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TRIPLE SEQUENCES AND DEFERRED STATISTICAL CONVERGENCE IN THE CONTEXT OF GRADUAL NORMED LINEAR SPACES

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Abstract. In this article, we introduce the concepts of "deferred statistical convergence" and "strong deferred convergence" concerning triple sequences in the framework of gradual normed linear spaces. Our work reveals significant insights into the distinctions between these two concepts. Furthermore, we conduct an in-depth examination of the analytical properties of these notions and establish multiple implication relationships.

1. Introduction and literature review

The concept of fuzzy sets was initially introduced by Zadeh [25] in 1965 as an extension of the traditional set theory. Today, it finds broad applications across various scientific and engineering fields. The term "fuzzy number" plays a pivotal role in fuzzy set theory but differs from classical numbers and doesn't adhere to certain algebraic properties, which has led to debate among authors about its behavior. To address this confusion, some authors prefer using the term "fuzzy intervals" instead of "fuzzy numbers".

To alleviate researcher's confusion, Fortin et al. [9] introduced the concept of gradual real numbers within fuzzy intervals in 2008. Gradual real numbers are primarily defined by their assignment function in the interval $(0,1]$, and every real number can be seen as a gradual real number with a constant assignment function. Unlike fuzzy numbers, gradual real numbers comply with all the algebraic properties of classical real numbers and find applications in computation and optimization problems.

In 2011, Sadeqi and Azari [18] were the pioneers in introducing the concept of gradual normed linear spaces. They conducted an in-depth analysis of various properties, considering both algebraic and topological aspects, and demonstrated that a gradual normed linear space can be classified as a locally convex space. This classification has significant implications, as it implies that the four fundamental theorems of locally convex spaces – the Hahn-Banach theorem, the uniform boundedness theorem, the open mapping theorem, and the closed graph theorem are applicable in the context of gradual normed spaces.

Considering that the scope of gradual normed linear spaces extends beyond that of classical spaces, researchers have recognized the importance of delving deeper into this direction. In recent years, this field has witnessed significant developments, driven by

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the contributions of researchers such as Ettefagh et al. [6, 7], Choudhury and Debnath [4], and many others. For a comprehensive exploration of gradual real numbers, readers can refer to [2, 5, 14], which offer extensive references for further study in this domain.

In 1951, the concept of statistical convergence emerged independently through the work of Fast [8] and Steinhaus [22], aiming to offer deeper insights into summability theory. Subsequently, it garnered significant attention within the realm of sequence spaces, with notable contributions from researchers like Fridy [10, 11], Šalát [20], and others. In 2003, Mursaleen and Edely [16] extended this concept to double sequences, focusing on its relationship with statistical Cauchy double sequences and strong Cesàro summable double sequences.

Additionally, Tripathy [23], in a study published in 2003, explored various properties of sequence spaces formed by statistical convergent double sequences and established a decomposition theorem. In 2007, Sahiner et al. [19] investigated statistical convergence for triple sequences, further expanding the reach of this concept. It's worth noting that statistical convergence finds applications in diverse branches of mathematics, including number theory, mathematical analysis, probability theory, and more.

In 1932, Agnew [1] introduced a generalization of the Cesàro mean, known as the deferred Cesàro mean, which offered enhanced features and utility. Building on this development, Küçükaslan and Yilmaztürk [13] in 2016 introduced the concept of deferred statistical convergence, utilizing the deferred Cesàro mean as a foundation. Their work involved proving fundamental properties and establishing several implication relationships between deferred statistical convergence, strong deferred Cesàro mean, and statistical convergence. For more in-depth information regarding deferred statistical convergence and its various generalizations, [3, 12, 15, 21] can be addressed where many more references can be found.

2. Definitions and preliminaries

Throughout the paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers respectively and by the convergence of a triple sequence, we mean the convergence in Pringsheim's [17] sense.

DEFINITION 1. [9] A gradual real number $\tilde{r}$ is defined by an assignment function $\mathscr{A}_{\tilde{r}} : (0,1] \to \mathbb{R}$. The set of all gradual real numbers is denoted by $G(\mathbb{R})$. A gradual real number $\tilde{r}$ is said to be non-negative if for every $\kappa \in (0,1]$, $\mathscr{A}_{\tilde{r}}(\kappa) \geqslant 0$. The set of all non-negative gradual real numbers is denoted by $G^*(\mathbb{R})$.

