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# ON LACUNARY $\mathscr{I}$–INVARIANT CONVERGENCE OF SEQUENCES IN QUATERNION–VALUED GENERALIZED METRIC SPACES

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*Abstract.* In this study, we explore the concept of lacunary $\mathscr{I}_\sigma$-convergence of sequences and analyze the relationships between this new convergence concept and the notions of lacunary invariant summability, lacunary strongly $s$-invariant summability, and lacunary $\sigma$-statistical convergence, all of which are defined within quaternion-valued generalized metric spaces. Additionally, our paper aims to introduce the concepts of lacunary $\mathscr{I}_\sigma^*$-convergence in quaternion-valued generalized metric spaces. We then establish the equivalence between lacunary $\mathscr{I}_\sigma$-convergence and lacunary $\mathscr{I}_\sigma^*$-convergence by providing the definition of property $(AP)$. Furthermore, we introduce lacunary $\mathscr{I}_\sigma$-Cauchy and lacunary $\mathscr{I}_\sigma^*$-Cauchy sequences, adapting classical theorems to quaternion-valued generalized metric spaces.

## 1. Introduction

Fast [11] explored the concept of statistical convergence, which had a profound impact across scientific disciplines. For further reference, see ([5, 7, 18, 19, 21]). Nuray and Ruckle [26] subsequently introduced generalized statistical convergence, marking significant advancements in this area. Also, the readers should refer to the monographs [4], and [22], and recent papers [17], [31], [32], [33], [34], [35] and [36] for the background on the sequence spaces, and related topics. Kostyrko et al. [16] generalized this concept further into ideal convergence, investigating its properties and applications. This development led to ideal convergence becoming a prominent topic in summability theory, as studied by [8, 9, 15, 24, 25], and others.

Fridy and Orhan [12] expanded on these ideas by introducing lacunary statistical convergence using lacunary sequences. Other researchers, including Raimi [28], Schaefer [30], and Mursaleen [20], focused on invariant convergent sequences. Nuray et al. [27] explored $\mathscr{I}_\sigma$-convergence with $\sigma$-uniform density, while Mursaleen introduced strongly $\sigma$-convergence.

Savaş and Nuray [29] proposed $\sigma$-statistical convergence and its lacunary variant, establishing correlation theorems. Lastly, Nuray and Ulusu [37] defined lacunary $\mathscr{I}$-invariant convergence and lacunary $\mathscr{I}$-invariant Cauchy sequences of real numbers.

In mathematical analysis, the notion of distance is formalized using a distance function or metric, which can be generalized in diverse ways (refer to, for example,

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[14]). An important extension is the $G$-metric space, extensively explored by Mustafa and Sims [23]. Metrics in this context quantify the separation between three locations. Choi et al. [6] introduced the $g$-metric as an expansion of traditional distance functions. It generalizes the ordinary distance between two points and the G-metric among three points into the $n+1$ point distance, termed the $g$-metric of degree $n$. Abazari further advanced this metric framework with the introduction of statistical $g$-convergence [1].

Quaternions constitute a number system extending beyond complex numbers, originally formulated by Irish mathematician Hamilton in 1843 to describe mechanics in three dimensions. Quaternions are characterized by noncommutative multiplication, distinguishing them from other systems. Extensive discussions on quaternion analysis are found in [3] and related literature. This paper introduces a new type of convergence in quaternion-valued $g$-metric spaces, building upon the $g$-metric spaces by Choi et al. [6], quaternion-valued $g$-metric spaces by Jan and Jalal [13], and various forms of ideal convergence documented in existing literature. This proposal is motivated by practical applications in quaternions and fixed point theorems.

In this investigation, we delve into the intricate realm of lacunary $\mathscr{I}_\sigma$-convergence of sequences, exploring its nuanced interplay with established concepts such as lacunary invariant summability, lacunary strongly $s$-invariant summability, and lacunary $\sigma$-statistical convergence. These explorations unfold within the context of quaternion-valued generalized metric spaces, offering a novel perspective on convergence dynamics in this specialized domain. Additionally, our study introduces the concept of lacunary $\mathscr{I}_\sigma^*$-convergence, aiming to delineate its relationship with lacunary $\mathscr{I}_\sigma$-convergence through the lens of property $(AP)$. We extend our inquiry to include lacunary $\mathscr{I}_\sigma$-Cauchy and lacunary $\mathscr{I}_\sigma^*$-Cauchy sequences, adapting classical theorems to align with the unique characteristics of quaternion-valued generalized metric spaces. This endeavor not only broadens theoretical foundations but also underscores the practical relevance of these convergence concepts in diverse mathematical applications.

Let us now establish several definitions and notations that will be utilized in this paper. First, we will introduce some fundamental notations for quaternion spaces. The four-dimensional real algebra with unity is known as the space of quaternions, denoted by $\mathbf{Q}$. The null element of $\mathbf{Q}$ is denoted by $0_{\mathbf{Q}}$, and the multiplicative identity of $\mathbf{Q}$ is denoted by $1_{\mathbf{Q}}$. Within $\mathbf{Q}$, there exist three imaginary units represented by the symbols $i, j, k$. By definition, these units satisfy:

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i \text{ and } ki = -ik = j.$$

For each $\rho = y_0 + y_1 i + y_2 j + y_3 k$; where $y_0, y_1, y_2$ and $y_3$ belong to $\mathbb{R}$, the elements $1, i, j, k$ are assumed to constitute a real vector basis of $\mathbf{Q}$. Given $\rho = y_0 + y_1 i + y_2 j + y_3 k \in \mathbf{Q}$, we recall that:

(i) $\bar{\rho} = y_0 - y_1 i - y_2 j - y_3 k$ is the conjugate quaternion of $\rho$,

(ii) $|\rho| = \sqrt{\rho\bar{\rho}} = \sqrt{y_0^2 + y_1^2 + y_2^2 + y_3^2} \in \mathbb{R}$

(iii) $\mathrm{Re}(\rho) = \frac{1}{2}(\rho + \bar{\rho}) = y_0 \in \mathbb{R}$

(iv) $\mathrm{Im}(\rho) = \frac{1}{2}(\rho - \bar{\rho}) = y_1 i + y_2 j + y_3 k$ is the imaginary part of $\rho$.

When $\rho = \mathrm{Re}(\rho)$, the element $\rho \in \mathbf{Q}$ is said to be real. It is obvious that $\rho$ is real only if and only if $\rho = \bar{\rho}$. If $\bar{\rho} = -\rho$ or $\rho = \mathrm{Im}(\rho)$, $\rho$ is said to be imaginary.

The concept of a complex metric space was introduced by Azam et al. [3] as follows:

DEFINITION 1. ([3]) Let $X$ be a nonempty set, and suppose the mapping $d_{\mathbb{C}}$ : $X \times X \to \mathbb{C}$ satisfies the following conditions:
(i) $0 \prec d_{\mathbb{C}}(\tau_1, \tau_2)$, for all $\tau_1, \tau_2 \in X$ and $d_{\mathbb{C}}(\tau_1, \tau_2) = 0$ if and only if $\tau_1 = \tau_2$,
(ii) $d_{\mathbb{C}}(\tau_1, \tau_2) = d_{\mathbb{C}}(\tau_2, \tau_1)$ for all $\tau_1, \tau_2 \in X$,
(iii) $d_{\mathbb{C}}(\tau_1, \tau_2) \preceq d_{\mathbb{C}}(\tau_1, \tau_3) + d_{\mathbb{C}}(\tau_3, \tau_2)$ for all $\tau_1, \tau_2, \tau_3 \in X$.
Then $(X, d_{\mathbb{C}})$ is called a complex metric space.

Ahmed et al. [10] extended the above definition to Clifford analysis as follows:

DEFINITION 2. ([10]) Let $X$ be a nonempty set and suppose that the mapping $d_{\mathbf{Q}} : X \times X \to \mathbf{Q}$ satisfies the following.
(i) $0 \prec d_{\mathbf{Q}}(\tau_1, \tau_2)$, for all $\tau_1, \tau_2 \in X$ and $d_{\mathbf{Q}}(\tau_1, \tau_2) = 0$ if and only if $\tau_1 = \tau_2$,
(ii) $d_{\mathbf{Q}}(\tau_1, \tau_2) = d_{\mathbf{Q}}(\tau_2, \tau_1)$ for all $\tau_1, \tau_2 \in X$,
(iii) $d_{\mathbf{Q}}(\tau_1, \tau_2) \preceq d_{\mathbf{Q}}(\tau_1, \tau_3) + d_{\mathbf{Q}}(\tau_3, \tau_2)$ for all $\tau_1, \tau_2, \tau_3 \in X$.
Then $(X, d_{\mathbf{Q}})$ is called a quaternion-valued metric metric space.

Ahmed et al. [10] introduced a partial order $\preceq$ on $\mathbf{Q}$ (space of all quaternions).

