_{s+z-1}(x) + \frac{1}{2}\mathrm{Li}_{z+s}(x) - \sum_{m=1}^{q} \frac{s^{\overline{2m-1}} B_{2m}}{(2m)!} \mathrm{Li}_{s+z+2m-1}(x)$$
$$+ \frac{s^{\overline{2q+1}}}{(2q+1)!} \sum_{n \geqslant 1} \frac{x^n}{n^z} \int_n^\infty \frac{\widetilde{B}_{2q+1}(t)}{t^{s+2q+1}}\, dt.$$

The condition listed is again from the convergence of the latter integral term.

$$\begin{aligned} \frac{\mathrm{Hl}_z^{(s)}(ix) - \mathrm{Hl}_z^{(s)}(-ix)}{2i} &= \mathrm{Ti}_z(x)\zeta(s) + \frac{1}{1-s}\mathrm{Ti}_{s+z-1}(x) + \frac{1}{2}\mathrm{Ti}_{z+s}(x) \\ &\quad - \sum_{m=1}^{q} \frac{s^{\overline{2m-1}} B_{2m}}{(2m)!} \mathrm{Ti}_{s+z+2m-1}(x) \\ &\quad + \frac{s^{\overline{2q+1}}}{(2q+1)!} \sum_{n \geqslant 1} \frac{(-1)^n x^{2n+1}}{(2n+1)^z} \int_{2n+1}^\infty \frac{\widetilde{B}_{2q+1}(t)}{t^{s+2q+1}}\, dt \end{aligned}$$

where $\mathrm{Ti}_z(x)$ denotes the generalized inverse tangent integral. Letting $x = 1$ gives the formula for our desired alternating odd harmonic zeta function $\sum_{n=0}^\infty \frac{H_{2n+1}^{(s)}}{(2n+1)^z}(-1)^n$.

Though closely related, since it is not mainly about the topic in this paper I will not go too deep investigating the function $\mathrm{Hl}_z^{(s)}(x)$. A curious reader may consider studying the function

$$\sum_{n=1}^\infty \frac{x^n}{n^s} \sum_{k=1}^n \frac{y^k}{k^z}.$$

This function elicits similar properties like the argument interchange formulas in Corollary 4 and Corollary 7.

4. Difference transform

Before proceeding to the Laurent series, I will define a mathematical object that favors our derivation.

DEFINITION 6. The difference transform (DT for short) on an analytic function $f(x)$, denoted as $\mathscr{D}\{f\}(a,b)$ is defined by

$$\mathscr{D}\{f\}(a,b) := \int_a^b f(\lfloor x \rfloor) - f(x)\, dx.$$

We will mostly consider a, b to be integers. Sometimes if there are multiple variables, one could emphasize their notation as

$$\mathscr{D}_x\{f(x)\}(a,b) := \int_a^b f(\lfloor x \rfloor) - f(x)\, dx.$$

The x below the $\mathscr{D}$ is to show which variable we are doing transformation on.

Note that the difference transform could be written as

$$\mathscr{D}\{f\}(a,b) = \sum_{n=a}^{b-1} f(n) - \int_a^b f(x)\,dx.$$

If $f(x)$ has a removable singularity at an integer $x = x_0$, then we take $f(x_0)$ as $\lim_{x\to x_0} f(x)$. To solidify the usage of notation, here are some examples.

4.1. Examples

If $f(x) = \frac{1}{x}$, $a = 1$, $b = N$, then

$$\mathscr{D}\{f\}(1,N) = \int_1^N \frac{1}{\lfloor x \rfloor} - \frac{1}{x}\,dx = \sum_{n=1}^{N-1} \frac{1}{n} - \int_1^N \frac{1}{x}\,dx = H_{N-1} - \log(N).$$

If $f(x) = \frac{\log^n(x)}{x}$, $a = 1$, $b \longrightarrow \infty$, then

$$\mathscr{D}\{f\}(1,\infty) = \int_1^\infty \frac{\log^n(\lfloor x \rfloor)}{\lfloor x \rfloor} - \frac{\log^n(x)}{x}\,dx = \lim_{N\to\infty}\left[\sum_{k=1}^{N-1} \frac{\log^n(k)}{k} - \int_1^N \frac{\log^n(x)}{x}\right] = \gamma_n$$

where γ_n denotes the n-th Stieltjes constant.

Let $f(x) = \frac{\sin(x)}{x}$, $a = 0$, $b \longrightarrow \infty$, $f(x)$ has a removable singularity at $x = 0$ which has the limit 1.

$$\begin{aligned}\mathscr{D}\{f\}(0,\infty) &= \sum_{k=0}^{\infty} f(k) - \int_0^\infty f(t)\,dt = 1 + \sum_{k=1}^{\infty} \frac{\sin n}{n} - \int_0^\infty \frac{\sin t}{t}\,dt \\ &= 1 + \left(\frac{\pi - 1}{2}\right) - \frac{\pi}{2} = \frac{1}{2}\end{aligned}$$

where we used the classical results $\sum_{k=1}^\infty \frac{\sin n}{n} = \frac{\pi-1}{2}$ and $\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$.