In [9], the gradual operations between the elements of $G(\mathbb{R})$ was defined as follows:

DEFINITION 2. Let "$*$" be any operation in $\mathbb{R}$ and suppose $\tilde{r}_1, \tilde{r}_2 \in G(\mathbb{R})$ with assignment functions $\mathscr{A}_{\tilde{r}_1}$ and $\mathscr{A}_{\tilde{r}_2}$ respectively. Then, $\tilde{r}_1 * \tilde{r}_2 \in G(\mathbb{R})$ is defined with the assignment function $\mathscr{A}_{\tilde{r}_1 * \tilde{r}_2}$ given by

$$\mathscr{A}_{\tilde{r}_1 * \tilde{r}_2}(\kappa) = \mathscr{A}_{\tilde{r}_1}(\kappa) * \mathscr{A}_{\tilde{r}_2}(\kappa),$$

for all $\kappa \in (0,1]$. Then, the gradual addition $\tilde{r}_1 + \tilde{r}_2$ and the gradual scalar multiplication $c\tilde{r}(c \in \mathbb{R})$ are defined by

$$\mathscr{A}_{\tilde{r}_1+\tilde{r}_2}(\kappa) = \mathscr{A}_{\tilde{r}_1}(\kappa) + \mathscr{A}_{\tilde{r}_2}(\kappa) \quad \text{and} \quad \mathscr{A}_{c\tilde{r}}(\kappa) = c\mathscr{A}_{\tilde{r}}(\kappa),$$

for all $\kappa \in (0,1]$.

DEFINITION 3. [18] Let X be a real vector space. The function $\|\cdot\|_{\mathscr{G}} : X \to G^*(\mathbb{R})$ is said to be a gradual norm on X, if for every $\kappa \in (0,1]$, the following conditions are true for any $x, y \in X$:
(G_1) $\mathscr{A}_{\|x\|_{\mathscr{G}}}(\kappa) = \mathscr{A}_{\tilde{0}}(\kappa)$ if and only if $x = 0$;
(G_2) $\mathscr{A}_{\|\lambda x\|_{\mathscr{G}}}(\kappa) = |\lambda| \mathscr{A}_{\|x\|_{\mathscr{G}}}(\kappa)$ for any $\lambda \in \mathbb{R}$;
(G_3) $\mathscr{A}_{\|x+y\|_{\mathscr{G}}}(\kappa) \leqslant \mathscr{A}_{\|x\|_{\mathscr{G}}}(\kappa) + \mathscr{A}_{\|y\|_{\mathscr{G}}}(\kappa)$.

The pair $(X, \|\cdot\|_{\mathscr{G}})$ is called a gradual normed linear space (GNLS).

EXAMPLE 1. [18] Let $X = \mathbb{R}^w$ and $x = (x_1, x_2, \ldots, x_w) \in \mathbb{R}^w$, define

$$\|\cdot\|_{\mathscr{G}} : \mathbb{R}^w \to G^*(\mathbb{R})$$

for $\kappa \in (0,1]$, as follows

$$\mathscr{A}_{\|x\|_{\mathscr{G}}}(\kappa) = e^{\kappa} \sum_{i=1}^{w} |x_i|.$$

Then, $\|\cdot\|_{\mathscr{G}}$ is a gradual norm on $\mathbb{R}^w$ and $(\mathbb{R}^w, \|\cdot\|_{\mathscr{G}})$ is a GNLS.

DEFINITION 4. [18] Let $x = (x_k)$ be a sequence in the GNLS $(X, \|\cdot\|_{\mathscr{G}})$. Then, x is said to be gradual convergent to $x_0 \in X$, if for every $\kappa \in (0,1]$ and $\varepsilon > 0$, there exists $N(= N_\varepsilon(\kappa)) \in \mathbb{N}$ such that

$$\mathscr{A}_{\|x_k - x_0\|_{\mathscr{G}}}(\kappa) < \varepsilon,$$

satisfies for all $k \geqslant N$. Symbolically, $x_k \to x_0(G)$.

Let $p = (p_n)$ and $q = (q_n)$ be the sequences of non-negative integers satisfying

$$p_n < q_n \text{ and } \lim_{n\to\infty} q_n = \infty.$$

DEFINITION 5. [1] A real-valued sequence $x = (x_k)$ is said to be strong deferred Cesàro convergent to $x_0 \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} |x_k - x_0| = 0.$$

Symbolically, it is denoted by $x_k \to x_0(D[p,q])$.

DEFINITION 6. [24] Let $K \subset \mathbb{N}$ and $K_{p,q}(n)$ denote the set $\{p_n + 1 \leqslant k \leqslant q_n : k \in K\}$. Then, the deferred density of K is denoted and defined as

$$\delta_{p,q}(K) = \lim_{n\to\infty} \frac{1}{(q_n - p_n)} |K_{p,q}(n)|$$

provided that the limit exists. Here, $|\cdot|$ indicates the cardinality of the inside set.

DEFINITION 7. [13] A real-valued sequence $x = (x_k)$ is said to be deferred statistical convergent to $x_0 \in \mathbb{R}$ if for every $\varepsilon > 0$,

$$\delta_{p,q}(B(\varepsilon)) = 0,$$

where $B(\varepsilon) = \{k \in \mathbb{N} : |x_k - x_0| \geqslant \varepsilon\}$.

Symbolically, it is represented as $x_k \to x_0(DS[p,q])$.

In particular, if we take $p(n) = 0$ and $q(n) = n$, then Definition 5, Definition 6, and Definition 7 reduces to the definition of strong Cesàro summability, natural density, and statistical convergence respectively.