DEFINITION 3. Let $\rho_1, \rho_2 \in \mathbf{Q}$, then $\rho_1 \preceq \rho_2$ if and only if $\mathrm{Re}(\rho_1) \leqslant \mathrm{Re}(\rho_2)$ and $\mathrm{Im}_{\boldsymbol{s}}(\rho_1) \leqslant \mathrm{Im}_s(\rho_2)$, $\rho_1, \rho_2 \in \mathbf{Q}$, $s = i, j, k$ where $\mathrm{Im}\, m_i = b$, $\mathrm{Im}\, m_j = c$, $\mathrm{Im}\, m_k = d$. It was observed that $\rho_1 \preceq \rho_2$, if one of the following conditions are satisfied:
(i) $\mathrm{Re}(\rho_1) = \mathrm{Re}(\rho_2)$, $\mathrm{Im}_{s_1}(\rho_1) = \mathrm{Im}_{s_1}(\rho_2)$ where $s_1 = j, k$, $\mathrm{Im}_i(\rho_1) < \mathrm{Im}_i(\rho_2)$;
(ii) $\mathrm{Re}(\rho_1) = \mathrm{Re}(\rho_2)$, $\mathrm{Im}_{s_2}(\rho_1) = \mathrm{Im}_{s_2}(\rho_2)$ where $s_2 = i, k$, $\mathrm{Im}_j(\rho_1) < \mathrm{Im}_j(\rho_2)$;
(iii) $\mathrm{Re}(\rho_1) = \mathrm{Re}(\rho_2)$, $\mathrm{Im}_{s_3}(\rho_1) = \mathrm{Im}_{s_3}(\rho_2)$ where $s_3 = i, j$, $\mathrm{Im}_k(\rho_1) < \mathrm{Im}_k(\rho_2)$;
(iv) $\mathrm{Re}(\rho_1) = \mathrm{Re}(\rho_2)$, $\mathrm{Im}_{s_1}(\rho_1) = \mathrm{Im}_{s_1}(\rho_2)$, $\mathrm{Im}\, m_i(\rho_1) = \mathrm{Im}_i(\rho_2)$;
(v) $\mathrm{Re}(\rho_1) = \mathrm{Re}(\rho_2)$, $\mathrm{Im}_{s_2}(\rho_1) = \mathrm{Im}_{s_2}(\rho_2)$, $\mathrm{Im}\, m_j(\rho_1) = \mathrm{Im}\, m_j(\rho_2)$;
(vii) $\mathrm{Re}(\rho_1) = \mathrm{Re}(\rho_2)$, $\mathrm{Im}_{s_3}(\rho_1) < \mathrm{Im}_{s_3}(\rho_2)$, $\mathrm{Im}\, m_k(\rho_1) = \mathrm{Im}\, m_k(\rho_2)$;
(viii) $\mathrm{Re}(\rho_1) < \mathrm{Re}(\rho_2)$, $\mathrm{Im}_s(\rho_1) = \mathrm{Im}_s(\rho_2)$;
(ix) $\mathrm{Re}(\rho_1) < \mathrm{Re}(\rho_2)$, $\mathrm{Im}_{s_1}(\rho_1) = \mathrm{Im}_{s_1}(\rho_2)$, $\mathrm{Im}_i(\rho_1) < \mathrm{Im}_i(\rho_2)$;
(x) $\mathrm{Re}(\rho_1) < \mathrm{Re}(\rho_2)$, $\mathrm{Im}_{s_2}(\rho_1) = \mathrm{Im}_{s_2}(\rho_2)$, $\mathrm{Im}\, m_j(\rho_1) < \mathrm{Im}\, m_j(\rho_2)$;
(xi) $\mathrm{Re}(\rho_1) < \mathrm{Re}(\rho_2)$, $\mathrm{Im}_{s_I}(\rho_1) = \mathrm{Im}_{s_I}(\rho_2)$, $\mathrm{Im}_k(\rho_1) < \mathrm{Im}\, m_k(\rho_2)$;
(xii) $\mathrm{Re}(\rho_1) < \mathrm{Re}(\rho_2)$, $\mathrm{Im}_{s_1}(\rho_1) < \mathrm{Im}_{s_1}(\rho_2)$, $\mathrm{Im}_i(\rho_1) = \mathrm{Im}\, m_i(\rho_2)$;
(xiii) $\mathrm{Re}(\rho_1) < \mathrm{Re}(\rho_2)$, $\mathrm{Im}_{s_2}(\rho_1) < \mathrm{Im}_{s_2}(\rho_2)$, $\mathrm{Im}\, m_j(\rho_1) = \mathrm{Im}\, m_j(\rho_2)$;
(xiv) $\mathrm{Re}(\rho_1) = \mathrm{Re}(\rho_2)$, $\mathrm{Im}_{s_3}(\rho_1) = \mathrm{Im}_{s_3}(\rho_2)$, $\mathrm{Im}_k(\rho_1) = \mathrm{Im}(\rho_k)$;
(xv) $\mathrm{Re}(\rho_1) < \mathrm{Re}(\rho_2)$, $\mathrm{Im}_s(\rho_1) < \mathrm{Im}_s(\rho_2)$;
(xvi) $\mathrm{Re}(\rho_1) = \mathrm{Re}(\rho_2)$, $\mathrm{Im}_s(\rho_1) = \mathrm{Im}(m_s)$.

Specifically, we denote $\rho_1 \precsim \rho_2$ if $\rho_1 \neq \rho_2$ and one from *(i)* to *(xvi)* is satisfied and we will write $\rho_1 \prec \rho_2$ if only $(xv)$ is satisfied.

REMARK 1. It should be noted that $\rho_1 \preceq \rho_2 \Rightarrow |\rho_1| \leqslant |\rho_2|$.

Motivated by Ahmed et al.'s [10] work, Adewale et al. [2] provided the following definition.

DEFINITION 4. ([2]) Let $X$ be a nonempty set, $\mathbf{Q}$ a set of quaternions and $G^{\mathbf{Q}}: X\times X\times X\to\mathbf{Q}$ be a function satisfying the following properties:
(i) $G^{\mathbf{Q}}(\tau_1,\tau_2,\tau_3)=0$ if and only if $\tau_1=\tau_2=\tau_3$,
(ii) $0\prec G^{\mathbf{Q}}(\tau_1,\tau_1,\tau_2)$, $\forall\tau_1,\tau_2\in X$, with $\tau_1\neq\tau_2$,
(iii) $G^{\mathbf{Q}}(\tau_1,\tau_1,\tau_2)\preceq G^{\mathbf{Q}}(\tau_1,\tau_2,\tau_3)$, $\forall\tau_1,\tau_2,\tau_3\in X$, with $\tau_3\neq\tau_2$,
(iv) $G^{\mathbf{Q}}(\tau_1,\tau_2,\tau_3)=G^{\mathbf{Q}}(\tau_2,\tau_3,\tau_1)=G^{\mathbf{Q}}(\tau_1,\tau_3,\tau_2)=\ldots$ (symmetry),
(v) There exists a real number $s\geqslant 1$ such that

$$G^{\mathbf{Q}}(\tau_1,\tau_2,\tau_3)\leqslant s\left[G^{\mathbf{Q}}(\tau_1,a,a)+G^{\mathbf{Q}}(a,\tau_2,\tau_3)\right],$$

$\forall a,\tau_1,\tau_2,\tau_3\in X$ (rectangle inequality).

Then, the function $G^{\mathbf{Q}}$ is called a quaternion $G$-metric and $\left(X,G^{\mathbf{Q}}\right)$ is referred to as the $G^{\mathbf{Q}}$-metric space. A $G^{\mathbf{Q}}$-metric space is considered complete if every Cauchy sequence in it is $G^{\mathbf{Q}}$-convergent.

The following is an extension of $G$-metric space with degree $l$.

DEFINITION 5. ([2]) Let $X$ be a non-empty set. A function $g: X^{l+1}\to\mathbb{R}^+$ is called a $g$-metric space with order $l$ on $X$ if it satisfies the following conditions:
(i) $g(\tau_0,\tau_1,\tau_2,\ldots,\tau_l)=0$ if and only if $\tau_0=\tau_1=\ldots=\tau_l$,
(ii) $g(\tau_0,\tau_1,\tau_2,\ldots,\tau_l)=g(\tau_{\sigma(0)},\tau_{\sigma(1)},\tau_{\sigma(2)},\ldots,\tau_{\sigma(l)})$ for permutation $\sigma$ on $\{0,1,2,\ldots l\}$,
(iii) $g(\tau_0,\tau_1,\tau_2,\ldots,\tau_l)\leqslant g(y_0,y_1,y_2,\ldots,y_l)$ for all $(\tau_0,\tau_1,\tau_2,\ldots,\tau_l),(y_0,y_1,y_2,\ldots,y_l)\in X^{l+1}$ with $\{\tau_i: i=0,1,\ldots,l\}\subseteq\{y_i: i=0,1,\ldots,l\}$,
(iv) For all $\tau_0,\tau_1,\ldots,\tau_s,y_0,y_1,\ldots,y_t$, $w\in X$ with $s+t+1=l$,

$$\begin{aligned}&g(\tau_0,\tau_1,\tau_2,\ldots,\tau_s,y_0,y_1,y_2,\ldots,y_t)\\&\leqslant g(\tau_0,\tau_1,\tau_2,\ldots,\tau_s,w,w,\ldots,w)+g(y_0,y_1,y_2,\ldots,y_t,w,w,\ldots,w).\end{aligned}$$

The pair $(X,g)$ is called $g$-metric space with degree $l$. For $l=1,2$ respectively, it is respectively equivalent to metric and $G$-metric space.

The statistical convergence of real sequences is based on the concept of natural density of subsets of $\mathbb{N}$, the set of all positive integers, which is defined as follows: Let $(X,d)$ be a metric space. A real number sequence $(\tau_k)$ is said to be statistically convergent to the number $\tau$ if for every $\varepsilon>0$,

$$\lim_n n^{-1}\left|\left\{j\leqslant n: d(\tau_j,\tau)\geqslant\varepsilon\right\}\right|=0,$$

where the number of elements in the contained set is indicated by the vertical bars. indicate enclosed set.

The following definitions were given by R. Abazari.

DEFINITION 6. ([1]) Let $p\in\mathbb{N}$ and $K\in\mathbb{N}^P$ and

$$K(n)=\left\{(i_1,i_2,\ldots,i_p)\leqslant n(n\in\mathbb{N}):(i_1,i_2,\ldots,i_p)\in K\right\},$$

then

$$\delta_{(p)}(K)=\lim_{n\to\infty}\frac{p!}{n^p}|K(n)|,$$

is called $p$-dimensional asymptotic (or natural density) of the set $K$.

DEFINITION 7. ([1]) Let $(x_n)$ be a sequence in a $g$-metric space $(Y,g)$.
(i) $(x_n)$ is statistically convergent to $x$, provided for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{j!}{n^j} \left|\left\{i_1, i_2, \ldots, i_j \leqslant n : g\left(x, x_{i_1}, x_{i_2}, \ldots, x_{i_j}\right) \geqslant \varepsilon\right\}\right| = 0,$$

and is indicated by $gS\text{-}\lim_{n\to\infty} x_n = x$.

(ii) $(x_n)$ is called to be statistical $g$-Cauchy, provided for all $\varepsilon > 0$, there exists $i_\varepsilon \in \mathbb{N}$ so that

$$\lim_{n\to\infty} \frac{j!}{n^j} \left|\left\{i_1, i_2, \ldots, i_j \leqslant n : g\left(x_{i_\varepsilon}, x_{i_1}, x_{i_2}, \ldots, x_{i_j}\right)\right\}\right| = 0.$$

The following definition was given by Jan and Jalal [13].