Note that the difference transform could be written in terms of the Euler-Macluarin formula:

$$\begin{aligned}\mathscr{D}\{f\}(a,b) = &-\frac{f(b) - f(a)}{2} + \sum_{j=1}^{k} \frac{B_{2j}}{(2j)!}\left(f^{(2j-1)}(b) - f^{(2j-1)}(a)\right) \\ &- \frac{1}{(2k)!}\int_a^b \widetilde{B}_{2k}(t) f^{(2k)}(t)\,dt.\end{aligned} \tag{1}$$

5. Laurent series

DEFINITION 7. (Generalized harmonic zeta function) We define the generalized harmonic zeta function as

$$\mathscr{H}(s,z) = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{k=1}^{n} \frac{1}{k^z}$$

where the special case $z=1$ is denoted as

$$\mathscr{H}(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}\sum_{k=1}^{n}\frac{1}{k}.$$

This function is already thoroughly investigated in [2]. However, since it will be used later, I will state the formulas that we will use in the upcoming sections.

$$\mathscr{H}(s)=\frac{\zeta(s)}{s-1}+\frac{\zeta(1+s)}{2}-\sum_{n\geqslant 1}\frac{s}{n}\int_n^{\infty}\frac{\{t\}-\frac{1}{2}}{t^{s+1}}\,dt \tag{2}$$

$$\begin{aligned}\mathscr{H}(s,z)=\frac{\zeta(s+z-1)}{s-1}+\frac{\zeta(z+s)}{2}-\sum_{r=1}^{k}\binom{s+2r-2}{2r-1}\zeta(1-2r)\zeta(s+z+2r-1)\\ -\binom{s+2k}{2k+1}\sum_{n=1}^{\infty}\frac{1}{n^z}\int_n^{\infty}\frac{\widetilde{B}_{2k+1}(x)}{x^{s+2k+1}}\,dx.\end{aligned} \tag{3}$$

Formula 2 can be proved by putting $k=1$ and $z=1$ in formula 3.

This section will be split into three parts, the series expansion of $\mathscr{H}(s)$, $\mathscr{H}(s,z)$ and $\overline{\mathscr{H}}(s,z)$. Note that the case where $z=1$ is split up into different cases because $\mathscr{H}(s)$ has a double pole instead of a single pole at $s=1$.

LEMMA 4. *For integers n and complex s for where the integral converges, the following formulas hold:*

$$\int_n^{\infty}\frac{\{t\}-\frac{1}{2}}{t^2}\,dt=H_n-\log(n)-\gamma-\frac{1}{2n}$$

$$\int_n^{\infty}s\frac{\{x\}-\frac{1}{2}}{x^{s+1}}\,dx=\frac{n^{1-s}}{s-1}-\frac{1}{2n^s}-\zeta(s)+H_n^{(s)}.$$

I will only show the proof of the former formula, the latter one follows a similar procedure.

Proof. We start by considering the fact that $\{t\}=t-\lfloor t\rfloor$,

$$\begin{aligned}\int_N^{\infty}\frac{t-\lfloor t\rfloor-\frac{1}{2}}{t^2}\,dt&=\lim_{M\to\infty}\int_N^{M}\frac{1}{t}-\frac{\lfloor t\rfloor}{t^2}-\frac{1}{2t^2}\,dt\\&=\lim_{M\to\infty}\left[\int_N^{M}\frac{1}{t}\,dt-\sum_{k=N}^{M-1}\int_k^{k+1}\frac{k}{t^2}\,dt-\int_N^{M}\frac{1}{2t^2}\,dt\right]\\&=\lim_{M\to\infty}\left[\log(M)-\log(N)+\sum_{k=N}^{M-1}k\left(\frac{1}{k+1}-\frac{1}{k}\right)+\frac{1}{2M}-\frac{1}{2N}\right]\\&=\lim_{M\to\infty}\left[\log(M)-\log(N)-(H_M-H_N)+\frac{1}{2M}-\frac{1}{2N}\right]\\&=\lim_{M\to\infty}\left[\log(M)-H_M+\frac{1}{2M}-\log(N)+H_N-\frac{1}{2N}\right].\end{aligned}$$

Writing $\gamma=\lim_{M\to\infty}H_M-\log(M)$ finishes the proof. □

THEOREM 9. *For* $\Re(z) > -1$*:*

$$\int_n^\infty \frac{\{x\}-\frac{1}{2}}{x^2}\left(\frac{z+1}{x^z}\right)dx = \frac{1}{2n^{z+1}} - \mathscr{D}_t\left\{\frac{1}{t^{z+1}}\right\}(n,\infty).$$