DEFINITION 8. [17] A real valued triple sequence $x = (x_{ijk})$ is said to be convergent to a real number x_0, if for any $\varepsilon > 0$, there exists a positive integer $k_0 = k_0(\varepsilon)$ such that for all $i, j, k \geqslant k_0$,

$$\left|x_{ijk} - x_0\right| < \varepsilon.$$

DEFINITION 9. [19] Let $K \subseteq \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and $K_{l,m,n}$ denote the set

$$\{(i,j,k) \in K : i \leqslant l, j \leqslant m, k \leqslant n\}.$$

The triple natural density of K is denoted and defined by

$$\delta^3(K) = \lim_{l,m,n\to\infty} \frac{\left|K_{l,m,n}\right|}{lmn},$$

provided that the limit exists.

DEFINITION 10. Let $(X, \|\cdot\|_{\mathscr{G}})$ be a GNLS. Then, a triple sequence (x_{ijk}) in X is said to be gradual bounded if there exists $M > 0$ such that

$$\mathscr{A}_{\|x_{ijk}\|_{\mathscr{G}}}(\kappa) < M$$

holds for all $\kappa \in (0,1]$ and $(i,j,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$.

DEFINITION 11. A triple sequence $x = (x_{ijk})$ in the GNLS $(X, \|\cdot\|_{\mathscr{G}})$ is said to be statistical convergent to $x_0 \in X$ if for every $\varepsilon > 0$ and $\kappa \in (0,1]$,

$$\delta^3\left(\left\{(i,j,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon\right\}\right) = 0.$$

In this case, we write $x_{ijk} \to x_0(GS)$.

3. Main results

In this section, we are going to define strong gradual deferred convergence and gradual deferred statistical convergence of triple sequences. Throughout the paper, $\mathbf{0} \in \mathbb{R}^w$ denotes the w-tuple $(0,0,\dots,0)$ and $\varpi_l = b_l - a_l$, $\varpi'_l = b'_l - a'_l$, $\rho_m = q_m - p_m$, $\rho'_m = q'_m - p'_m$, $\varsigma_n = v_n - u_n$, $\varsigma'_n = v'_n - u'_n$, where $a = (a_l)$, $a' = (a'_l)$, $b = (b_l)$, $b' = (b'_l)$, $p = (p_m)$, $p' = (p'_m)$, $q = (q_m)$, $q' = (q'_m)$, $u = (u_n)$, $u' = (u'_n)$, $v = (v_n)$ and $v' = (v'_n)$ be the sequences of nonnegative integers satisfying the conditions $a_l < b_l$, $a'_l < b'_l$, $p_m < q_m$, $p'_m < q'_m$, $u_n < v_n$, $u'_n < v'_n$ and

$$\lim_{l\to\infty} b_l = \infty, \ \lim_{m\to\infty} q_m = \infty, \ \lim_{n\to\infty} v_n = \infty \text{ and } \lim_{l\to\infty} b'_l = \infty, \ \lim_{m\to\infty} q'_m = \infty, \ \lim_{n\to\infty} v'_n = \infty. \quad (1)$$

DEFINITION 12. Let $x = (x_{ijk})$ be a triple sequence in the GNLS $(X, \|\cdot\|_{\mathscr{G}})$. Then, x is said to be strong gradual deferred convergent to $x_0 \in X$ if for every $\kappa \in (0,1]$,

$$\lim_{l,m,n\to\infty} \frac{1}{\varpi_l \rho_m \varsigma_n} \sum_{i=a_l+1}^{b_l} \sum_{j=p_m+1}^{q_m} \sum_{k=u_n+1}^{v_n} \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) = 0.$$

In this case, we write $x_{ijk} \to x_0(GD_{apu}^{bqv})$.

DEFINITION 13. Let $K \subseteq \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and $K_{apu}^{bqv}(l,m,n)$ denote the set

$$\{(i,j,k) \in K : a_l < i \leqslant b_l, p_m < j \leqslant q_m, u_n < k \leqslant v_n\}.$$

The triple deferred density of K is denoted and defined by

$$\delta_{apu}^{3,bqv}(K) = \lim_{l,m,n\to\infty} \frac{\left|K_{apu}^{bqv}(l,m,n)\right|}{\varpi_l \rho_m \varsigma_n},$$

provided that the limit exists.

In particular if $b_l = l$, $a_l = 0$, $q_m = m$, $p_m = 0$, $v_n = n$ and $u_n = 0$ then the above definition reduces to the Definition 9.

DEFINITION 14. Let $x = (x_{ijk})$ be a triple sequence in the GNLS $(X, \|\cdot\|_{\mathscr{G}})$. Then, x is said to be gradual deferred statistical convergent to $x_0 \in X$ if for every $\varepsilon > 0$ and $\kappa \in (0,1]$,

$$\delta_{apu}^{3,bqv}\left(\left\{(i,j,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon\right\}\right) = 0,$$

i.e.,

$$\lim_{l,m,n\to\infty} \frac{1}{\varpi_l \rho_m \varsigma_n} \left|\left\{a_l < i \leqslant b_l, p_m < j \leqslant q_m, u_n < k \leqslant v_n : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon\right\}\right| = 0.$$

In this case, we write $x_{ijk} \to x_0(GDS_{apu}^{bqv})$.