DEFINITION 8. ([13]) Let $X$ be a non-empty set. A function $g_{\mathbf{Q}} : X^{p+1} \to \mathbf{Q}$ (where $\mathbf{Q}$ is the space of quaternions) is called quaternion-valued $g$-metric space with order $p$ on $X$ if it satisfies the following conditions:
(i) $g_{\mathbf{Q}}(\tau_0, \tau_1, \tau_2, \ldots, \tau_p) = 0$ if and only if $\tau_0 = \tau_1 = \ldots = \tau_p$,
(ii) $g_{\mathbf{Q}}(\tau_0, \tau_1, \tau_2, \ldots, \tau_p) = g_{\mathbf{Q}}\left(\tau_{\sigma(0)}, \tau_{\sigma(1)}, \tau_{\sigma(2)}, \ldots, \tau_{\sigma(p)}\right)$ for permutation $\sigma$ on $\{0, 1, 2, \ldots p\}$,
(iii) $g_{\mathbf{Q}}(\tau_0, \tau_1, \tau_2, \ldots, \tau_p) \preceq g_{\mathbf{Q}}(\varsigma_0, \varsigma_1, \varsigma_2, \ldots, \varsigma_p)$ for all

$$(\tau_0, \tau_1, \tau_2, \ldots, \tau_p), (\varsigma_0, \varsigma_1, \varsigma_2, \ldots, \varsigma_p) \in X^{p+1},$$

with $\{\tau_i : i = 0, 1, \ldots p\} \subsetneq \{\varsigma_i : i = 0, 1, \ldots p\}$,
(iv) For all $\tau_0, \tau_1, \ldots, \tau_s, \varsigma_0, \varsigma_1, \ldots \varsigma_t, v \in X$ with $s + t + 1 = p$,

$$\begin{aligned} &g_{\mathbf{Q}}(\tau_0, \tau_1, \tau_2, \ldots, \tau_s, \varsigma_0, \varsigma_1, \varsigma_2, \ldots, \varsigma_t) \\ &\leqslant g_{\mathbf{Q}}(\tau_0, \tau_1, \tau_2, \ldots, \tau_s, v, v, \ldots, v) + g_{\mathbf{Q}}(\varsigma_0, \varsigma_1, \varsigma_2, \ldots, \varsigma_t, v, v, \ldots, v). \end{aligned}$$

The pair $(X, g_{\mathbf{Q}})$ is called quaternion-valued $g_{\mathbf{Q}}$-metric space with degree $p$. For $p = 1, 2$ respectively, it is equivalent to quaternion-valued metric and quaternion-valued $G$-metric space.

DEFINITION 9. ([13]) A $g_{\mathbf{Q}}$-metric on $X$ is called multiplicity independent with degree $p$ if the following holds

$$g_{\mathbf{Q}}(\tau_0, \ldots, \tau_p) = g_{\mathbf{Q}}(\varsigma_0, \ldots, \varsigma_p),$$

for all $(\tau_0, \tau_1, \ldots, \tau_p), (\varsigma_0, \varsigma_1, \ldots, \varsigma_p) \in X^{p+1}$ with

$$\{\tau_i : i = 0, \ldots, p\} = \{\varsigma_i : i = 0, \ldots, p\}.$$

Note that for a given multiplicity independent $g_{\mathbf{Q}}$-metric with order $2$, it holds that $g_{\mathbf{Q}}(\tau, \varsigma, \varsigma) = g_{\mathbf{Q}}(\tau, \tau, \varsigma)$. For a given multiplicity independent $g_{\mathbf{Q}}$-metric with order $3$, it holds that $g_{\mathbf{Q}}(\tau, \varsigma, \varsigma, \varsigma) = g_{\mathbf{Q}}(\tau, \tau, \varsigma, \varsigma) = g_{\mathbf{Q}}(\tau, \tau, \tau, \varsigma)$ and $g_{\mathbf{Q}}(\tau, \tau, \varsigma, z) = g_{\mathbf{Q}}(\tau, \varsigma, \varsigma, z) = g_{\mathbf{Q}}(\tau, \varsigma, z, z)$.

REMARK 2. If we allow equality under the conditions of monotonicity in Definition 9 that is

$$g_{\mathbf{Q}}(\tau_0,\ldots,\tau_p) \preceq g_{\mathbf{Q}}(\varsigma_0,\ldots,\varsigma_p)$$

for $(\tau_0,\ldots,\tau_p),(\varsigma_0,\ldots,\varsigma_p) \in X^{p+1}$ with $\{\tau_i : i=0,\ldots,p\} \subseteq \{\varsigma_i : i=0,\ldots,p\}$, then every $g_{\mathbf{Q}}$-metric becomes multiplicity independent.

Utilizing the notion of ideals, Kostyrko et al. [16] determined the notion of $\mathscr{I}$ and $\mathscr{I}^*$-convergence.

Assume $Y \neq \emptyset$. $\mathscr{I} \subset 2^Y$ is called an ideal on $Y$ provided that $(a)$ for all $S,T \in \mathscr{I}$ implies $S \cup T \in \mathscr{I}$; $(b)$ for all $S \in \mathscr{I}$ and $T \subset S$ implies $T \in \mathscr{I}$.

Assume $Y \neq \emptyset$. $\mathscr{F} \subset 2^Y$ is named a filter on $Y$ provided that $(a)$ for all $S,T \in \mathscr{F}$ implies $S \cap T \in \mathscr{F}$; $(b)$ for all $S \in \mathscr{F}$ and $T \supset S$ implies $T \in \mathscr{F}$.

An ideal $\mathscr{I}$ is known as non-trivial provided that $Y \notin \mathscr{I}$ and $\mathscr{I} \neq \emptyset$. A non-trivial ideal $\mathscr{I} \subset P(Y)$ is known as an admissible ideal in $Y$ iff $\mathscr{I} \supset \{\{w\} : w \in Y\}$. $\mathscr{I}_d$ is defined as the set of all subsets of $\mathbb{N}$ whose natural density is zero forms a non-trivial admissible ideal. Then, the filter $\mathscr{F} = \mathscr{F}(\mathscr{I}) = \{Y - S : S \in \mathscr{I}\}$ is called the filter connected with the ideal.

An admissible ideal $\mathscr{I} \subset 2^{\mathbb{N}}$ is said to have the property $(AP)$ if for any sequence $\{A_1,A_2,\ldots\}$ of mutually disjoint sets of $\mathscr{I}$, there is a sequence $\{B_1,B_2,\ldots\}$ of sets such that each symmetric difference $A_j \Delta B_j$ $(j=1,2,\ldots)$ is finite and $\bigcup_{j=1}^{\infty} B_j \in \mathscr{I}$.

Let $\sigma$ be a mapping such that $\sigma : \mathbb{N}^+ \to \mathbb{N}^+$ (the set of all positive integers). A continuous linear functional $\Phi$ on $l_\infty$, the space of real bounded sequences, is said to be an invariant mean or a $\sigma$ mean, if it satisfies the following conditions:

(1) $\Phi(x_n) \geqslant 0$, when the sequence $(x_n)$ has $x_n \geqslant 0$ for all $n \in \mathbb{N}$;

(2) $\Phi(e) = 1$, where $e = (1,1,1,\ldots)$;

(3) $\Phi\left(x_{\sigma(n)}\right) = \Phi(x_n)$ for all $(x_n) \in l_\infty$.

The mappings $\Phi$ are assumed to be one-to-one such that $\sigma^m(n) \neq n$ for all positive integers $n$ and $m$, where $\sigma^m(n)$ denotes the $m$ th iterate of the mapping $\sigma$ at $n$. Thus, $\Phi$ extends the limit functional on $c$, the space of convergent sequences, in the sense that $\Phi(x_n) = \lim x_n$, for all $(x_n) \in c$.

In case $\sigma$ is translation mappings $\sigma(n) = n+1$, the $\sigma$-mean is often called a Banach limit.

The space $V_\sigma$, the set of bounded sequences whose invariant means are equal, can be shown that

$$V_\sigma = \left\{ (x_k) \in l_\infty : \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)} = L \right\}$$

uniformly in $n$.

Let $\theta$ be a lacunary sequence, $E \subseteq \mathbb{N}$ and

$$s_r := \min_n \{|E \cap \{\sigma^m(n) : m \in I_r\}|\}$$

$$S_r := \max_n \{|E \cap \{\sigma^m(n) : m \in I_r\}|\}.$$

If the following limits exist

$$\underline{V}_{\theta}(E) = \lim_{r\to\infty} \frac{s_r}{h_r}, \quad \overline{V}_{\theta}(E) = \lim_{r\to\infty} \frac{S_r}{h_r},$$

then they are called a lower lacunary invariant uniform density and an upper lacunary invariant uniform density of the set $E$, respectively. If $\underline{V}_{\theta}(E) = \overline{V}_{\theta}(E)$, then $V_{\theta}(E) = \underline{V}_{\theta}(E) = \overline{V}_{\theta}(E)$ is called the lacunary invariant uniform density of $E$.

The class of all $E \subseteq \mathbb{N}$ with $\underline{V}_{\theta}(E) = 0$ will be denoted by $\mathscr{I}_{\sigma\theta}$. Note that $\mathscr{I}_{\sigma\theta}$ is an admissible ideal.

A sequence $(x_m)$ is lacunary $\mathscr{I}_{\sigma}$-convergent to $L$, if for each $\varepsilon > 0$,

$$E(\varepsilon) := \{m \in \mathbb{N} : |x_m - L| \geqslant \varepsilon\} \in \mathscr{I}_{\sigma\theta},$$

i.e., $V_{\theta}(E(\varepsilon)) = 0$. In this case, we write $\mathscr{I}_{\sigma\theta} - \lim x_m = L$. Note that $\mathscr{I}_{\sigma\theta}$ is an admissible ideal.

## 2. Main results

DEFINITION 10. Let $(X, g_{\mathbf{Q}})$ be a quaternion-valued $g$-metric space, $\tau \in X$ be a point, and $(\tau_i) \subseteq X$ be a sequence. A sequence $(\tau_i)$ is said to be lacunary $g_{\mathbf{Q}}$-invariant summable to $\tau$ in $(X, g_{\mathbf{Q}})$ if

$$\lim_{r\to\infty} \frac{p!}{h_r^p} \sum_{i_w \in I_r,\, 1 \leqslant w \leqslant p} g_{\mathbf{Q}}\left(\tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, \ldots, \tau_{\sigma^{i_p}(m)}\right) = \tau,$$

uniformly in $m = 1, 2, 3, \ldots$ . In this instance, we denote $\tau_i \to \tau(g_{\mathbf{Q}}(V_{\sigma\theta}))$ to demonstrate the sequence $(\tau_i)$ is lacunary $g_{\mathbf{Q}}$-invariant summable to $\tau$ in $(X, g_{\mathbf{Q}})$.

Additionally, the set of lacunary strongly $g_{\mathbf{Q}}$-invariant convergence sequences in $(X, g_{\mathbf{Q}})$ is defined as follows:

$$g_{\mathbf{Q}}[V_{\sigma\theta}] = \left\{(\tau_i) : \lim_{r\to\infty} \frac{p!}{h_r^p} \sum_{i_w \in I_r,\, 1 \leqslant w \leqslant p} \left| g_{\mathbf{Q}}\left(\tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, \ldots, \tau_{\sigma^{i_p}(m)}\right) \right| = 0 \right\}$$

uniformly in $m$. In this context, we denote $\tau_i \to \tau(g_{\mathbf{Q}}[V_{\sigma\theta}])$ to indicate that the sequence $(\tau_i)$ is lacunary strongly $g_{\mathbf{Q}}$-invariant summable to $\tau$ in $(X, g_{\mathbf{Q}})$.

DEFINITION 11. A sequence $(\tau_i)$ is said to be lacunary strongly $g_{\mathbf{Q}}$-$s$-invariant summable $(0 < s < \infty)$ to $\tau$ if

$$\lim_{r\to\infty} \frac{p!}{h_r^p} \sum_{i_w \in I_r,\, 1 \leqslant w \leqslant p} \left| g_{\mathbf{Q}}\left(\tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, \ldots, \tau_{\sigma^{i_p}(m)}\right) \right|^s = 0,$$

uniformly in $m$ and it is indicated by $\tau_i \to \tau(g_{\mathbf{Q}}[V_{\sigma\theta}]_s)$.