Proof. Consider Lemma 4

$$\begin{aligned}\int_n^\infty \frac{\{x\}-\frac{1}{2}}{x^2}\left(\frac{z+1}{x^z}\right)dx &= \frac{n^{-z}}{z} + \frac{1}{2n^{z+1}} - \sum_{k\geqslant n}\frac{1}{k^{z+1}}\\ &= \frac{1}{2n^{z+1}} + \int_n^\infty \frac{dx}{x^{1+z}} - \sum_{k\geqslant n}\frac{1}{k^{z+1}}.\end{aligned}$$

Writing the latter term in terms of the difference transform gives the theorem. □

LEMMA 5.

$$\sum_{n\geqslant 1}\frac{1}{n}\int_n^\infty \frac{\{t\}-\frac{1}{2}}{t^2}\,dt = -\gamma_1 - \frac{\gamma^2}{2}.$$

Proof. Since we have Lemma 4, all we have to do is prove that

$$\sum_{n=1}^\infty \frac{H_n - \log(n) - \gamma}{n} = -\gamma_1 - \frac{\gamma^2}{2} + \frac{\pi^2}{12}$$

and subtract off $\frac{\pi^2}{12}$. Write

$$\begin{aligned}S = \sum_{n=1}^\infty \frac{H_n - \log(n) - \gamma}{n} &= \lim_{z\to 0}\sum_{n=1}^\infty \frac{H_n - \log(n) - \gamma}{n^{1+z}}\\ &= \lim_{z\to 0}\mathscr{H}(z+1) + \zeta'(z+1) - \gamma\zeta(z+1).\end{aligned}$$

Notice that the summand is $O(\frac{1}{n^2})$, by formula 2:

$$\mathscr{H}(s) = \frac{\zeta(s)}{s-1} + \frac{\zeta(s+1)}{2} - \sum_{n\geqslant 1}\frac{s}{n}\int_n^\infty \frac{\{x\}-\frac{1}{2}}{t^{s+1}}\,dt.$$

Let $s = z+1$, we can plug it into the limit

$$\begin{aligned}S &= \lim_{z\to 0}\zeta'(z+1) - \gamma\zeta(z+1) + \frac{\zeta(z+1)}{z} + \frac{\zeta(z+2)}{2} + (-z-1)\sum_{N\geqslant 1}\int_N^\infty \frac{\{t\}-\frac{1}{2}}{Nt^{z+2}}\,dt\\ &= \lim_{z\to 0}\zeta'(z+1) + \zeta(z+1)\left(\frac{1}{z}-\gamma\right) + \frac{\zeta(2)}{2} - \sum_{N\geqslant 1}\int_N^\infty \frac{\{t\}-\frac{1}{2}}{Nt^{z+2}}\,dt\end{aligned}$$

to evaluate the limit, we use the Laurent expansion of ζ:

$$\zeta(1+s) = \frac{1}{s} + \gamma - \gamma_1 s + O(s^2) \quad\text{and}\quad \zeta'(1+s) = -\frac{1}{s^2} - \gamma_1 + O(s),$$

$$S = \lim_{z\to 0} -\frac{1}{z^2} - \gamma_1 + O(z) + \left(\frac{1}{z} + \gamma - \gamma_1 z + O(z^2)\right)\left(\frac{1}{z} - \gamma\right)$$
$$+ \frac{\zeta(2)}{2} - \sum_{N\geqslant 1}\int_N^\infty \frac{\{t\}-\frac{1}{2}}{Nt^{z+2}}\,dt$$
$$= -2\gamma_1 - \gamma^2 + \frac{\pi^2}{12} - \lim_{z\to 0}\sum_{N\geqslant 1}\int_N^\infty \frac{\{t\}-\frac{1}{2}}{Nt^{z+2}}\,dt.$$

By Lemma 4,

$$S = -2\gamma_1 - \gamma^2 + \frac{\pi^2}{12} - \left(S - \frac{\pi^2}{12}\right)$$

$$S = \sum_{n=1}^{\infty}\frac{H_n - \log(n) - \gamma}{n} = -\gamma_1 - \frac{\gamma^2}{2} + \frac{\pi^2}{12}$$

subtracting $\frac{\pi^2}{12}$ proves the lemma. □

LEMMA 6. *For integers $m \geqslant 0$*

$$\frac{d^m}{dz^m}\frac{\zeta(z+1)}{z} = \frac{(-1)^m}{z^{m+2}}(m+1)! + \frac{\gamma(-1)^m}{z^{m+1}}m! + \sum_{k\geqslant 0}\frac{\gamma_{k+m+1}}{(k+m+1)k!}z^k(-1)^{k+m+1}$$

where γ_n denotes the n-th Stieltjes constant.