THEOREM 1. *Let* (x_{ijk}) *be any triple sequence in the GNLS* $(X, \|\cdot\|_{\mathscr{G}})$ *such that* $x_{ijk} \to x_0(\mathscr{G})$ *in* X. *Then,* $x_k \to x_0(GDS_{apu}^{bqv})$ *for any* a, p, u, b, q *and* v.

Proof. Since $x_{ijk} \to x_0(\mathscr{G})$, then the set

$$\left\{(i,j,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : A_{\|x_{ijk}-x_0\|_G}(\kappa) \geqslant \varepsilon\right\}$$

contains a finite number of elements and consequently has deferred triple density zero. □

The converse of the above theorem is not true, in general.

EXAMPLE 2. Let $X = \mathbb{R}^w$ and $\|\cdot\|_{\mathscr{G}}$ be the norm given in Example 1. Suppose, b_l, q_m, v_n are strictly increasing sequences and a_l, p_m, u_n satisfy the conditions $0 < a_l < [|\sqrt{b_l}|] - 1$, $0 < p_m < [|\sqrt{q_m}|] - 1$, $0 < u_n < [|\sqrt{v_n}|] - 1$ for all $(l,m,n) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$, where $[|\cdot|]$ denotes the greatest integer function. Consider the triple sequence (x_{ijk}) in $\mathbb{R}^w$ as follows:

$$x_{ijk} = \begin{cases} (0,0,\ldots,0,(ijk)^2), & \text{if } \begin{array}{l} [|\sqrt{b_l}|]-1 < i \leqslant [|\sqrt{b_l}|], \\ [|\sqrt{q_m}|]-1 < j \leqslant [|\sqrt{q_m}|], \\ [|\sqrt{v_n}|]-1 < k \leqslant [|\sqrt{v_n}|], \\ l,m,n = 1,2,3,\ldots \end{array} \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

Then, for any $\varepsilon > 0$ and $\kappa \in (0,1]$,

$$\frac{\left|\left\{a_l < i \leqslant b_l, p_m < j \leqslant q_m, u_n < k \leqslant v_n : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon\right\}\right|}{\varpi_l \rho_m \varsigma_n} = \frac{1}{\varpi_l \rho_m \varsigma_n}. \tag{2}$$

Letting $l,m,n \to \infty$ on both sides of (2) we obtain $x_{ijk} \to \mathbf{0}(GDS_{apu}^{bqv})$. But it is clear that $x_{ijk} \nrightarrow \mathbf{0}(\mathscr{G})$.

THEOREM 2. *Let* (x_{ijk}) *be any triple sequence in the GNLS* $(X, \|\cdot\|_{\mathscr{G}})$ *such that* $x_{ijk} \to x_0(GDS_{apu}^{bqv})$ *in* X. *Then,* x_0 *is uniquely determined.*

Proof. If possible suppose $x_{ijk} \to x_0(GDS_{apu}^{bqv})$ and $x_{ijk} \to y_0(GDS_{apu}^{bqv})$ for some $x_0 \neq y_0$ in X. Let $\varepsilon > 0$ be arbitrary. Then, by Definition 14 we have, for any $\varepsilon > 0$ and $\kappa \in (0,1]$,

$$\delta_{apu}^{3,bqv}(B_1(\kappa,\varepsilon)) = \delta_{apu}^{3,bqv}(B_2(\kappa,\varepsilon)) = 1,$$

where

$$B_1(\kappa,\varepsilon) = \left(\left\{(i,j,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) < \varepsilon\right\}\right)$$

and

$$B_2(\kappa,\varepsilon)=\left(\left\{(i,j,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\mathscr{A}_{\|x_{ijk}-y_0\|_{\mathscr{G}}}(\kappa)<\varepsilon\right\}\right).$$

Choose $(i_0,j_0,k_0)\in B_1(\kappa,\varepsilon)\cap B_2(\kappa,\varepsilon)$, then

$$\mathscr{A}_{\|x_{i_0j_0k_0}-x_0\|_{\mathscr{G}}}(\kappa)<\varepsilon \text{ and } \mathscr{A}_{\|x_{i_0j_0k_0}-y_0\|_{\mathscr{G}}}(\kappa)<\varepsilon.$$

Hence,

$$\mathscr{A}_{\|x_0-y_0\|_{\mathscr{G}}}(\kappa)\leqslant\mathscr{A}_{\|x_{i_0j_0k_0}-x_0\|_{\mathscr{G}}}(\kappa)+\mathscr{A}_{\|x_{i_0j_0k_0}-y_0\|_{\mathscr{G}}}(\kappa)<\varepsilon+\varepsilon=2\varepsilon.$$