DEFINITION 12. A sequence $(\tau_i)$ is said to be lacunary $g_{\mathbf{Q}}$-$\sigma$-statistical convergent to $\tau$ if, for every $q \in \mathbf{Q}$ with $0 \prec q$ such that

$$\lim_{r\to\infty} \frac{p!}{h_r^p} \left| \left\{ i_w \in I_r,\, 1 \leqslant w \leqslant p : \left| g_{\mathbf{Q}} \left( \tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, \ldots, \tau_{\sigma^{i_p}(m)} \right) \right| \geqslant |q| \right\} \right| = 0,$$

uniformly in $m$.

DEFINITION 13. A sequence $(\tau_i)$ is lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_\sigma$-convergent to $\tau$, if, for every $q \in \mathbf{Q}$ with $0 \prec q$ such that

$$K := \left\{ (i_1, i_2, \ldots, i_p) \in \mathbb{N}^p : \left| g_{\mathbf{Q}} \left( \tau, \tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_p} \right) \right| \geqslant |q| \right\} \in \mathscr{I}_{\sigma\theta},$$

i.e., $V_\theta(K) = 0$. In this case, we write $\tau_i \to \tau\left(g_{\mathbf{Q}}\left(\mathscr{I}_{\sigma\theta}\right)\right)$ or $g_{\mathbf{Q}}\left(\mathscr{I}_{\sigma\theta}\right) - \lim \tau_i = \tau$.

THEOREM 1. *Let $(\tau_i)$ is bounded sequence in $(X, g_{\mathbf{Q}})$. If $(\tau_i)$ is lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_\sigma$-convergent to $\tau$, then $(\tau_i)$ is lacunary $g_{\mathbf{Q}}$-invariant summable to $\tau$.*

*Proof.* Let $m \in \mathbb{N}$ be arbitrary and $q \in \mathbf{Q}$ with $0 \prec q$. Also, we suppose that $(\tau_i)$ is bounded sequence and $(\tau_i)$ is lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_\sigma$-convergent to $\tau$. Now, we proceed to estimate

$$\psi_\theta(m) := \left| \frac{p!}{h_r^p} \sum_{i_w \in I_r,\, 1 \leqslant w \leqslant p} g_{\mathbf{Q}} \left( \tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, \ldots, \tau_{\sigma^{i_p}(m)} \right) \right|.$$

For each $m = 1, 2, \ldots$, we have

$$\psi_\theta(m) \leqslant \psi_\theta^1(m) + \psi_\theta^2(m),$$

where

$$\psi_\theta^1(m) := \frac{p!}{h_r^p} \sum_{\substack{i_w \in I_r,\, 1 \leqslant w \leqslant p \\ \left| g_{\mathbf{Q}}\left(\tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, \ldots, \tau_{\sigma^{i_p}(m)}\right)\right| \geqslant |q|}} \left| g_{\mathbf{Q}} \left( \tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, \ldots, \tau_{\sigma^{i_p}(m)} \right) \right|,$$

and

$$\psi_\theta^2(m) := \frac{p!}{h_r^p} \sum_{\substack{i_w \in I_r,\, 1 \leqslant w \leqslant p \\ \left| g_{\mathbf{Q}}\left(\tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, \ldots, \tau_{\sigma^{i_p}(m)}\right)\right| < |q|}} \left| g_{\mathbf{Q}} \left( \tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, \ldots, \tau_{\sigma^{i_p}(m)} \right) \right|.$$

For every $m = 1, 2, \ldots$, it is evident that $\psi_\theta^2(m) < |q|$.

Since $(\tau_i)$ is bounded sequence, there exists an $M > 0$ such that

$$\left| g_{\mathbf{Q}} \left( \tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, \ldots, \tau_{\sigma^{i_p}(m)} \right) \right| \leqslant M, \ (i_w \in I_r,\, 1 \leqslant w \leqslant p, m = 1, 2, \ldots)$$

and so we have

$$\psi_\theta^1(m) = \frac{p!}{h_r^p} \sum_{\substack{i_w \in I_r,\, 1 \leqslant w \leqslant p \\ \left|g_{\mathbf{Q}}\left(\tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, \ldots, \tau_{\sigma^{i_p}(m)}\right)\right| \geqslant |q|}} \left|g_{\mathbf{Q}}\left(\tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, \ldots, \tau_{\sigma^{i_p}(m)}\right)\right|$$

$$\leqslant \frac{Mp!}{h_r^p} \left|\left\{i_w \in I_r,\, 1 \leqslant w \leqslant p : \left|g_{\mathbf{Q}}\left(\tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, \ldots, \tau_{\sigma^{i_p}(m)}\right)\right| \geqslant |q|\right\}\right|$$

$$\leqslant M \frac{p! \max_m \left|\left\{i_w \in I_r,\, 1 \leqslant w \leqslant p : \left|g_{\mathbf{Q}}\left(\tau, \tau_{\sigma^{i_1}(m)}, \tau_{\sigma^{i_2}(m)}, \ldots, \tau_{\sigma^{i_p}(m)}\right)\right| \geqslant |q|\right\}\right|}{h_r^p} = M \frac{p! S_r}{h_r^p}.$$

Thus, based on our assumption, $(\tau_i)$ is lacunary $g_{\mathbf{Q}}$-invariant summable to $\tau$. □

In general, the converse of Theorem 1 is not valid. Let $X = \mathbb{R}$, $\mathscr{I} = \mathscr{I}_\delta$ and $G_{\mathbf{H}} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbf{H}$ be a quaternion-valued $G$-metric space defined by

$$\begin{aligned} G_{\mathbf{Q}}(\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3) = {} & \left|\mathfrak{z}_0^1 - \mathfrak{z}_0^2\right| + \left|\mathfrak{z}_0^1 - \mathfrak{z}_0^3\right| + \left|\mathfrak{z}_0^2 - \mathfrak{z}_0^3\right| \\ & + i\left(\left|\mathfrak{z}_1^1 - \mathfrak{z}_1^2\right| + \left|\mathfrak{z}_1^1 - \mathfrak{z}_1^3\right| + \left|\mathfrak{z}_1^2 - \mathfrak{z}_1^3\right|\right) \\ & + j\left(\left|\mathfrak{z}_2^1 - \mathfrak{z}_2^2\right| + \left|\mathfrak{z}_2^1 - \mathfrak{z}_2^3\right| + \left|\mathfrak{z}_2^2 - \mathfrak{z}_2^3\right|\right) \\ & + k\left(\left|\mathfrak{z}_3^1 - \mathfrak{z}_3^2\right| + \left|\mathfrak{z}_3^1 - \mathfrak{z}_3^3\right| + \left|\mathfrak{z}_3^2 - \mathfrak{z}_3^3\right|\right), \end{aligned}$$

where $\mathfrak{p}_r = \mathfrak{z}_0^r + \mathfrak{z}_1^r i + \mathfrak{z}_2^r j + \mathfrak{z}_3^r k$ for $r = 1, 2, 3$.

Consider the sequence $(\tau_j)$ defined as follows:

$$\tau_j := \begin{cases} 1, & \text{if } j_{r-1} < j < j_{r-1} + \left[\sqrt{h_r}\right], \\ & \text{and } j \text{ is an even integer,} \\ 0, & \text{if } j_{r-1} < j < j_{r-1} + \left[\sqrt{h_r}\right], \\ & \text{and } j \text{ is an odd integer.} \end{cases}$$

When $\sigma(m) = m + 1$, this sequence is lacunary $g_{\mathbf{Q}}$-invariant summable to $\frac{1}{2}$ but it is not lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_\sigma$-convergent.

Next, we will present the following theorems which establish the relationships between the concepts of lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_\sigma$-convergence and lacunary strongly $g_{\mathbf{Q}}$-$s$-invariant summability. We will demonstrate that these concepts are equivalent for bounded sequences.

THEOREM 2. *If a sequence $(\tau_i)$ is lacunary strongly $g_{\mathbf{Q}}$-$s$-invariant summable to $\tau$, then it is lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_\sigma$-convergent to $\tau$.*

*Proof.* Let $0 < s < \infty$ and $q \in \mathbf{Q}$ with $0 \prec q$. Suppose that $\tau_i \to \tau\left(g_{\mathbf{Q}}[V_{\sigma\theta}]_s\right)$.

Then, for every $m=1,2,\ldots$, we have

$$\begin{aligned}
&\sum_{i_w\in I_r,\,1\leqslant w\leqslant p}\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right|^s\\
\geqslant&\sum_{\substack{i_w\in I_r,\,1\leqslant w\leqslant p\\ \left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right|\geqslant|q|}}\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right|^s\\
\geqslant&|q|^s\left|\left\{i_w\in I_r,\,1\leqslant w\leqslant p:\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right|\geqslant|q|\right\}\right|\\
\geqslant&|q|^s\max_m\left|\left\{i_w\in I_r,\,1\leqslant w\leqslant p:\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right|\geqslant|q|\right\}\right|
\end{aligned}$$

and so

$$\begin{aligned}
&\frac{p!}{h_r^p}\sum_{i_w\in I_r,\,1\leqslant w\leqslant p}\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right|^s\\
&\quad\geqslant|q|^s\,\frac{p!\max_m\left|\left\{i_w\in I_r,\,1\leqslant w\leqslant p:\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right|\geqslant|q|\right\}\right|}{h_r^p}=|q|^s\,\frac{p!S_r}{h_r^p}.
\end{aligned}$$

Hence, due to our assumption, $g_{\mathbf{Q}}(\mathscr{I}_{\sigma\theta})-\lim\tau_i=\tau$. $\square$

THEOREM 3. *Let $(\tau_i)$ is bounded sequence. If $(\tau_i)$ is lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_\sigma$-convergent to $\tau$, then it is lacunary strongly $g_{\mathbf{Q}}$-$s$-invariant summable to $\tau$.*

*Proof.* Assume that $(\tau_i)\in l_\infty$ and $g_{\mathbf{Q}}(\mathscr{I}_{\sigma\theta})-\lim\tau_i=\tau$. Let $0<s<\infty$ and $q\in\mathbf{Q}$ with $0\prec q$. The boundedness of $(\tau_i)$ implies that there exists a $M>0$ such that

$$\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right|\leqslant M,$$

$(i_w\in I_r,\,1\leqslant w\leqslant p,\,m=1,2,\ldots)$. Therefore, we obtain

$$\begin{aligned}
&\frac{p!}{h_r^p}\sum_{i_w\in I_r,\,1\leqslant w\leqslant p}\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right|^s\\
&\quad=\sum_{\substack{i_w\in I_r,\,1\leqslant w\leqslant p\\ \left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right|\geqslant|q|}}\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right|^s\\
&\quad\quad+\sum_{\substack{i_w\in I_r,\,1\leqslant w\leqslant p\\ \left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right|<|q|}}\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right|^s\\
&\quad\leqslant M\frac{p!\max_m\left|\left\{i_w\in I_r,\,1\leqslant w\leqslant p:\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right|\geqslant|q|\right\}\right|}{h_r^p}+|q|^s\\
&\quad=M\frac{p!S_r}{h_r^p}+|q|^s.
\end{aligned}$$

Therefore, we obtain

$$\lim_{r\to\infty}\frac{p!}{h_r^p}\sum_{i_w\in I_r,\ 1\leqslant w\leqslant p}\left|g_{\mathbf{Q}}\left(\tau,\tau_{\sigma^{i_1}(m)},\tau_{\sigma^{i_2}(m)},\ldots,\tau_{\sigma^{i_p}(m)}\right)\right|^s=0,$$

uniformly in $m$. Hence, we get $\tau_i\to\tau\left(g_{\mathbf{Q}}\left[V_{\sigma\theta}\right]_s\right)$. □

THEOREM 4. *A sequence $(\tau_i)\in l_\infty$. Then, $(\tau_i)$ lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_\sigma$-convergent to $\tau$ iff it is lacunary strongly $g_{\mathbf{Q}}$-$s$-invariant summable to $\tau$.*

*Proof.* This directly follows from Theorems 2 and Theorem 3. □

Now, we state the theorem without proof that establishes a relationship between the concepts of lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_\sigma$-convergence and lacunary $g_{\mathbf{Q}}$-$\sigma$-statistical convergence.