Proof. Consider the Laurent series of $\zeta(s)$ centered at 1:

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{n\geqslant 1}(-1)^n\frac{\gamma_n}{n!}(s-1)^n$$

divide by z and differentiate n times gives the lemma. □

5.1. Harmonic zeta function

Consider formula 2

$$\mathscr{H}(s) = \frac{\zeta(s)}{s-1} + \frac{\zeta(1+s)}{2} - \sum_{n\geqslant 1}\frac{s}{n}\int_n^\infty \frac{\{t\}-\frac{1}{2}}{t^{s+1}}\,dt$$

and the Laurent series of $\zeta(s)$

$$\Longrightarrow \mathscr{H}(s) = \frac{1}{s-1}\left(\frac{1}{s-1} + \gamma + \sum_{n\geqslant 1}(-1)^n\frac{\gamma_n}{n!}(s-1)^n\right) + \frac{\zeta(s+1)}{2}$$
$$- \sum_{n\geqslant 1}\frac{s}{n}\int_n^\infty \frac{\{t\}-\frac{1}{2}}{t^{s+1}}\,dt$$

where γ_n denotes the n-th Stieltjes constant

$$= \frac{1}{(s-1)^2} + \frac{\gamma}{s-1} - \gamma_1 + O(s-1) + \frac{\zeta(1+s)}{2} - \sum_{n \geqslant 1} \frac{s}{n} \int_n^\infty \frac{\{t\} - \frac{1}{2}}{t^{s+1}} dt$$

$$\implies \lim_{s \to 1} \left[\mathscr{H}(s) - \frac{1}{(s-1)^2} - \frac{\gamma}{s-1} \right] = -\gamma_1 + \frac{\zeta(2)}{2} - \sum_{n \geqslant 1} \frac{1}{n} \int_n^\infty \frac{\{t\} - \frac{1}{2}}{t^2} dt.$$

The infinite sum is evaluated in Lemma 4

$$\lim_{s \to 1} \left[\mathscr{H}(s) - \frac{1}{(s-1)^2} - \frac{\gamma}{s-1} \right] = \frac{\gamma^2 + \zeta(2)}{2}.$$

Therefore we now have the Laurent expansion centered at $s = 1$ up to the constant term

$$\implies \mathscr{H}(s) = \frac{1}{(s-1)^2} + \frac{\gamma}{s-1} + \frac{\gamma^2 + \zeta(2)}{2} + O(s-1).$$

We now compute the Taylor expansion, letting $z = s+1$ and differentiating formula 2 on both sides gives

$$\mathscr{H}^{(m)}(z+1) = \frac{d^m}{dz^m} \frac{\zeta(z+1)}{z} + \frac{\zeta^{(m)}(z+2)}{2} - \sum_{N \geqslant 1} \frac{1}{N} \frac{d^m}{dz^m} \int_N^\infty \frac{\{x\} - \frac{1}{2}}{x^2} \left(\frac{z+1}{x^z} \right) dx.$$

Using Lemma 6 gives

$$\begin{aligned} &\mathscr{H}^{(m)}(z+1) - \frac{(-1)^m}{z^{m+2}}(m+1)! - \frac{\gamma(-1)^m}{z^{m+1}} m! \\ &= \frac{\zeta^{(m)}(z+2)}{2} + \sum_{k \geqslant 0} \frac{\gamma_{k+m+1}}{(k+m+1)k!} z^k (-1)^{k+m+1} \\ &\quad - \sum_{N \geqslant 1} \frac{1}{N} \frac{d^m}{dz^m} \int_N^\infty \frac{\{x\} - \frac{1}{2}}{x^2} \left(\frac{z+1}{x^z} \right) dx. \end{aligned} \tag{4}$$

Let $f(z+1) = \mathscr{H}(z+1) - \frac{1}{z^2} - \frac{\gamma}{z}$, then $f(z)$ has the expansion

$$f(z+1) = \frac{\gamma^2 + \zeta(2)}{2} + \sum_{n=1}^\infty \frac{f^{(n)}(1)}{n!} z^n.$$

The left hand side of formula 4 is $f^{(m)}(z+1)$, taking z to 0 gives

$$f^{(m)}(1) = \frac{\zeta^{(m)}(2)}{2} + \frac{(-1)^{m+1}\gamma_{m+1}}{m+1} - \sum_{N \geqslant 1} \frac{1}{N} \frac{d^m}{dz^m} \int_N^\infty \frac{\{x\} - \frac{1}{2}}{x^2} \left(\frac{z+1}{x^z} \right) dx \bigg|_{z=0}.$$

Using Theorem 9 gives

$$= \frac{\zeta^{(m)}(2)}{2} + \frac{(-1)^{m+1}\gamma_{m+1}}{m+1} - \sum_{N \geqslant 1} \frac{1}{N} \left[(-1)^m \left[\frac{\log^m N}{2N} - \mathscr{D}\left(\frac{\log^m t}{t} \right)(N, \infty) \right] \right]$$

$$= \frac{\zeta^{(m)}(2)}{2} + \frac{(-1)^{m+1}\gamma_{m+1}}{m+1} - \frac{\zeta^{(m)}(2)}{2} + \sum_{N \geqslant 1} \frac{(-1)^m}{N} \mathscr{D}\left(\frac{\log^m t}{t} \right)(N, \infty)$$

$$\implies f^{(m)}(1) = \frac{(-1)^{m+1}\gamma_{m+1}}{m+1} + (-1)^m \sum_{N\geqslant 1} \frac{1}{N}\mathscr{D}\left(\frac{\log^m t}{t}\right)(N,\infty).$$

This gives the general term of the Laurent expansion of the harmonic zeta function.

