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DUALITY AND STABILITY OF OPERATOR VALUED FRAMES FOR QUATERNIONIC HILBERT SPACES

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Abstract. This paper aims to prove some significant properties and relations between operator valued frames (OV-frames) in quaternionic Hilbert spaces. Various properties concerning the dual of an OV-frame are proved and the precise form of the family of duals of an OV-frame is given. Moreover, we try to construct OV-frames with the help of some partial isometries and finally, the stability of OV-frames under some perturbation conditions is established in quaternionic Hilbert spaces.

1. Introduction and preliminaries

The notion of frames and their generalizations like fusion frames, g-frames etc. have been studied and developed rapidly in the past decade mainly due to its significant applications in signal processing and coding theory [1, 4, 5, 13, 14, 15]. Further, a generalization of the concept of vector-valued frame, that enables us to deal with the operators in a Hilbert space instead of its elements, was introduced by L. Găvruta [6]. The notion of OV-frames, provides a more general way of series expansion of elements that is very similar to frame decomposition and have immense applications in quantum computing, packets encoding and many more. For more details on OV-frames, readers can refer [7, 9, 10, 11].

OV-frames in quaternionic Hilbert spaces are defined in [2]. This paper aims to provide a few more results on OV-frames in a right-quaternionic Hilbert space and is structured as follows: With the aim of making this paper self-contained, we recall some basic definitions and results concerning OV-frames and quaternions in the remaining part of this section. In Section 2, we prove some properties and relations between OV-frames. Section 3 concerns mainly about the properties of the duals of an OV-frame. In Section 4, construction of some OV-frames with the help of some partial isometries is given and finally, the stability of OV-frames under some perturbation conditions is established in Section 5.

All through the paper, H denotes a right quaternionic Hilbert space and K a two sided quaternionic Hilbert space, $\mathbb{H}$ the set of all real quaternions, $\mathbb{N}$ the set of all natural numbers and $\mathscr{I}$ an index set. The set of all the bounded operators from H to K is denoted by $\mathscr{B}(\mathsf{H},\mathsf{K})$ and the identity operator on H is denoted by I_{H}. The set of all the quaternionic valued square summable sequences is given by the set $\ell^2_{\mathbb{H}} =$

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$\left\{\{\mathfrak{q}_n\}_{n\in\mathbb{N}} \subseteq \mathbb{H} : \sum_{n\in\mathbb{N}} |\mathfrak{q}_n|^2 < \infty\right\}$. Moreover, the term "operator valued frames (OV-frames)" refers to quaternionic operator valued frames (quaternionic OV-frames).

Quaternions are a non-commutative extension of the complex numbers to four dimensions, defined over the real numbers $\mathbb{R}$. The quaternionic algebra $\mathbb{H}$ is given by the set

$$\mathbb{H} := \{\mathfrak{q} = \mathfrak{q}_0 + i\mathfrak{q}_1 + j\mathfrak{q}_2 + k\mathfrak{q}_3 : \mathfrak{q}_0, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3 \in \mathbb{R} \text{ and } i^2 = j^2 = k^2 = ijk = -1\}.$$

For any $\mathfrak{q} = \mathfrak{q}_0 + i\mathfrak{q}_1 + j\mathfrak{q}_2 + k\mathfrak{q}_3 \in \mathbb{H}$, $\mathfrak{q}_0$ is called the scalar or real part and $i\mathfrak{q}_1 + j\mathfrak{q}_2 + k\mathfrak{q}_3$ is called the imaginary or vector part of $\mathfrak{q}$. Moreover, it can be expressed as $\mathfrak{q} = s + v$, where s is the scalar part and v is the imaginary part of $\mathfrak{q}$ and its conjugate $\bar{\mathfrak{q}}$ is given by

$$\bar{\mathfrak{q}} = \mathfrak{q}_0 - i\mathfrak{q}_1 - j\mathfrak{q}_2 - k\mathfrak{q}_3.$$

This leads to a norm of $\mathfrak{q} \in \mathbb{H}$ defined as

$$|\mathfrak{q}| = \sqrt{\mathfrak{q}\bar{\mathfrak{q}}} = \sqrt{\mathfrak{q}_0^2 + \mathfrak{q}_1^2 + \mathfrak{q}_2^2 + \mathfrak{q}_3^2}.$$