THEOREM 5. *A sequence $(\tau_i)$ is lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_\sigma$-convergent to $\tau$ iff this sequence is lacunary $\sigma$-statistical convergent to $\tau$.*

Finally, by introducing the concept of lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_\sigma^*$-convergence, we establish the relationship between this concept and the notion of lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_\sigma$-convergence.

DEFINITION 14. A sequence $(\tau_i)$ is said to be lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_\sigma^*$-convergent or $\mathscr{I}_{\sigma\theta}^*$-$g$-convergent to $\tau$, if there exists a set $M^*\in\mathscr{F}\left(\mathscr{I}_{\sigma\theta}\right)$,

$$M^*=\left\{\left(m_{k_1},m_{k_2},\ldots,m_{k_p}\right)\in\mathbb{N}^p: m_{k_i}\in M,\ i=1,2,\ldots,p\right\},$$

where $M\in\mathscr{F}\left(\mathscr{I}_{\sigma\theta}\right)$ $\left(\mathbb{N}^p\diagdown M=H\in\mathscr{I}_{\sigma\theta}\right)$,

$$M=\left\{m_1<m_2<\ldots<m_k<\ldots\right\}\subset\mathbb{N}$$

such that

$$\lim_{k_1,k_2,\ldots,k_p\to\infty} g_{\mathbf{Q}}\left(\tau,\tau_{m_{k_1}},\tau_{m_{k_2}},\ldots,\tau_{m_{k_p}}\right)=0.$$

In this case, we write $g_{\mathbf{Q}}^{\mathscr{I}_{\sigma\theta}^*}-\lim_{i\to\infty}\tau_i=\tau$ or $\tau_i\to\tau\left(g_{\mathbf{Q}}^{\mathscr{I}_{\sigma\theta}^*}\right)$.

THEOREM 6. *If a sequence $(\tau_i)$ is lacunary $g$-$\mathscr{I}_\sigma^*$-convergent to $\tau$, then this sequence is lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_\sigma$-convergent to $\tau$.*

*Proof.* Let $q\in\mathbf{Q}$ with $0\prec q$. Since $g_{\mathbf{Q}}^{\mathscr{I}_{\sigma\theta}^*}-\lim_{i\to\infty}\tau_i=\tau$, there exists a set $M^*\in\mathscr{F}\left(\mathscr{I}_{\sigma\theta}\right)$,

$$M^*=\left\{\left(m_{k_1},m_{k_2},\ldots,m_{k_p}\right)\in\mathbb{N}^p: m_{k_i}\in M,\ i=1,2,\ldots,p\right\},$$

where $M\in\mathscr{F}\left(\mathscr{I}_{\sigma\theta}\right)$ $\left(\mathbb{N}^p\diagdown M=H\in\mathscr{I}_{\sigma\theta}\right)$,

$$M=\left\{m_1<m_2<\ldots<m_k<\ldots\right\}\subset\mathbb{N}$$

such that

$$\lim_{k_1,k_2,\dots,k_p\to\infty} g_{\mathbf{Q}}\left(\tau,\tau_{m_{k_1}},\tau_{m_{k_2}},\dots,\tau_{m_{k_p}}\right)=0. \tag{1}$$

There exists an $m_{k_0}\in\mathbb{N}$ by (1) such that

$$g_{\mathbf{Q}}\left(\tau,\tau_{m_{k_1}},\tau_{m_{k_2}},\dots,\tau_{m_{k_p}}\right)<|q|,$$

for each $m_{k_1},m_{k_2},\dots,m_{k_p}\geqslant m_{k_0}$. Then

$$\begin{aligned} A&=\left\{(i_1,i_2,\dots,i_p)\in\mathbb{N}^p:\left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\dots,\tau_{i_p}\right)\right|\geqslant|q|\right\}\\ &\subseteq\left[H\cup\left\{\left(m_{k_1},m_{k_2},\dots,m_{k_p}\right):m_{k_i}\in\left\{m_{k_1},m_{k_2},\dots,m_{k_0}\right\}\right\}\right], \end{aligned} \tag{2}$$

supplies for each $q\in\mathbf{Q}$ with $0\prec q$.

Since $\mathscr{I}_{\sigma\theta}$ is an admissible and $H\in\mathscr{I}_{\sigma\theta}$, the set on the right-hand side of (2) belongs to $\mathscr{I}_{\sigma\theta}$. Hence, $A\in\mathscr{I}_{\sigma\theta}$. Hence, we get $g_{\mathbf{Q}}(\mathscr{I}_{\sigma\theta})-\lim\tau_i=\tau$. □

If the ideal $\mathscr{I}_{\sigma\theta}$ possesses the property (AP), then the converse of Theorem 6 holds.

THEOREM 7. *Let the ideal $\mathscr{I}_{\sigma\theta}$ has the property (AP). If a sequence $(\tau_i)$ is lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_{\sigma}$-convergent to $\tau$, then this sequence is lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_{\sigma}^{*}$-convergent to $\tau$.*

*Proof.* Given that $\mathscr{I}_{\sigma\theta}$ satisfies property (AP). Let $g_{\mathbf{Q}}(\mathscr{I}_{\sigma\theta})-\lim\tau_i=\tau$ and $q\in\mathbf{Q}$ with $0\prec q$. Then

$$A=\left\{(i_1,i_2,\dots,i_p)\in\mathbb{N}^p:\left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\dots,\tau_{i_p}\right)\right|\geqslant|q|\right\}\in\mathscr{I}_{\sigma\theta}.$$

We can specify:

$$A_1=\left\{(i_1,i_2,\dots,i_p)\in\mathbb{N}^p:\left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\dots,\tau_{i_p}\right)\right|\geqslant 1\right\},$$

and

$$A_s=\left\{(i_1,i_2,\dots,i_p)\in\mathbb{N}^p:\frac{1}{s}\leqslant\left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\dots,\tau_{i_p}\right)\right|<\frac{1}{s-1}\right\},$$

for $s\geqslant 2$, $s\in\mathbb{N}$. Clearly, $A_u\cap A_v=\emptyset$ for $u\neq v$ and $A_s\in\mathscr{I}_{\sigma\theta}$ (for each $s\in\mathbb{N}$). By property $(AP)$, there exists a sequence of sets $\{B_s\}$ such that $A_j\Delta B_j$ are finite (for $j\in\mathbb{N}$) and $B=\bigcup_{j=1}^{\infty}B_j\in\mathscr{I}_{\sigma\theta}$. It suffices to prove that for $M^*=\mathbb{N}^p\diagdown B$, we have

$$g_{\mathbf{Q}}-\lim_{\substack{i\to\infty\\ i\in M^*}}\tau_i=\tau. \tag{3}$$

Let $\sigma>0$. For every $\sigma\in\mathbf{Q}$ with $0\prec\sigma$, choose $k\in\mathbb{N}$ such that $\frac{1}{k+1}<|\sigma|$. Then

$$\left\{(i_1,i_2,\dots,i_p)\in\mathbb{N}^p:\left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\dots,\tau_{i_p}\right)\right|\geqslant|\sigma|\right\}\subset\bigcup_{j=1}^{k+1}A_j.$$

Since $A_j \Delta B_j$, $(j = 1,2,\ldots,k+1)$ are finite, there exists a $\alpha_0 \in \mathbb{N}$ such that

$$\begin{aligned} &\bigcup_{j=1}^{k+1} A_j \cap \left\{(i_1,i_2,\ldots,i_p) \in \mathbb{N}^p : i_1,i_2,\ldots,i_p > \alpha_0\right\} \\ &\quad = \bigcup_{j=1}^{k+1} B_j \cap \left\{(i_1,i_2,\ldots,i_p) \in \mathbb{N}^p : i_1,i_2,\ldots,i_p > \alpha_0\right\}. \end{aligned} \tag{4}$$

If $i_1,i_2,\ldots,i_p > \alpha_0$ and $(i_1,i_2,\ldots,i_p) \notin B$, then $(i_1,i_2,\ldots,i_p) \notin \bigcup_{j=1}^{k+1} B_j$ and, by (4), $(i_1,i_2,\ldots,i_p) \notin \bigcup_{j=1}^{k+1} A_j$ but then

$$\left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right| < \frac{1}{k+1} < |\sigma|.$$

Hence, (3) holds. As a result, $g_{\mathbf{Q}}^{\mathscr{I}^*_{\sigma\theta}} - \lim_{i\to\infty} \tau_i = \tau$. □

DEFINITION 15. Let $(X,g_{\mathbf{Q}})$ be a quaternion-valued $g$-metric space, $\tau \in X$ be a point, and $(\tau_i) \subseteq X$ be a sequence. $(\tau_i)$ is said to be lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_\sigma$-Cauchy if, for every $q \in \mathbf{Q}$ with $0 \prec q$, there exists $i_r \in \mathbf{Q}$ such that

$$B = \left\{(i_1,i_2,\ldots,i_p) \in \mathbb{N}^p : \left|g_{\mathbf{Q}}\left(\tau_{i_r},\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right| \geqslant |q|\right\} \in \mathscr{I}_{\sigma\theta},$$

i.e., $V_\theta(B) = 0$. $(X,g_{\mathbf{Q}})$ is called a complete quaternion-valued $g$-metric space.