THEOREM 10. (Laurent series of the harmonic zeta function) *For* $|z|<1$*:*

$$\mathscr{H}(z+1) = \frac{1}{z^2} + \frac{\gamma}{z} + \frac{\gamma^2+\zeta(2)}{2} + \sum_{n\geqslant 1} \frac{(-1)^n\tilde{\gamma}_n}{n!} z^n$$

where

$$\tilde{\gamma}_n = -\frac{\gamma_{n+1}}{n+1} + \sum_{N\geqslant 1} \frac{1}{N}\mathscr{D}_t\left(\frac{\log^n t}{t}\right)(N,\infty)$$

denotes the n*-th harmonic Stieljes constants.*

Note that $\mathscr{H}(s)$ has poles at $1,0,-1,-3,\ldots$, so the radius of convergence of its expansion at $s=1$ is 1.

REMARK 1. We can also write the formula as

$$\mathscr{H}(z+1) = \frac{\zeta(1+z)}{z} + \gamma_1 + \frac{\gamma^2+\zeta(2)}{2} + \sum_{n\geqslant 1} \frac{z^n}{n!}(-1)^n \sum_{N\geqslant 1} \frac{1}{N}\mathscr{D}\left(\frac{\log^n t}{t}\right)(N,\infty).$$

COROLLARY 9. *For natural numbers* m

$$\sum_{n\geqslant 1} \frac{H_n - \log(n) - \gamma}{n} \log^m(n) = \tilde{\gamma}_m - \gamma_{m+1} - \gamma\gamma_m.$$

Proof. This can be proved by considering the Laurent expansion of Riemann zeta and harmonic zeta function, differentiating both sides m times and taking the limit z to 0. □

5.2. Generalized harmonic zeta function

In this section, we consider $z \neq 1$ so that the pole at $s=1$ are all simple poles. Consider letting $k=0$ in formula 3:

$$\mathscr{H}(s,z) = \frac{\zeta(s+z)}{2} + \frac{\zeta(s+z-1)}{s-1} - s\sum_{n\geqslant 1} \frac{1}{n^z} \int_n^\infty \frac{\{x\}-\frac{1}{2}}{x^{s+1}}\,dx.$$

Consider $z\neq 1$, then $\mathscr{H}(s,z)$ only have simple poles, more specifically, it has a simple pole at $s=1$

$$\begin{aligned}
&\lim_{s\to 1}\left(\mathscr{H}(s,z) - \frac{\zeta(z)}{s-1}\right)\\
&= \lim_{s\to 1}\left[\frac{\zeta(s+z)}{2} + \frac{\zeta(s+z-1)-\zeta(z)}{s-1} - s\sum_{n\geqslant 1}\frac{1}{n^z}\int_n^\infty \frac{\{x\}-\frac{1}{2}}{x^{s+1}}\,dx\right]\\
&= \frac{\zeta(z+1)}{2} + \zeta'(z) - \sum_{n\geqslant 1}\frac{1}{n^z}\int_n^\infty \frac{\{x\}-\frac{1}{2}}{x^2}\,dx.
\end{aligned}$$

From Lemma 4, we have

$$\int_n^\infty \frac{\{x\}-\frac{1}{2}}{x^2}\,dx = H_n - \log n - \gamma - \frac{1}{2n}$$
$$\Longrightarrow \lim_{s\to 1}\left(\mathscr{H}(s,z)-\frac{\zeta(z)}{s-1}\right) = \frac{\zeta(z+1)}{2}+\zeta'(z)-\sum_{n\geqslant 1}\frac{1}{n^z}\left(H_n-\log n-\gamma-\frac{1}{2n}\right)$$
$$=\gamma\zeta(z)-\mathscr{H}(z)+\zeta(z+1).$$

This shows that

$$\mathscr{H}(s,z)=\frac{\zeta(z)}{s-1}+\gamma\zeta(z)-\mathscr{H}(z)+\zeta(z+1)+O(s-1).$$

Before deriving the general term of $\mathscr{H}(s,z)$, we prove the following limit which will arise in our later derivation

LEMMA 7. *The following limit holds:*

$$\frac{\partial^n}{\partial s^n}\left(\frac{\zeta(s+z)-\zeta(z)}{s}\right)\Big|_{s\to 0}=\frac{\zeta^{(n+1)}(z)}{n+1}.$$

Proof. Leibniz product rule gives

$$\frac{\partial^m}{\partial s^m}\left(\frac{\zeta(s+z)-\zeta(z)}{s}\right)\Big|_{s\to 0}=\lim_{s\to 0}\left[\frac{(-1)^m}{s^{m+1}}\left(m!(\zeta(s+z)-\zeta(z))\right.\right.$$
$$\left.\left.+\sum_{j=1}^{m}\binom{m}{j}(-1)^j(m-j)!s^j\zeta^{(j)}(s+z)\right)\right].$$