For basic definitions and information about quaternionic Hilbert spaces and two-sided quaternionic Hilbert spaces and related terms, one may refer [8, 12] and references therein. We recall the definition of a quaternionic OV-frame as follows:

DEFINITION 1. A sequence $\{\mathscr{F}_n\}_{n\in\mathbb{N}} \subseteq \mathscr{B}(\mathsf{H}, \mathsf{K})$ is termed as a *quaternionic operator valued frame (or OV-frame)* of H having range contained in K if there exist two real constants A, $B > 0$ such that

$$AI_{\mathsf{H}} \leqslant \sum_{n\in\mathbb{N}} \mathscr{F}_n^* \mathscr{F}_n \leqslant BI_{\mathsf{H}},$$

or equivalently

$$A\|\mathsf{h}\|^2 \leqslant \sum_{n\in\mathbb{N}} \|\mathscr{F}_n(\mathsf{h})\|^2 \leqslant B\|\mathsf{h}\|^2, \quad \mathsf{h} \in \mathsf{H}. \tag{1}$$

The real constants A and B are known as lower and upper frame bounds of $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ respectively and inequality (1) is known as the *quaternionic OV-frame inequality*. The sequence $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ is called as *quaternionic Bessel OV-sequence* of H having range contained in K if the right hand side of the OV-frame inequality holds. Also, $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ is said to be a *quaternionic Parseval OV-frame* if $A = B = 1$ or $\sum_{n\in\mathbb{N}} \mathscr{F}_n^* \mathscr{F}_n = I_{\mathsf{H}}$.

Let $\{\mathfrak{e}_n\}_{n\in\mathbb{N}}$ denotes the standard orthonormal basis of $\ell^2_{\mathbb{H}}$. For each $n \in \mathbb{N}$, define the partial isometry $|\mathfrak{e}_n\rangle_{\ell^2_{\mathbb{H}}} : \mathsf{K} \to \ell^2_{\mathbb{H}} \otimes \mathsf{K}$ such that

$$|\mathfrak{e}_n\rangle_{\ell^2_{\mathbb{H}}}(k) = \mathfrak{e}_n \otimes k, \quad k \in \mathsf{K},$$

and its adjoint $|\mathfrak{e}_n\rangle^*_{\ell^2_{\mathbb{H}}} := {}_{\ell^2_{\mathbb{H}}}\langle\mathfrak{e}_n| : \ell^2_{\mathbb{H}} \otimes \mathsf{K} \to \mathsf{K}$ is given by

$$ {}_{\ell^2_{\mathbb{H}}}\langle\mathfrak{e}_n|(\mathfrak{q}\otimes k) = \mathfrak{q}_n k, \quad k \in \mathsf{K},\ \mathfrak{q} = \{\mathfrak{q}_n\}_{n\in\mathbb{N}} \in \ell^2_{\mathbb{H}}. $$

Moreover, the *analysis operator* of a quaternionic OV-frame $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ of H having range contained in K is given by $\mathscr{T}_{\mathscr{F}} : \mathsf{H} \to \ell^2_{\mathbb{H}} \otimes \mathsf{K}$ such that

$$ \mathscr{T}_{\mathscr{F}}(\mathsf{h}) = \sum_{n\in\mathbb{N}} \mathfrak{e}_n \otimes \mathscr{F}_n(\mathsf{h}), \quad \mathsf{h} \in \mathsf{H}, $$

and its adjoint $\mathscr{T}^*_{\mathscr{F}}$, the *synthesis operator* is given by $\mathscr{T}^*_{\mathscr{F}} : \ell^2_{\mathbb{H}} \otimes \mathsf{K} \to \mathsf{K}$ such that

$$ \mathscr{T}^*_{\mathscr{F}}(\mathfrak{q}\otimes k) = \sum_{n\in\mathbb{N}} \mathscr{F}^*_n(\mathfrak{q}_n k), \quad k \in \mathsf{K},\ \mathfrak{q} = \{\mathfrak{q}_n\}_{n\in\mathbb{N}} \in \ell^2_{\mathbb{H}}. $$

Consequently, the *frame operator* of $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ is given by $\mathscr{S}_{\mathscr{F}} : \mathsf{H} \to \mathsf{H}$ such that $\mathscr{S}_{\mathscr{F}} = \mathscr{T}^*_{\mathscr{F}}\mathscr{T}_{\mathscr{F}} = \sum_{n\in\mathbb{N}} \mathscr{F}^*_n\mathscr{F}_n$. Frame operator of a quaternionic OV-frame is a bounded right-linear, self-adjoint, positive invertible operator. For more details, one may refer [2].