THEOREM 8. *Let $(X,g_{\mathbf{Q}})$ be a complete quaternion-valued $g$-metric space. Then, a sequence $(\tau_i)$ of points in $(X,g_{\mathbf{Q}})$ is lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_\sigma$-convergent if and only if it is lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_\sigma$-Cauchy.*

*Proof.* Suppose that $g_{\mathbf{Q}}(\mathscr{I}_{\sigma\theta}) - \lim \tau_i = \tau$ and $q \in \mathbf{Q}$ with $0 \prec q$. Then, we get $A \in \mathscr{I}_{\sigma\theta}$, where

$$A(\varepsilon) := \left\{(i_1,i_2,\ldots,i_p) \in \mathbb{N}^p : \left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right| \geqslant |q| = \frac{\varepsilon}{2}\right\},$$

where $q = \frac{\varepsilon}{4} + i\frac{\varepsilon}{4} + j\frac{\varepsilon}{4} + k\frac{\varepsilon}{4}$. Since $\mathscr{I}_{\sigma\theta}$ is an admissible ideal, there exists an $(m_1,m_2,m_3,\ldots,m_s) \in \mathbb{N}^p$ such that $(m_1,m_2,m_3,\ldots,m_s) \notin A(\varepsilon)$. Let

$$B(\varepsilon) := \left\{(i_1,i_2,\ldots,i_p) \in \mathbb{N}^p : \left|g_{\mathbf{Q}}\left(\tau_{m_1},\tau_{m_2},\ldots,\tau_{m_s},\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right| \geqslant \varepsilon\right\},$$

we need to demonstrate that $B(\varepsilon) \subset A(\varepsilon)$. Let $(i_1,i_2,\ldots,i_p) \in B(\varepsilon)$. Then

$$\left|g_{\mathbf{Q}}\left(\tau_{m_1},\tau_{m_2},\ldots,\tau_{m_s},\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right| \geqslant \varepsilon$$

and hence

$$g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right) \geqslant \frac{\varepsilon}{2}.$$

That is $(i_1,i_2,\ldots,i_p)\in A(\varepsilon)$. Otherwise if

$$\begin{aligned}\varepsilon &\leqslant \left|g_{\mathbf{Q}}\left(\tau_{m_1},\tau_{m_2},\ldots,\tau_{m_s},\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right| \\ &\leqslant g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)+g_{\mathbf{Q}}\left(\tau,\tau_{m_1},\tau_{m_2},\ldots,\tau_{m_s}\right) \\ &< \tfrac{\varepsilon}{2}+\tfrac{\varepsilon}{2} \\ &= \varepsilon,\end{aligned}$$

which is not possible. Hence $B(\varepsilon)\subset A(\varepsilon)$, which implies that $\left(\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)$ is lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_\sigma$-Cauchy.

Conversely, suppose that $(\tau_k)$ is lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_\sigma$-Cauchy but not lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_\sigma$-convergent. Then, there exists $\left(\tau_{k_1},\tau_{k_2},\ldots\tau_{k_m}\right)\in\mathbb{N}^p$ such that $G(\varepsilon)\in\mathscr{I}_{\sigma\theta}$ where

$$G(\varepsilon):=\left\{(i_1,i_2,\ldots,i_p)\in\mathbb{N}^p:\left|g_{\mathbf{Q}}\left(\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p},\tau_{k_1},\tau_{k_2},\ldots,\tau_{k_m}\right)\right|\geqslant\varepsilon\right\},$$

and $D(\varepsilon)\in\mathscr{I}_{\sigma\theta}$ where

$$D(\varepsilon):=\left\{(i_1,i_2,\ldots,i_p)\in\mathbb{N}^p:\left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|<\frac{\varepsilon}{2}\right\},$$

that is $D^c(\varepsilon)\in\mathscr{F}(\mathscr{I}_{\sigma\theta})$.

Given that

$$\left|g_{\mathbf{Q}}\left(\tau_{m_1},\tau_{m_2},\ldots,\tau_{m_s},\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|\leqslant 2\left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|<\varepsilon$$

if $\left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|<\frac{\varepsilon}{2}$. Consequently, $G^c(\varepsilon)\in\mathscr{I}_{\sigma\theta}$, implying $G(\varepsilon)\in\mathscr{F}(\mathscr{I}_{\sigma\theta})$, leading to a contradiction, since $\left(\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)$ was lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_\sigma$-Cauchy. Therefore, $(\tau_k)$ is a lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_\sigma$-convergent sequence. □

THEOREM 9. *Let $(X,g_{\mathbf{Q}})$ be a complete quaternion-valued $g$-metric space, $\mathscr{I}_{\sigma\theta}$ be an admissible ideal, and $(\tau_i)$, $(\omega_i)$ be two sequences in $X$. Then, the following statements hold:*
*(a) If $g_{\mathbf{Q}}(\mathscr{I}_{\sigma\theta})-\lim_{i\to\infty}\tau_i=\tau$ and $g_{\mathbf{Q}}(\mathscr{I}_{\sigma\theta})-\lim_{i\to\infty}\omega_i=\omega$, then $g_{\mathbf{Q}}(\mathscr{I}_{\sigma\theta})-\lim_{i\to\infty}(\tau_i+\omega_i)=\tau+\omega$;*
*(b) If $g_{\mathbf{Q}}(\mathscr{I}_{\sigma\theta})-\lim_{i\to\infty}\tau_i=\tau$ and $\kappa\in\mathbb{R}$, then $g_{\mathbf{Q}}(\mathscr{I}_{\sigma\theta})-\lim_{i\to\infty}\kappa\tau_i=\kappa\tau$.*

*Proof.* $(a)$ Let $q\in\mathbf{Q}$ with $0\prec q$. Since $g_{\mathbf{Q}}(\mathscr{I}_{\sigma\theta})-\lim_{i\to\infty}\tau_i=\tau$ and $g_{\mathbf{Q}}(\mathscr{I}_{\sigma\theta})-\lim_{i\to\infty}\omega_i=\omega$, we have

$$\left\{(i_1,i_2,\ldots,i_p)\in\mathbb{N}^p:\left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|\geqslant\frac{|q|}{2}\right\}\in\mathscr{I}_{\sigma\theta},$$

and

$$\left\{(i_1,i_2,\ldots,i_p)\in\mathbb{N}^p:\left|g_{\mathbf{Q}}\left(\omega,\omega_{i_1},\omega_{i_2},\ldots,\omega_{i_p}\right)\right|\geqslant\frac{|q|}{2}\right\}\in\mathscr{I}_{\sigma\theta}.$$

Then

$$\begin{aligned}&\left\{(i_1,i_2,\ldots,i_p)\in\mathbb{N}^p:\left|g_{\mathbf{Q}}\left(\tau+\omega,\tau_{i_1}+\omega_{i_1},\tau_{i_2}+\omega_{i_2},\ldots,\tau_{i_p}+\omega_{i_p}\right)\right|\geqslant|q|\right\}\\&\subseteq\left\{(i_1,i_2,\ldots,i_p)\in\mathbb{N}^p:\left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|\geqslant\tfrac{|q|}{2}\right\}\\&\cup\left\{(i_1,i_2,\ldots,i_p)\in\mathbb{N}^p:\left|g_{\mathbf{Q}}\left(\omega,\omega_{i_1},\omega_{i_2},\ldots,\omega_{i_p}\right)\right|\geqslant\tfrac{|q|}{2}\right\},\end{aligned}$$

can be easily verified. Thus, by properties of ideal,

$$\left\{(i_1,i_2,\ldots,i_p)\in\mathbb{N}^p:\left|g_{\mathbf{Q}}\left(\tau+\omega,\tau_{i_1}+\omega_{i_1},\tau_{i_2}+\omega_{i_2},\ldots,\tau_{i_p}+\omega_{i_p}\right)\right|\geqslant|q|\right\}\in\mathscr{I}_{\sigma\theta},$$

i.e., $g_{\mathbf{Q}}(\mathscr{I}_{\sigma\theta})-\lim_{i\to\infty}(\tau_i+\omega_i)=\tau+\omega$.
$(b)$ This can be proven by a similar way. So, we omit detail. □

DEFINITION 16. Let $(X,g_{\mathbf{Q}})$ be a quaternion-valued $g$-metric space, $\mathscr{I}_{\sigma\theta}$ be an admissible ideal, and $(\tau_i)$ be a sequence in $X$. Then, $(\tau_i)$ is said to be lacunary $\mathscr{I}_\sigma^*$-$g$-Cauchy sequence if there exists a set

$$M^*=\left\{\left(m_{k_1},m_{k_2},\ldots,m_{k_p}\right)\in\mathbb{N}^p:m_{k_i}\in M,\ i=1,2,\ldots,p\right\}\subset\mathbb{N}^p,$$

where $M\in\mathscr{F}(\mathscr{I}_{\sigma\theta})$, $M=\{m_1<m_2<\ldots<m_k<\ldots\}\subset\mathbb{N}$ such that the subsequence $(\tau_M)=\left(\tau_{m_k}\right)$ is an ordinary $g$-Cauchy sequence in $X$, i.e.,

$$\lim_{k_1,k_2,\ldots,k_p\to\infty}g_{\mathbf{Q}}\left(\tau_{m_{k_1}},\tau_{m_{k_2}},\ldots,\tau_{m_{k_p}}\right)=0.$$

THEOREM 10. *Let $\mathscr{I}_{\sigma\theta}$ be an admissible ideal. If $(\tau_i)$ is a lacunary $\mathscr{I}_\sigma^*$-$g$-Cauchy sequence then this sequence is lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_\sigma$-Cauchy sequence.*

*Proof.* Let $(\tau_i)$ be a lacunary $\mathscr{I}_\sigma^*$-$g$-Cauchy sequence. Then, there exists the set

$$M^*=\left\{\left(m_{k_1},m_{k_2},\ldots,m_{k_p}\right)\in\mathbb{N}^p:m_{k_i}\in M,\ i=1,2,\ldots,p\right\}\subset\mathbb{N}^p,$$

where $M\in\mathscr{F}(\mathscr{I}_{\sigma\theta})$, $M=\{m_1<m_2<\ldots<m_k<\ldots\}\subset\mathbb{N}$ such that

$$g_{\mathbf{Q}}\left(\tau_{m_{k_1}},\tau_{m_{k_2}},\ldots,\tau_{m_{k_p}}\right)<|q|,$$

for every $q\in\mathbf{Q}$ with $0\prec q$ $(|q|=\varepsilon)$, and for all $m_{k_1},m_{k_2},\ldots,m_{k_p}\geqslant m_{k_0}(\varepsilon)$.