Since ε is arbitrary, so $\mathscr{A}_{\|x_0-y_0\|_{\mathscr{G}}}(\kappa)=\mathscr{A}_{\tilde{0}}(\kappa)$ and so we must have $x_0=y_0$. □

THEOREM 3. *Let (x_{ijk}) and (y_{ijk}) be two triple sequences in the GNLS $(X,\|\cdot\|_{\mathscr{G}})$ such that $x_{ijk}\to x_0(GDS_{apu}^{bqv})$ and $y_{ijk}\to y_0(GDS_{apu}^{bqv})$. Then,*

(i) $x_{ijk}+y_{ijk}\to x_0+y_0(GDS_{apu}^{bqv})$ and

(ii) $cx_{ijk}\to cx_0(GDS_{apu}^{bqv})$, $c\in\mathbb{R}$.

Proof. (i) Suppose $x_{ijk}\to x_0(GDS_{apu}^{bqv})$ and $y_{ijk}\to y_0(GDS_{apu}^{bqv})$. Then, by Definition 14, for given $\varepsilon>0$,

$$\delta_{apu}^{3,bqv}(C_1)=\delta_{apu}^{3,bqv}(C_2)=0,$$

where

$$C_1=\left\{(i,j,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\mathscr{A}_{\|x_{ijk}-x_0\|_G}(\kappa)\geqslant\tfrac{\varepsilon}{2}\right\}$$

and

$$C_2=\left\{(i,j,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\mathscr{A}_{\|y_{ijk}-y_0\|_G}(\kappa)\geqslant\tfrac{\varepsilon}{2}\right\}.$$

Now as the inclusion

$$\begin{gathered}((\mathbb{N}\times\mathbb{N}\times\mathbb{N})\setminus C_1)\cap((\mathbb{N}\times\mathbb{N}\times\mathbb{N})\setminus C_2)\\ \subseteq\{(i,j,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\mathscr{A}_{\|x_{ijk}+y_{ijk}-x_0-y_0\|_G}(\kappa)<\varepsilon\}\end{gathered}$$

holds, so we must have

$$\delta_{apu}^{3,bqv}(\{(i,j,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\mathscr{A}_{\|x_{ijk}+y_{ijk}-x_0-y_0\|_G}(\kappa)\geqslant\varepsilon\})\leqslant\delta_{apu}^{3,bqv}(C_1\cup C_2)=0;$$

and consequently,

$$x_{ijk}+y_{ijk}\to x_0+y_0(GDS_{apu}^{bqv}).$$

(ii) If $c=0$, then there is nothing to prove. So let us assume $c\neq 0$. Then, since $x_{ijk}\to x_0(GDS_{apu}^{bqv})$, we have for given $\varepsilon>0$, $\delta_{apu}^{3,bqv}(C_1)=0$, where

$$C_1=\left\{(i,j,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\mathscr{A}_{\|x_{ijk}-x_0\|_G}(\kappa)\geqslant\tfrac{\varepsilon}{|c|}\right\}.$$

Now since

$$\mathscr{A}_{\|cx_{ijk}-cx_0\|_G}(\kappa)=|c|\mathscr{A}_{\|x_{ijk}-x_0\|_G}(\kappa)$$

holds for any $c\in\mathbb{R}$, we must have $C_2\subseteq C_1$, where

$$C_2=\{(i,j,k)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\mathscr{A}_{\|cx_{ijk}-cx_0\|_G}(\kappa)\geqslant\varepsilon\},$$

which as a consequence implies $\delta^{3,bqv}_{apu}(C_2)=0$. This completes the proof. □

THEOREM 4. *Let* (x_{ijk}) *be any triple sequence in the GNLS* $(X,\|\cdot\|_{\mathscr{G}})$. *Then,* $x_{ijk}\to x_0(GD^{bqv}_{apu})$ *implies* $x_{ijk}\to x_0(GDS^{bqv}_{apu})$.

Proof. Let $x_{ijk}\to x_0(GD^{bqv}_{apu})$ and $\varepsilon>0$ be arbitrary. Then, for any $\kappa\in(0,1]$,

$$\lim_{l,m,n\to\infty}\frac{1}{\varpi_l\rho_m\varsigma_n}\sum_{i=a_l+1}^{b_l}\sum_{j=p_m+1}^{q_m}\sum_{k=u_n+1}^{v_n}\mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa)=0, \tag{3}$$

holds. Now,

$$\begin{aligned}
&\frac{1}{\varpi_l\rho_m\varsigma_n}\sum_{i=a_l+1}^{b_l}\sum_{j=p_m+1}^{q_m}\sum_{k=u_n+1}^{v_n}\mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa)\\
&\quad\geqslant\frac{1}{\varpi_l\rho_m\varsigma_n}\underbrace{\sum_{i=a_l+1}^{b_l}\sum_{j=p_m+1}^{q_m}\sum_{k=u_n+1}^{v_n}}_{\mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa)\geqslant\varepsilon}\mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa)\\
&\quad\geqslant\frac{1}{\varpi_l\rho_m\varsigma_n}\underbrace{\sum_{i=a_l+1}^{b_l}\sum_{j=p_m+1}^{q_m}\sum_{k=u_n+1}^{v_n}}_{\mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa)\geqslant\varepsilon}\varepsilon\\
&\quad\geqslant\frac{\varepsilon}{\varpi_l\rho_m\varsigma_n}\left|\left\{a_l<i\leqslant b_l,p_m<j\leqslant q_m,u_n<k\leqslant v_n:\mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa)\geqslant\varepsilon\right\}\right|.
\end{aligned}$$