Quaternionic OV-frames are further classified as follows:

DEFINITION 2. A quaternionic OV-frame $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ of H having range contained in K is known as

(i) a *quaternionic Riesz OV-frame* if the analysis operator $\mathscr{T}_{\mathscr{F}}$ is surjective, i.e., $\mathscr{T}_{\mathscr{F}}(\mathsf{H}) = \ell^2_{\mathbb{H}} \otimes \mathsf{K}$.

(ii) a *quaternionic orthonormal OV-frame* if it is both Parseval and Riesz OV-frame.

REMARK 1. A more general characterization of Riesz and orthonormal OV-frames is given in [2] in the form of a theorem which states that an OV-frame $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ forms a Riesz OV-frame if and only if $\mathscr{F}_n\mathscr{S}^{-1}_{\mathscr{F}}\mathscr{F}^*_m = \delta_{m,n}I_{\mathsf{K}}$, where $\mathscr{S}_{\mathscr{F}}$ is the frame operator of $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$.

2. Properties of quaternionic operator valued frames

In the following result, it is shown that a quaternionic orthonormal OV-frame composed with a bounded invertible operator turns out to be a Riesz OV-frame.

THEOREM 1. *For a given orthonormal OV-frame $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ of H having range contained in K and a bounded invertible operator $\mathscr{U}$ on H, the system $\{\mathscr{F}_n\mathscr{U}\}_{n\in\mathbb{N}}$ forms a Riesz OV-frame of H having range contained in K.*

Proof. Since $\mathscr{U}$ is given to be a bounded and invertible operator, there exist two real constants C_1, $C_2 > 0$ such that

$$ C_1\|\mathsf{h}\|^2 \leqslant \|\mathscr{U}(\mathsf{h})\|^2 \leqslant C_2\|\mathsf{h}\|^2, \quad \mathsf{h} \in \mathsf{H}. \tag{2} $$

For $n \in \mathbb{N}$, let $\mathscr{G}_n := \mathscr{F}_n \mathscr{U}$. Then, we have

$$\sum_{n\in\mathbb{N}} \mathscr{G}_n^* \mathscr{G}_n = \sum_{n\in\mathbb{N}} \mathscr{U}^* \mathscr{F}_n^* \mathscr{F}_n \mathscr{U} = \mathscr{U}^* \Big(\sum_{n\in\mathbb{N}} \mathscr{F}_n^* \mathscr{F}_n \Big) \mathscr{U} = \mathscr{U}^* \mathscr{U}. \tag{3}$$

From (2) and (3), we get

$$C_1 I_{\mathsf{H}} \leqslant \sum_{n\in\mathbb{N}} \mathscr{G}_n^* \mathscr{G}_n = \mathscr{U}^* \mathscr{U} \leqslant C_2 I_{\mathsf{H}}.$$

Let $\mathscr{S}_{\mathscr{G}} := \mathscr{U}^* \mathscr{U}$. Then for n, $m \in \mathbb{N}$,

$$\mathscr{G}_n \mathscr{S}_{\mathscr{G}}^{-1} \mathscr{G}_m^* = \mathscr{F}_n \mathscr{U} \mathscr{U}^{-1} (\mathscr{U}^*)^{-1} \mathscr{U}^* \mathscr{F}_m^* = \mathscr{F}_n \mathscr{F}_m^* = \delta_{m,n} I_{\mathsf{K}},$$

which implies that $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ forms a Riesz OV-frame of H having range contained in K with frame operator $\mathscr{S}_{\mathscr{G}}$. $\square$

The following result illustrates that the converse of the above theorem also holds good.

THEOREM 2. *For a given Riesz OV-frame* $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ *of* H *having range contained in* K*, there always exists a bounded invertible operator* $\mathscr{U}$ *on* H *and an orthonormal OV-frame* $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ *such that for each* $n \in \mathbb{N}$, $\mathscr{G}_n = \mathscr{F}_n \mathscr{U}$.