To evaluate this limit, we do LHospital $m+1$ times

$$=\lim_{s\to 0}\left[\frac{(-1)^m}{(m+1)!}\left(m!\zeta^{(m+1)}(s+z)+\sum_{j=1}^{m}\binom{m}{j}(-1)^j(m-j)!\frac{\partial^{m+1}}{\partial s^{m+1}}s^j\zeta^{(j)}(s+z)\right)\right]$$
$$=\lim_{s\to 0}\left[\frac{(-1)^m}{(m+1)!}\left(m!\zeta^{(m+1)}(s+z)+\sum_{j=1}^{m}\binom{m}{j}(-1)^j(m-j)!\right.\right.$$
$$\left.\left.\times\sum_{i=0}^{m+1}\binom{m+1}{i}j^{\underline{i}}s^{j-i}\zeta^{(j+m+1-i)}(z+s)\right)\right].$$

As we are taking $s\to 0$, the latter inner sum is non zero only if $i=j$ (because of the s^{j-i} term and the falling factorial term).

$$
\begin{aligned}
&= \lim_{s\to 0}\left[\frac{(-1)^m}{(m+1)!}\left(m!\zeta^{(m+1)}(s+z)+\sum_{j=1}^{m}\binom{m}{j}(-1)^j(m-j)!\binom{m+1}{j}j!\zeta^{(m+1)}(z+s)\right)\right]\\
&= \lim_{s\to 0}\left[\frac{(-1)^m}{m+1}\left(\zeta^{(m+1)}(s+z)+\sum_{j=1}^{m}(-1)^j\binom{m+1}{j}\zeta^{(m+1)}(z+s)\right)\right]\\
&= \frac{(-1)^m}{m+1}\left(\sum_{j=0}^{m}(-1)^j\binom{m+1}{j}\zeta^{(m+1)}(z)\right)\\
&= \frac{(-1)^m}{m+1}\left(\underbrace{\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}}_{(1+(-1))^{m+1}=0}\zeta^{(m+1)}(z)-(-1)^{m+1}\zeta^{(m+1)}(z)\right)\\
&= \frac{\zeta^{(m+1)}(z)}{m+1}
\end{aligned}
$$

$$
\Longrightarrow \frac{\partial^n}{\partial s^n}\left(\frac{\zeta(s+z)-\zeta(z)}{s}\right)\Big|_{s\to 0}=\frac{\zeta^{(n+1)}(z)}{n+1}.
$$

This proves the limit. □

THEOREM 11. (Laurent expansion of the generalized harmonic zeta function)

$$
\begin{aligned}
\mathscr{H}(1+s,z) = &\frac{\zeta(z)}{s}+\gamma\zeta(z)-\mathscr{H}(z)+\zeta(1+z)\\
&+\sum_{n\geqslant 1}\frac{s^n}{n!}\left[\frac{\zeta^{(n+1)}(z)}{n+1}+(-1)^n\sum_{k\geqslant 1}\frac{1}{k^z}\mathscr{D}\left(\frac{\log^n t}{t}\right)(k,\infty)\right].
\end{aligned}
$$

Proof. Let $f(s+1)=\mathscr{H}(s+1,z)-\frac{\zeta(z)}{s}$. Then by Taylor series expansion: $f(1+s)=\gamma\zeta(z)-\mathscr{H}(z)+\zeta(z+1)+\sum_{n\geqslant 1}\frac{f^{(n)}(1)}{n!}s^n$.

Our goal is to compute $f^{(n)}(1)$. We again utilize formula 3, consider

$$
\mathscr{H}(s,z)=\frac{\zeta(s+z)}{2}+\frac{\zeta(s+z-1)}{s-1}-s\sum_{k\geqslant 1}\frac{1}{k^z}\int_n^\infty\frac{\{x\}-\frac{1}{2}}{x^{s+1}}\,dx
$$

$$
\begin{aligned}
f^{(n)}(1+s)=&\frac{\partial^n}{\partial s^n}\left(\frac{\zeta(s+z)-\zeta(z)}{s}+\frac{\zeta(s+z+1)}{2}\right)\\
&-\sum_{k\geqslant 1}\frac{1}{k^z}\frac{\partial^n}{\partial s^n}\int_n^\infty\frac{\{x\}-\frac{1}{2}}{x^2}\frac{s+1}{x^s}\,dx.
\end{aligned}
$$