Taking $l,m,n\to\infty$ on both sides of the above inequation and using (3), we obtain

$$\lim_{l,m,n\to\infty}\frac{\left|\left\{a_l<i\leqslant b_l,p_m<j\leqslant q_m,u_n<k\leqslant v_n:\mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa)\geqslant\varepsilon\right\}\right|}{\varpi_l\rho_m\varsigma_n}=0.$$

Hence, $x_{ijk}\to x_0(GDS^{bqv}_{apu})$. □

It should be noted that the converse of Theorem 4 is not necessarily true. Consider Example 2. It was shown that $x_{ijk}\to\mathbf{0}(GDS^{bqv}_{apu})$. But since the right-hand side of the following inequation

$$\frac{1}{\varpi_l\rho_m\varsigma_n}\sum_{i=a_l+1}^{b_l}\sum_{j=p_m+1}^{q_m}\sum_{k=u_n+1}^{v_n}\mathscr{A}_{\|x_{ijk}-\mathbf{0}\|_{\mathscr{G}}}(\kappa)\geqslant\frac{([|\sqrt{b_l}|]-1)^2([|\sqrt{q_m}|]-1)^2([|\sqrt{v_n}|]-1)^2}{\varpi_l\rho_m\varsigma_n}$$

tends to 1 as $l,m,n \to \infty$, so the left-hand side never approaches zero. In other words, $x_{ijk} \nrightarrow \mathbf{0}(GD_{apu}^{bqv})$.

From the above remark, naturally, a question arises under which condition the converse of Theorem 4 holds. The next theorem answers.

THEOREM 5. *Let* (x_{ijk}) *be a gradual bounded triple sequence in the GNLS* $(X, \|\cdot\|_{\mathscr{G}})$. *Then,* $x_{ijk} \to x_0(GDS_{apu}^{bqv})$ *implies* $x_{ijk} \to x_0(GD_{apu}^{bqv})$.

Proof. Let $x_k \to x_0(GDS_{apu}^{bqv})$. Since (x_{ijk}) is gradual bounded, so there exists a $M > 0$ such that for all $\kappa \in (0,1]$ and $i,j,k \in \mathbb{N}$,

$$\mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \leqslant M.$$

Now

$$\begin{aligned}
&\frac{1}{\varpi_l \rho_m \varsigma_n} \sum_{i=a_l+1}^{b_l} \sum_{j=p_m+1}^{q_m} \sum_{k=u_n+1}^{v_n} \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \\
&= \frac{1}{\varpi_l \rho_m \varsigma_n} \left(\underbrace{\sum_{i=a_l+1}^{b_l} \sum_{j=p_m+1}^{q_m} \sum_{k=u_n+1}^{v_n}}_{\mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon} \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) + \underbrace{\sum_{i=a_l+1}^{b_l} \sum_{j=p_m+1}^{q_m} \sum_{k=u_n+1}^{v_n}}_{\mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) < \varepsilon} \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \right) \\
&\leqslant \frac{1}{\varpi_l \rho_m \varsigma_n} \left(\underbrace{\sum_{i=a_l+1}^{b_l} \sum_{j=p_m+1}^{q_m} \sum_{k=u_n+1}^{v_n}}_{\mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon} M + \underbrace{\sum_{i=a_l+1}^{b_l} \sum_{j=p_m+1}^{q_m} \sum_{k=u_n+1}^{v_n}}_{\mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) < \varepsilon} \varepsilon \right) \\
&\leqslant \frac{M}{\varpi_l \rho_m \varsigma_n} \left| \left\{ a_l < i \leqslant b_l, p_m < j \leqslant q_m, u_n < k \leqslant v_n : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon \right\} \right| \\
&\qquad + \frac{\varepsilon}{\varpi_l \rho_m \varsigma_n} \left| \left\{ a_l < i \leqslant b_l, p_m < j \leqslant q_m, u_n < k \leqslant v_n : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) < \varepsilon \right\} \right|.
\end{aligned}$$

From the assumption and the above inequation, we conclude that for any $\kappa \in (0,1]$,

$$\lim_{l,m,n\to\infty} \frac{1}{\varpi_l \rho_m \varsigma_n} \sum_{i=a_l+1}^{b_l} \sum_{j=p_m+1}^{q_m} \sum_{k=u_n+1}^{v_n} \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) = 0.$$