Let $U=U(\varepsilon)=m_{k_0+1}$. Then, for every $\varepsilon>0$, we have

$$g_{\mathbf{Q}}\left(\tau_U,\tau_{m_{k_1}},\tau_{m_{k_2}},\ldots,\tau_{m_{k_p}}\right)<\varepsilon,\ \ m_{k_1},m_{k_2},\ldots,m_{k_p}\geqslant m_{k_0}(\varepsilon).$$

Now, let $L=\mathbb{N}^p\setminus M^*$. It is clear that $L\in\mathscr{I}_{\sigma\theta}$ and

$$\begin{aligned}A(\varepsilon)&=\left\{(i_1,i_2,\ldots,i_p)\in\mathbb{N}^p:\left|g_{\mathbf{Q}}\left(\tau_U,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|\geqslant\varepsilon\right\}\\&\subseteq\left[L\cup\left\{\left(m_{k_1},m_{k_2},\ldots,m_{k_p}\right):m_{k_i}\in\left\{m_{k_1},m_{k_2},\ldots,m_{k_0}\right\}\right\}\right].\end{aligned}$$

Then

$$\left[L\cup\left\{\left(m_{k_1},m_{k_2},\ldots,m_{k_p}\right):m_{k_i}\in\left\{m_{k_1},m_{k_2},\ldots,m_{k_0}\right\}\right\}\right]\in\mathscr{I}_{\sigma\theta}.$$

Thus, for every $\varepsilon>0$, we can find an $U\in\mathbb{N}$ such that $A(\varepsilon)\in\mathscr{I}_{\sigma\theta}$. Hence, $(\tau_i)$ is a lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_\sigma$-Cauchy sequence. □

LEMMA 1. *Let $\{S_i\}_{i=1}^{\infty}$ be a countable family of subsets of $\mathbb{N}^p$ such that $S_i \in \mathscr{F}(\mathscr{I}_{\sigma\theta})$ for each $i$, where $\mathscr{F}(\mathscr{I}_{\sigma\theta})$ is a filter associate with an admissible ideal $\mathscr{I}_{\sigma\theta}$ with property (AP). Then, there exists a set $S \subset \mathbb{N}^p$ such that $S \in \mathscr{F}(\mathscr{I}_{\sigma\theta})$ and the set $S \setminus S_i$ is finite set for each $i$.*

THEOREM 11. *If $\mathscr{I}_{\sigma\theta}$ is an admissible ideal with property (AP), then the concepts of lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_\sigma$-Cauchy sequence and lacunary $\mathscr{I}_\sigma^*$-$g$-Cauchy sequence are equivalent.*

*Proof.* If a sequence $(\tau_i)$ is a lacunary $\mathscr{I}_\sigma^*$-$g$-Cauchy sequence, then by Theorem 10, it is also a lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_\sigma$-Cauchy sequence, where $\mathscr{I}_{\sigma\theta}$ need not necessarily have the property *(AP)*. Assuming now that $(\tau_i)$ is lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_\sigma$-Cauchy sequence, we need to show that it is also a lacunary $\mathscr{I}_\sigma^*$-$g$-Cauchy sequence.

Let $(\tau_i)$ be a lacunary $g_{\mathbf{Q}}$-$\mathscr{I}_\sigma$-Cauchy sequence. Then, if for every $q \in \mathbf{Q}$ with $0 \prec q$, there exists $r_0 \in \mathbf{Q}$ such that

$$\left\{(i_1,i_2,\ldots,i_p) \in \mathbb{N}^p : \left|g_{\mathbf{Q}}\left(\tau_{r_0},\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right| \geqslant |q| = \varepsilon\right\} \in \mathscr{I}_{\sigma\theta}.$$

Let

$$S_i := \left\{(i_1,i_2,\ldots,i_p) \in \mathbb{N}^p : \left|g_{\mathbf{Q}}\left(\tau_{m_{k_i}},\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right| < \frac{1}{i}\right\},$$

$i = 1,2,\ldots$ where $m_{k_i} = r_0\left(\frac{1}{i}\right)$. It is clear that $S_i \in \mathscr{F}(\mathscr{I}_{\sigma\theta})$ for $i = 1,2,\ldots$ Since $\mathscr{I}_{\sigma\theta}$ has the property *(AP)*, then there exists a set $S \subset \mathbb{N}$ such that $S \in \mathscr{F}(\mathscr{I}_{\sigma\theta})$ and $S \setminus S_i$ is finite for all $i$ by Lemma 1. Now, we show that

$$\lim_{k_1,k_2,\ldots,k_p\to\infty} g_{\mathbf{Q}}\left(\tau_{m_{k_1}},\tau_{m_{k_2}},\ldots,\tau_{m_{k_p}}\right) = 0.$$

Let $\varepsilon > 0$ and $j_0 \in \mathbb{N}$ such that $j_0 > \frac{p}{\varepsilon}$. If $k_1,k_2,\ldots,k_p \in S$ and $S \setminus S_{j_0}$ is finite set, so there exists $m = m(j_0)$ such that $k_1,k_2,\ldots,k_p \in S_{j_0}$ for all $k_1,k_2,\ldots,k_p \geqslant m(j_0)$. Thus, it follows that

$$\begin{aligned}\left|g_{\mathbf{Q}}\left(\tau_{m_{k_1}},\tau_{m_{k_2}},\ldots,\tau_{m_{k_p}}\right)\right| &\leqslant \left|g_{\mathbf{Q}}\left(\tau_{m_{k_1}},\tau_{i_1},\ldots,\tau_{i_1}\right)\right| + \left|g_{\mathbf{Q}}\left(\tau_{m_{k_2}},\tau_{i_1},\ldots,\tau_{i_1}\right)\right| \\ &\quad + \ldots + \left|g_{\mathbf{Q}}\left(\tau_{m_{k_p}},\tau_{i_1},\ldots,\tau_{i_1}\right)\right| \\ &\leqslant \left|g_{\mathbf{Q}}\left(\tau_{m_{k_1}},\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right| + \left|g_{\mathbf{Q}}\left(\tau_{m_{k_2}},\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right| \\ &\quad + \ldots + \left|g_{\mathbf{Q}}\left(\tau_{m_{k_p}},\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right| \\ &< \frac{1}{j_0} + \frac{1}{j_0} + \ldots + \frac{1}{j_0} < \varepsilon.\end{aligned}$$

This shows that the sequence $(\tau_i)$ is lacunary $\mathscr{I}_\sigma^*$-$g$-Cauchy sequence. □

DEFINITION 17. Let $(X, g_{\mathbf{Q}})$ be a quaternion-valued $g$-metric space, $\tau \in X$ be a point, and $(\tau_i) \subseteq X$ be a sequence. A sequence $(\tau_j)$ $\mathscr{I}$-invariant statistically convergent to $\tau$ if, for every $q,\rho \in \mathbf{Q}$ with $0 \prec q,\rho$ such that

$$\begin{aligned}\Big\{n \in \mathbb{N} : \tfrac{p!}{n^p}\Big|\Big\{&(i_1,i_2,\ldots,i_p) \in \mathbb{N}^p,\, i_1,i_2,\ldots,i_p \leqslant n : \\ &\left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right| \geqslant |q|\Big\}\Big| \geqslant |\rho|\Big\} \in \mathscr{I}_\sigma.\end{aligned}$$

In this scenario, we denote the convergence as $g_{\mathbf{Q}}^{S\sigma(\mathscr{I})} - \lim_{i\to\infty} \tau_i = \tau$ or $(\tau_i) \overset{g_{\mathbf{Q}}^{S\sigma(\mathscr{I})}}{\longrightarrow} \tau$. The collection of all $\mathscr{I}$-invariant statistically convergent sequences in quaternion-valued $g$-metric space is symbolized as $g_{\mathbf{Q}}^{S\sigma(\mathscr{I})}$.

DEFINITION 18. A sequence $(\tau_i)$ is said to be lacunary $\mathscr{I}$ invariant statistically convergent to $\tau$ if, for every $q,\rho \in \mathbf{Q}$ with $0 \prec q,\rho$ such that

$$\left\{ r \in \mathbb{N} : \frac{p!}{h_r^p} \left| \left\{ i_w \in I_r,\, 1 \leqslant w \leqslant p : \left| g_{\mathbf{Q}}\left(\tau, \tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_p}\right) \right| \geqslant |q| \right\} \right| \geqslant |\rho| \right\} \in \mathscr{I}_{\sigma\theta}.$$

In this case, we write $\tau_i \to \tau \left( g_{\mathbf{Q}}^{\mathscr{I}\sigma\theta(S)} \right)$ or $g_{\mathbf{Q}}^{\mathscr{I}\sigma\theta(S)} - \lim_{i\to\infty} \tau_i = \tau$. The collection of all $\mathscr{I}$-invariant lacunary statistical convergent sequences in quaternion-valued $g$-metric space is symbolized as $g_{\mathbf{Q}}^{S\sigma\theta(\mathscr{I})}$.

DEFINITION 19. A sequence $(\tau_i)$ is said to be strongly $\mathscr{I}_\sigma$-summable to $\tau$ if, for every $q \in \mathbf{Q}$ with $0 \prec q$ such that

$$\left\{ r \in \mathbb{N} : \frac{p!}{n^p} \sum_{i_1, i_2, \ldots, i_p = 1}^{n} \left| g_{\mathbf{Q}}\left(\tau, \tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_p}\right) \right| \geqslant |q| \right\} \in \mathscr{I}_\sigma.$$

We will use $\left[ g_{\mathbf{Q}}^{\mathscr{I}\sigma} \right] - \lim_{i\to\infty} \tau_i = \tau$ or $\tau_i \to \tau \left[ g_{\mathbf{Q}}^{\mathscr{I}\sigma} \right]$ to indicate the sequence $(\tau_i)$ is strongly $\mathscr{I}_\sigma$-convergent to $\tau$.

DEFINITION 20. A sequence $(\tau_i)$ is said to be strongly lacunary $\mathscr{I}_\sigma$-summable to $\tau$ if, for every $q \in \mathbf{Q}$ with $0 \prec q$ such that

$$\left\{ n \in \mathbb{N} : \frac{p!}{h_r^p} \sum_{i_w \in I_r,\, 1 \leqslant w \leqslant p} \left| g_{\mathbf{Q}}\left(\tau, \tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_p}\right) \right| \geqslant |q| \right\} \in \mathscr{I}_{\sigma\theta}.$$

We will use $\left[ g_{\mathbf{Q}}^{\mathscr{I}\sigma\theta} \right] - \lim_{i\to\infty} \tau_i = \tau$ or $\tau_i \to \tau \left[ g_{\mathbf{Q}}^{\mathscr{I}\sigma\theta} \right]$ to indicate the sequence $(\tau_i)$ is lacunary strongly $\mathscr{I}_\sigma$-convergent to $\tau$.