We take $s \to 0$,

$$f^{(n)}(1) = \frac{\partial^n}{\partial s^n}\left(\frac{\zeta(s+z)-\zeta(z)}{s} + \frac{\zeta(s+z+1)}{2}\right)\Big|_{s\to 0} - \sum_{k\geqslant 1}\frac{1}{k^z}\frac{\partial^n}{\partial s^n}\int_k^\infty \frac{\{x\}-\frac{1}{2}}{x^2}\frac{s+1}{x^s}\,dx\Big|_{s\to 0}$$
$$= \frac{\partial^n}{\partial s^n}\left(\frac{\zeta(s+z)-\zeta(z)}{s}\right)\Big|_{s\to 0} + \frac{\zeta^{(n)}(z+1)}{2} - \sum_{k\geqslant 1}\frac{1}{k^z}\frac{\partial^n}{\partial s^n}\int_k^\infty \frac{\{x\}-\frac{1}{2}}{x^2}\frac{s+1}{x^s}\,dx\Big|_{s\to 0}$$

using Theorem 9:

$$f^{(n)}(1) = \frac{\partial^n}{\partial s^n}\left(\frac{\zeta(s+z)-\zeta(z)}{s}\right)\Big|_{s\to 0} + \frac{\zeta^{(n)}(z+1)}{2} - \sum_{N\geqslant 1}\frac{1}{N^z}(-1)^n\left[\frac{\log^n N}{2N} - \mathscr{D}\left(\frac{\log^n t}{t}\right)(N,\infty)\right]$$
$$= \frac{\partial^n}{\partial s^n}\left(\frac{\zeta(s+z)-\zeta(z)}{s}\right)\Big|_{s\to 0} + (-1)^n\sum_{N\geqslant 1}\frac{\mathscr{D}\left(\frac{\log^n t}{t}\right)(N,\infty)}{N^z}.$$

The limit is computed in Lemma 6

$$f^{(n)}(1) = \frac{\zeta^{(n+1)}(z)}{n+1} + (-1)^n\sum_{N\geqslant 1}\frac{\mathscr{D}\left(\frac{\log^n t}{t}\right)(N,\infty)}{N^z}.$$

Putting $f^{(n)}(1)$ back into its expansion, we have that

$$\mathscr{H}(1+s,z) = \frac{\zeta(z)}{s} + \gamma\zeta(z) - \mathscr{H}(z) + \zeta(1+z) + \sum_{n\geqslant 1}\frac{s^n}{n!}\left[\frac{\zeta^{(n+1)}(z)}{n+1} + (-1)^n\sum_{k\geqslant 1}\frac{1}{k^z}\mathscr{D}\left(\frac{\log^n t}{t}\right)(k,\infty)\right].$$

This finishes the proof. □

Note that since $\mathscr{H}(s,z)$ has poles at $1, 2-z, 1-z, -2n-z$ for $n = 0,1,2,3,\ldots$, the radius of convergence is dependent on z. This also applies to Theorem 12.

REMARK 2. If we move the infinite sum into the difference transform, we have

$$\mathscr{H}(1+s,z) = \gamma\zeta(z) - \mathscr{H}(z) + \zeta(1+z) + \frac{\zeta(z+s)}{s} - \zeta'(z) + \sum_{k\geqslant 1}\frac{1}{k^z}\mathscr{D}\left(\frac{1}{t^{s+1}}\right)(k,\infty).$$

5.3. Generalized skew harmonic zeta function

THEOREM 12. (Laurent series of the generalized skew harmonic zeta function)

$$\overline{\mathscr{H}}(s+1,z) = \frac{\eta(z)}{s} + \gamma\eta(z) + \eta(z+1) - \mathscr{J}(z) + \sum_{m\geqslant 1}\frac{s^m}{m!}\left[\frac{\eta^{(m+1)}(z)}{m+1} + (-1)^m \sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n^z}\mathscr{D}\left\{\frac{\log^m t}{t}\right\}(n,\infty)\right].$$

Proof. We start with Theorem 2, putting $q=0$ gives

$$\overline{\mathscr{H}}(s,z) = \frac{\eta(s+z-1)}{s-1} + \frac{\eta(s+z)}{2} - \sum_{n\geqslant 1}\frac{(-1)^{n+1}s}{n^z}\int_n^\infty \frac{\{x\}-\frac{1}{2}}{x^{s+1}}\,dx.$$

This is similar to formula 3, finding the residue and the constant term

$$\lim_{s\to 1}(s-1)\overline{\mathscr{H}}(s,z) = \lim_{s\to 1}\eta(s+z-1) + O(s-1) = \eta(z)$$

$$\implies \overline{\mathscr{H}}(s,z) = \frac{\eta(z)}{s-1} + \frac{\eta(s+z-1)-\eta(z)}{s-1} + \frac{\eta(s+z)}{2} - \sum_{n\geqslant 1}\frac{(-1)^{n+1}s}{n^z}\int_n^\infty \frac{\{x\}-\frac{1}{2}}{x^{s+1}}\,dx.$$

The constant term is

$$\lim_{s\to 1}\left(\overline{\mathscr{H}}(s,z) - \frac{\eta(z)}{s-1}\right) = \eta'(z) + \frac{\eta(z+1)}{2} - \sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n^z}\int_n^\infty \frac{\{x\}-\frac{1}{2}}{x^2}\,dx$$

we use Lemma 4 to obtain

$$\lim_{s\to 1}\left(\overline{\mathscr{H}}(s,z) - \frac{\eta(z)}{s-1}\right) = \gamma\eta(z) + \eta(z+1) - \mathscr{J}(z).$$

Now we find the general coefficient of its Taylor series. We use the same technique used in the prior section, let $f(s) = \overline{\mathscr{H}}(s,z) - \frac{\eta(z)}{s-1}$.