Hence, $x_{ijk} \to x_0(GD_{apu}^{bqv})$. $\square$

THEOREM 6. *Let* (x_{ijk}) *be a triple sequence in the GNLS* $(X,\|\cdot\|_{\mathscr{G}})$ *such that* $x_{ijk} \to x_0(GS)$. *Then,* $x_{ijk} \to x_0(GDS^{bqv}_{apu})$ *provided that the sequences* $\left(\frac{a_l}{\varpi_l}\right), \left(\frac{p_m}{\rho_m}\right)$, *and* $\left(\frac{u_n}{\varsigma_n}\right)$ *are bounded.*

Proof. Since, $x_{ijk} \to x_0(GS)$, then for any $\varepsilon > 0$ and $\kappa \in (0,1]$,

$$\lim_{l,m,n\to\infty} \frac{1}{lmn} \left|\left\{i \leqslant l, j \leqslant m, k \leqslant n : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon\right\}\right| = 0.$$

Again since the sequences $b = (b_l)$, $q = (q_m)$ and $v = (v_n)$ satisfy (1), from the above limit, we must have

$$\lim_{l,m,n\to\infty} \frac{1}{b_l q_m v_n} \left|\left\{i \leqslant b_l, j \leqslant q_m, k \leqslant v_n : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon\right\}\right| = 0. \tag{4}$$

Clearly the inclusion

$$\left\{a_l < i \leqslant b_l, p_m < j \leqslant q_m, u_n < k \leqslant v_n : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon\right\} \subseteq \left\{i \leqslant b_l, j \leqslant q_m, k \leqslant v_n : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon\right\}$$

yields the inequation

$$\left|\left\{a_l < i \leqslant b_l, p_m < j \leqslant q_m, u_n < k \leqslant v_n : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon\right\}\right| \leqslant \left|\left\{i \leqslant b_l, j \leqslant q_m, k \leqslant v_n : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon\right\}\right|$$

which subsequently gives the following inequation

$$\begin{aligned} \frac{1}{\varpi_l \rho_m \varsigma_n} &\left|\left\{a_l < i \leqslant b_l, p_m < j \leqslant q_m, u_n < k \leqslant v_n : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon\right\}\right| \\ &\leqslant \left(1 + \frac{a_l}{\varpi_l}\right)\left(1 + \frac{p_m}{\rho_m}\right)\left(1 + \frac{u_n}{\varsigma_n}\right) \frac{1}{b_l q_m v_n} \\ &\quad \times \left|\left\{i \leqslant b_l, j \leqslant q_m, k \leqslant v_n : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon\right\}\right|. \end{aligned} \tag{5}$$

Since the sequences $\left(\frac{a_l}{\varpi_l}\right), \left(\frac{p_m}{\rho_m}\right)$, and $\left(\frac{u_n}{\varsigma_n}\right)$ are bounded, so letting $l,m,n \to \infty$ on both sides of (5) and using (4) we obtain,

$$\lim_{l,m,n\to\infty} \frac{\left|\left\{a_l < i \leqslant b_l, p_m < j \leqslant q_m, u_n < k \leqslant v_n : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon\right\}\right|}{\varpi_l \rho_m \varsigma_n} = 0.$$

Hence, $x_{ijk} \to x_0(GDS^{bqv}_{apu})$ holds and the proof is complete. □

THEOREM 7. *Let* $a' = (a'_l)$, $b' = (b'_l)$, $p' = (p'_m)$, $q' = (q'_m)$, $u' = (u'_n)$, *and* $v' = (v'_n)$ *be sequences of positive integers such that*

$$a_l \leqslant a'_l < b'_l \leqslant b_l, p_m \leqslant p'_m < q'_m \leqslant q_m, \text{ and } u_n \leqslant u'_n < v'_n \leqslant v_n$$

holds for all $l,m,n \in \mathbb{N}$. *Then,*

(i) $x_{ijk} \to x_0(GDS^{b'q'v'}_{a'p'u'})$ *implies* $x_{ijk} \to x_0(GDS^{bqv}_{apu})$ *provided that the sets*

$$\left\{(i,j,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : a_l < i \leqslant a'_l, p_m < j \leqslant p'_m, u_n < k \leqslant u'_n\right\}$$

and

$$\left\{(i,j,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : b'_l < i \leqslant b_l, q'_m < j \leqslant q_m, v'_n < k \leqslant v_n\right\}$$

are finite sets for all $(l,m,n) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$.