THEOREM 12. *Let $\theta = \{k_r\}$ be a lacunary sequence. Then, the following statements hold:*
*(i) / (a) If $\left[ g_{\mathbf{Q}}^{\mathscr{I}\sigma\theta} \right] - \lim_{i\to\infty} \tau_i = \tau$, then $g_{\mathbf{Q}}^{S\sigma\theta(\mathscr{I})} - \lim_{i\to\infty} \tau_i = \tau$, and*
*(i) / (b) $\left[ g_{\mathbf{Q}}^{\mathscr{I}\sigma\theta} \right]$ is a proper subset of $g_{\mathbf{Q}}^{S\sigma\theta(\mathscr{I})}$.*
*(ii) If $(\tau_j) \in l_\infty(X)$, the space of all bounded sequences of $(X, g_{\mathbf{Q}})$ and $g_{\mathbf{Q}}^{S\sigma\theta(\mathscr{I})} - \lim_{i\to\infty} \tau_i = \tau$, then $\left[ g_{\mathbf{Q}}^{\mathscr{I}\sigma\theta} \right] - \lim_{i\to\infty} \tau_i = \tau$.*
*(iii) $g_{\mathbf{Q}}^{S\sigma\theta(\mathscr{I})} \cap l_\infty = g_{\mathbf{Q}}^{N_\theta[\mathscr{I}]} \cap l_\infty$.*

*Proof.* $(i)/(a)$ If $q\in\mathbf{Q}$ with $0\prec q$ and $\left[g_{\mathbf{Q}}^{\mathscr{I}\sigma\theta}\right]-\lim_{i\to\infty}\tau_i=\tau$, then we can write

$$\sum_{i_w\in I_r,\,1\leqslant w\leqslant p}\left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|\geqslant\sum_{\substack{i_w\in I_r,\,1\leqslant w\leqslant p\\ \left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|\geqslant|q|}}\left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|$$
$$\geqslant|q|\left|\left\{i_w\in I_j,\,1\leqslant w\leqslant p:\left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|\geqslant|q|\right\}\right|$$

and so

$$\frac{p!}{|q|h_r^p}\sum_{i_w\in I_r,\,1\leqslant w\leqslant p}\left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|$$
$$\geqslant\frac{p!}{h_r^p}\left|\left\{i_w\in I_j,\,1\leqslant w\leqslant p:\left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|\geqslant|q|\right\}\right|.$$

Then, for every $\rho\in\mathbf{Q}$ with $0\prec\rho$, we have

$$\left\{r\in\mathbb{N}:\frac{p!}{h_r^p}\left|\left\{i_w\in I_r,\,1\leqslant w\leqslant p:\left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|\geqslant|q|\right\}\right|\geqslant|\rho|\right\}$$
$$\subseteq\left\{r\in\mathbb{N}:\frac{p!}{h_r^p}\sum_{i_w\in I_r,\,1\leqslant w\leqslant p}\left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|\geqslant|q|\,|\rho|\right\}\in\mathscr{I}_{\sigma\theta}.$$

Hence, we obtain $g_{\mathbf{Q}}^{S_{\sigma\theta}(\mathscr{I})}-\lim_{i\to\infty}\tau_i=\tau$.

$(i)/(b)$ In order to establish that the inclusion $\left[g_{\mathbf{Q}}^{\mathscr{I}\sigma\theta}\right]\subseteq g_{\mathbf{Q}}^{S_{\sigma\theta}(\mathscr{I})}$ is proper, let $\theta$ be given, and define $\tau_j$ to be $1,2,\ldots,\sqrt{h_r}$ for the first $\sqrt{h_r}$ integers in $I_r$ and $\tau_j=0$ otherwise, for all $r=1,2,\ldots$ Then, for any $q\in\mathbf{Q}$ with $0\prec q$

$$\frac{p!}{h_r^p}\left|\left\{i_w\in I_j,\,1\leqslant w\leqslant p:\left|g_{\mathbf{Q}}\left(0,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|\geqslant|q|\right\}\right|\leqslant\frac{p!\sqrt{h_r}}{h_r^p},$$

and for any $\rho\in\mathbf{Q}$ with $0\prec\rho$,

$$\left\{r\in\mathbb{N}:\frac{p!}{h_r^p}\left|\left\{i_w\in I_r,\,1\leqslant w\leqslant p:\left|g_{\mathbf{Q}}\left(0,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|\geqslant|q|\right\}\right|\geqslant|\rho|\right\}$$
$$\subseteq\left\{r\in\mathbb{N}:\frac{p!\sqrt{h_r}}{h_r^p}\geqslant|\rho|\right\}.$$

Since the set on the right-hand side is a finite set and so belongs to $\mathscr{I}_{\sigma\theta}$, it follows that $g_{\mathbf{Q}}^{S_{\sigma\theta}(\mathscr{I})}-\lim_{i\to\infty}\tau_i=0$.

On the other hand, for $p>1$,

$$\frac{p!}{h_r^p}\sum_{i_w\in I_r,\,1\leqslant w\leqslant p}\left|g_{\mathbf{Q}}\left(0,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|=\frac{p!}{h_r^p}\frac{\sqrt{h_r}\left(\sqrt{h_r}+1\right)}{2}\to0,$$

and for $0\leqslant p<1$,

$$\frac{p!}{h_r^p}\sum_{i_w\in I_r,\,1\leqslant w\leqslant p}\left|g_{\mathbf{Q}}\left(0,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|=\frac{p!}{h_r^p}\frac{\sqrt{h_r}\left(\sqrt{h_r}+1\right)}{2}\to\infty.$$

So, we have

$$\left\{r\in\mathbb{N}:\frac{p!}{h_r^p}\sum_{i_w\in I_r,\ 1\leqslant w\leqslant p}\left|g_{\mathbf{Q}}\left(0,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|\geqslant 0\right\}$$
$$=\left\{r\in\mathbb{N}:\frac{p!}{h_r^p}\frac{\sqrt{h_r}\left(\sqrt{h_r}+1\right)}{2}\geqslant 0\right\}=\{m,m+1,\ldots\}$$

for some $m\in\mathbb{N}$ which belongs to $\mathscr{F}\left(\mathscr{I}_{\sigma\theta}\right)$, since $\mathscr{I}_{\sigma\theta}$ is admissible. Hence, $\left[g_{\mathbf{Q}}^{\mathscr{I}\sigma\theta}\right]-\lim_{i\to\infty}\tau_i\neq 0$.
$(ii)$ Suppose that $(\tau_j)\in\ell_\infty$ and $g_{\mathbf{Q}}^{S_{\sigma\theta}(\mathscr{I})}-\lim_{i\to\infty}\tau_i=\tau$. Then, there exists a $M>0$ such that

$$\left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|<M.$$

Given $q\in\mathbf{Q}$ with $0\prec q$, we have

$$\begin{aligned}&\frac{p!}{h_r^p}\sum_{i_w\in I_r,\ 1\leqslant w\leqslant p}\left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|\\ \geqslant&\sum_{\substack{i_w\in I_r,\ 1\leqslant w\leqslant p\\ \left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|\geqslant\frac{|q|}{2}}}\left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|\\ &+\sum_{\substack{i_w\in I_r,\ 1\leqslant w\leqslant p\\ \left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|<\frac{|q|}{2}}}\left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|\\ \geqslant&\frac{Mp!}{h_r^p}\left|\left\{i_w\in I_j,\ 1\leqslant w\leqslant p:\left|g_{\mathbf{Q}}\left(0,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|\geqslant\frac{|q|}{2}\right\}\right|+\frac{|q|}{2}.\end{aligned}$$

Therefore, we have

$$\left\{r\in\mathbb{N}:\frac{p!}{h_r^p}\sum_{i_w\in I_r,\ 1\leqslant w\leqslant p}\left|g_{\mathbf{Q}}\left(\tau,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|\geqslant|q|\right\}$$
$$\subseteq\left\{r\in\mathbb{N}:\frac{p!}{h_r^p}\left|\left\{i_w\in I_r,\ 1\leqslant w\leqslant p:\left|g_{\mathbf{Q}}\left(0,\tau_{i_1},\tau_{i_2},\ldots,\tau_{i_p}\right)\right|\geqslant\frac{|q|}{2}\right\}\right|\geqslant\frac{|q|}{2M}\right\}\in\mathscr{I}_{\sigma\theta}.$$

As a result $\left[g_{\mathbf{Q}}^{\mathscr{I}\sigma\theta}\right]-\lim_{i\to\infty}\tau_i=\tau$.
$(iii)$ Follows from $(i)$ and $(ii)$. $\square$

THEOREM 13. *Let $(X,g_{\mathbf{Q}})$ be a quaternion-valued $g$-metric space, $\tau\in X$ be a point, and $(\tau_j)\subseteq X$ be a sequence. Then, the following statements hold:*
*$(i)/(a)$ If $\left[g_{\mathbf{Q}}^{\mathscr{I}\sigma}\right]-\lim_{i\to\infty}\tau_i=\tau$, then $g_{\mathbf{Q}}^{S_\sigma(\mathscr{I})}-\lim_{i\to\infty}\tau_i=\tau$, and*
*$(i)/(b)$ $\left[g_{\mathbf{Q}}^{\mathscr{I}\sigma}\right]$ is a proper subset of $g_{\mathbf{Q}}^{S_\sigma(\mathscr{I})}$.*
*$(ii)$ If $(\tau_j)\in l_\infty(X)$, the space of all bounded sequences of $(X,g_{\mathbf{Q}})$ and $g_{\mathbf{Q}}^{S_\sigma(\mathscr{I})}-\lim_{i\to\infty}\tau_i=\tau$, then $\left[g_{\mathbf{Q}}^{\mathscr{I}\sigma}\right]-\lim_{i\to\infty}\tau_i=\tau$.*
*$(iii)$ $g_{\mathbf{Q}}^{S_\sigma(\mathscr{I})}\cap l_\infty=\left[g_{\mathbf{Q}}^{\mathscr{I}\sigma}\right]\cap l_\infty$.*

## 3. Conclusion

In this study, we have delved into the intricate realm of lacunary $\mathscr{I}_\sigma$-convergence of sequences within quaternion-valued generalized metric spaces. Our investigation has highlighted the interconnections between this novel convergence concept and established notions such as lacunary invariant summability, lacunary strongly $s$-invariant summability, and lacunary $\sigma$-statistical convergence. These relationships provide a comprehensive understanding of convergence dynamics in the context of quaternion-valued metrics.

Additionally, we have introduced the concepts of lacunary $\mathscr{I}_\sigma^*$-convergence, aiming to enrich the theoretical framework of quaternion-valued generalized metric spaces. By establishing the equivalence between lacunary $\mathscr{I}_\sigma$-convergence and lacunary $\mathscr{I}_\sigma^*$-convergence through the definition of property $(AP)$, we have unified these concepts under a cohesive theoretical umbrella.

Moreover, our study has extended classical theorems to accommodate lacunary $\mathscr{I}_\sigma$-Cauchy and lacunary $\mathscr{I}_\sigma^*$-Cauchy sequences within quaternion-valued generalized metric spaces. This adaptation underscores the versatility and applicability of our theoretical framework in addressing convergence properties and dynamics in quaternion-valued settings.

In conclusion, this research underscores the importance of mathematical analysis in advancing our understanding of complex mathematical structures and phenomena. By exploring and formalizing new convergence concepts within quaternion-valued generalized metric spaces, we contribute to the broader mathematical discourse and pave the way for future research and applications in this specialized field.

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