Then

$$f(s+1) = \gamma\eta(z) + \eta(z+1) - \mathscr{J}(z) + \sum_{n\geqslant 1}\frac{f^{(n)}(1)}{n!}s^n.$$

Since

$$\overline{\mathscr{H}}(s,z) - \frac{\eta(z)}{s-1} = \frac{\eta(s+z-1)-\eta(z)}{s-1} + \frac{\eta(s+z)}{2} - \sum_{n\geqslant 1}\frac{(-1)^{n+1}s}{n^z}\int_n^\infty \frac{\{x\}-\frac{1}{2}}{x^{s+1}}\,dx$$

we have

$$\begin{aligned} f^{(m)}(1) &= \frac{\partial^m}{\partial s^m}\left[\frac{\eta(s+z-1)-\eta(z)}{s-1} + \frac{\eta(s+z)}{2} - \sum_{n\geqslant 1}\frac{(-1)^{n+1}s}{n^z}\int_n^\infty \frac{\{x\}-\frac{1}{2}}{x^{s+1}}\,dx\right]_{s\to 1} \\ &= \frac{\partial^m}{\partial s^m}\frac{\eta(s+z)-\eta(z)}{s}\bigg|_{s\to 0} + \frac{\eta^{(m)}(1+z)}{2} \\ &\quad - \sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n^z}\frac{\partial^m}{\partial s^m}\int_n^\infty \frac{\{x\}-\frac{1}{2}}{x^2}\frac{s+1}{x^s}\,dx\bigg|_{s\to 0}. \end{aligned}$$

Again, we use Theorem 9

$$f^{(m)}(1) = \frac{\partial^m}{\partial s^m} \frac{\eta(s+z)-\eta(z)}{s}\Big|_{s\to 0} + \frac{\eta^{(m)}(1+z)}{2} - (-1)^m \sum_{n\geqslant 1} \frac{(-1)^{n+1}}{n^z}\left(\frac{\log^m n}{2n} - \mathscr{D}\left\{\frac{\log^m t}{t}\right\}(n,\infty)\right)$$

$$\implies f^{(m)}(1) = \frac{\partial^m}{\partial s^m} \frac{\eta(s+z)-\eta(z)}{s}\Big|_{s\to 0} + (-1)^m \sum_{n\geqslant 1} \frac{(-1)^{n+1}}{n^z}\mathscr{D}\left\{\frac{\log^m t}{t}\right\}(n,\infty).$$

The last step is to compute the former limit, we do it similar to how we did it to the zeta function in Lemma 6. The limit works out to be

$$\frac{\partial^m}{\partial s^m} \frac{\eta(s+z)-\eta(z)}{s}\Big|_{s\to 0} = \frac{\eta^{(m+1)}(z)}{m+1}.$$

In fact, we can replace zeta with eta in the derivation of Lemma 6 and the lemma still holds. This shows that

$$f^{(m)}(1) = \frac{\eta^{(m+1)}(z)}{m+1} + (-1)^m \sum_{n\geqslant 1} \frac{(-1)^{n+1}}{n^z}\mathscr{D}\left\{\frac{\log^m t}{t}\right\}(n,\infty).$$

Substituting $f^{(m)}(1)$ the formula below

$$\overline{\mathscr{H}}(s+1,z) = \frac{\eta(z)}{s} + \gamma\eta(z) + \eta(z+1) - \mathscr{J}(z) + \sum_{n\geqslant 1} \frac{f^{(n)}(1)}{n!}s^n$$

proves the theorem. □

REMARK 3.

$$\overline{\mathscr{H}}(s+1,z) = \frac{\eta(s+z)}{s} - \eta'(z) + \gamma\eta(z) + \eta(z+1) - \mathscr{J}(z) + \sum_{m\geqslant 1} \frac{(-s)^m}{m!} \sum_{n\geqslant 1} \frac{(-1)^{n+1}}{n^z}\mathscr{D}\left\{\frac{\log^m t}{t}\right\}(n,\infty).$$

Furthermore, one can prove that

$$\frac{\partial^m}{\partial s^m}\mathrm{Hl}_z^{(s)}(x)\Big|_{s=1} = (-1)^m\gamma_m \mathrm{Li}_z(x) - \frac{\partial^{m+1}}{\partial z^{m+1}}\frac{\mathrm{Li}_z(x)}{m+1} + \frac{\partial^m}{\partial s^m}\mathrm{Li}_{s+z}(x)\Big|_{s=1} + (-1)^{m+1}\sum_{n\geqslant 1}\frac{x^n}{n^z}\mathscr{D}\left\{\frac{\log^m t}{t}\right\}(n,\infty)$$

but does not seem to have any significant use for now.

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