(ii) $x_{ijk} \to x_0(GDS^{bqv}_{apu})$ *implies* $x_{ijk} \to x_0(GDS^{b'q'v'}_{a'p'u'})$ *provided that*

$$\lim_{l,m,n\to\infty} \frac{\varpi_l \rho_m \varsigma_n}{\varpi'_l \rho'_m \varsigma'_n} = d > 0.$$

(iii) If $x_{ijk} \to x_0(GDS^{a'p'u'}_{apu})$ *and* $x_{ijk} \to x_0(GDS^{bqv}_{b'q'v'})$ *holds simultaneously, then,* $x_{ijk} \to x_0(GDS^{bqv}_{apu})$ *provided that the set*

$$\left\{(i,j,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : a'_l < i \leqslant b'_l, p'_m < j \leqslant q'_m, u'_n < k \leqslant v'_n\right\}$$

is finite for all $(l,m,n) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$.

Proof. (i) For any $\varepsilon > 0$ and $\kappa \in (0,1]$, the equality

$$\begin{aligned}
&\left\{a_l < i \leqslant b_l, p_m < j \leqslant q_m, u_n < k \leqslant v_n : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon\right\} \\
\subseteq &\left\{a_l < i \leqslant a'_l, p_m < j \leqslant p'_m, u_n < k \leqslant u'_n : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon\right\} \\
&\cup \left\{a'_l < i \leqslant b'_l, p'_m < j \leqslant q'_m, u'_n < k \leqslant v'_n : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon\right\} \\
&\cup \left\{b'_l < i \leqslant b_l, q'_m < j \leqslant q_m, v'_n < k \leqslant v_n : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon\right\}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\varpi_l \rho_m \varsigma_n} \left|\left\{a_l < i \leqslant b_l, p_m < j \leqslant q_m, u_n < k \leqslant v_n : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon\right\}\right| \\
\leqslant &\frac{1}{\varpi'_l \rho'_m \varsigma'_n} \left|\left\{a_l < i \leqslant a'_l, p_m < j \leqslant p'_m, u_n < k \leqslant u'_n : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon\right\}\right| \\
&+ \frac{1}{\varpi'_l \rho'_m \varsigma'_n} \left|\left\{a'_l < i \leqslant b'_l, p'_m < j \leqslant q'_m, u'_n < k \leqslant v'_n : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon\right\}\right| \\
&+ \frac{1}{\varpi'_l \rho'_m \varsigma'_n} \left|\left\{b'_l < i \leqslant b_l, q'_m < j \leqslant q_m, v'_n < k \leqslant v_n : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon\right\}\right|
\end{aligned}$$

holds.

On taking $l,m,n \to \infty$ we obtain

$$\lim_{l,m,n\to\infty} \frac{\left|\left\{a_l < i \leqslant b_l, p_m < j \leqslant q_m, u_n < k \leqslant v_n : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon\right\}\right|}{\varpi_l \rho_m \varsigma_n} = 0.$$

Hence, $x_{ijk} \to x_0(GDS^{bqv}_{apu})$.

(ii) For any $\varepsilon > 0$ and $\kappa \in (0,1]$, the inclusion

$$\begin{aligned} &\left\{a'_l < i \leqslant b'_l, p'_m < j \leqslant q'_m, u'_n < k \leqslant v'_n : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon\right\} \\ \subseteq &\left\{a_l < i \leqslant b_l, p_m < j \leqslant q_m, u_n < k \leqslant v_n : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon\right\} \end{aligned}$$

and the inequality

$$\begin{aligned} &\left|\left\{a'_l < i \leqslant b'_l, p'_m < j \leqslant q'_m, u'_n < k \leqslant v'_n : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon\right\}\right| \\ \leqslant &\left|\left\{a_l < i \leqslant b_l, p_m < j \leqslant q_m, u_n < k \leqslant v_n : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon\right\}\right| \end{aligned}$$

holds good. So we have,

$$\begin{aligned} &\frac{1}{\varpi'_l \rho'_m \varsigma'_n} \left|\left\{a'_l < i \leqslant b'_l, p'_m < j \leqslant q'_m, u'_n < k \leqslant v'_n : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon\right\}\right| \\ \leqslant &\frac{\varpi_l \rho_m \varsigma_n}{\varpi'_l \rho'_m \varsigma'_n} \frac{1}{\varpi_l \rho_m \varsigma_n} \left|\left\{a_l < i \leqslant b_l, p_m < j \leqslant q_m, u_n < k \leqslant v_n : \mathscr{A}_{\|x_{ijk}-x_0\|_{\mathscr{G}}}(\kappa) \geqslant \varepsilon\right\}\right| \end{aligned}$$

and by taking $l,m,n \to \infty$ the desired result is obtained.

(iii) The proof is easy, so omitted. □

4. Concluding remarks

In this paper, we have explored several fundamental characteristics of deferred statistical convergence of triple sequences in gradual normed linear spaces. Theorem 4 and Theorem 5 have uncovered the relationship between strong deferred convergence and gradual deferred statistical convergence of triple sequences. The realms of summability theory and sequence convergence find wide-ranging applications across various mathematical disciplines, particularly in the realm of mathematical analysis. Exploring this research avenue within gradual normed linear spaces remains relatively uncharted and is still in its early stages of development. The results obtained in this study could prove valuable to future researchers as they delve deeper into the various aspects of convergence within gradual normed linear spaces.

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