Proof. Let $\mathscr{S}_{\mathscr{G}}$ be the frame operator of $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ and take $\mathscr{F}_n := \mathscr{G}_n \mathscr{S}_{\mathscr{G}}^{-1/2}$, $n \in \mathbb{N}$. We have

$$\sum_{n\in\mathbb{N}} \mathscr{F}_n^* \mathscr{F}_n = \mathscr{S}_{\mathscr{G}}^{-1/2} \Big(\sum_{n\in\mathbb{N}} \mathscr{G}_n^* \mathscr{G}_n \Big) \mathscr{S}_{\mathscr{G}}^{-1/2} = \mathscr{S}_{\mathscr{G}}^{-1/2} \mathscr{S}_{\mathscr{G}} \mathscr{S}_{\mathscr{G}}^{-1/2} = I_{\mathsf{H}}.$$

Also, since $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ forms a Riesz OV-frame, therefore, for m, $n \in \mathbb{N}$, we have

$$\mathscr{F}_n \mathscr{F}_m^* = \mathscr{G}_n \mathscr{S}_{\mathscr{G}}^{-1/2} \mathscr{S}_{\mathscr{G}}^{-1/2} \mathscr{G}_m^* = \mathscr{G}_n \mathscr{S}_{\mathscr{G}}^{-1} \mathscr{G}_m^* = \delta_{m,n} I_{\mathsf{K}}.$$

Thus $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ forms an orthonormal OV-frame of H having range contained in K such that $\mathscr{G}_n = \mathscr{F}_n \mathscr{U}$, where $\mathscr{U} := \mathscr{S}_{\mathscr{G}}^{1/2}$. $\square$

REMARK 2. One may observe that, Theorem 1 and 2 together give the precise form of the Riesz OV-frames of H having range contained in K in terms of orthonormal OV-frames.

EXAMPLE 1. Let $\mathsf{H} := \ell^2_{\mathbb{H}}$, $\mathsf{K} := \mathbb{H}$ and $\{\mathfrak{e}_n\}_{n\in\mathbb{N}}$ be the standard orthonormal basis for $\ell^2_{\mathbb{H}}$. For each $n \in \mathbb{N}$, define $\mathscr{F}_n : \ell^2_{\mathbb{H}} \to \mathbb{H}$ such that

$$\mathscr{F}_n(\mathfrak{q}) = \langle \mathfrak{e}_n | \mathfrak{q} \rangle, \quad \mathfrak{q} = \{\mathfrak{q}_n\}_{n\in\mathbb{N}} \in \ell^2_{\mathbb{H}}.$$

It is easy to verify that

$$\sum_{n\in\mathbb{N}} \|\mathscr{F}_n(\mathfrak{q})\|^2 = \|\mathfrak{q}\|^2, \quad \mathfrak{q} = \{\mathfrak{q}_n\}_{n\in\mathbb{N}} \in \ell^2_{\mathbb{H}}.$$

Thus $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ forms a Parseval OV-frame for $\ell^2_{\mathbb{H}}$ with range in $\mathbb{H}$. This gives that the synthesis operator $\mathscr{T}_{\mathscr{F}}$ of $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ is a surjective bounded operator on $\ell^2_{\mathbb{H}}\otimes\mathbb{H}$ (by [2] Theorem 3.4).

Also, let $\mathscr{T}^*_{\mathscr{F}}(\mathfrak{q}\otimes k)=0$ for some $\mathfrak{q}\otimes k\in\ell^2_{\mathbb{H}}\otimes\mathbb{H}$. This gives that $\mathfrak{q}k=0$ and therefore $\mathfrak{q}\otimes k=0$. Thus, $\mathscr{T}^*_{\mathscr{F}}$ is injective and by ([2] Observation I), $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ forms an orthonormal OV-frame. Now, let $\lambda\neq 0$ be a given real number and $\mathscr{U}:\ell^2_{\mathbb{H}}\to\ell^2_{\mathbb{H}}$ be defined as

$$\mathscr{U}(\mathfrak{q})=\lambda\mathfrak{q},\quad \mathfrak{q}=\{\mathfrak{q}_n\}_{n\in\mathbb{N}}\in\ell^2_{\mathbb{H}}.$$

Clearly, $\mathscr{U}$ is a bounded invertible operator on $\ell^2_{\mathbb{H}}$. Therefore, by Theorem 1, $\{\mathscr{F}_n\mathscr{U}\}_{n\in\mathbb{N}}$ forms a Riesz OV-frame for $\ell^2_{\mathbb{H}}$ having range contained in $\mathbb{H}$. Note that $\mathscr{F}_n\mathscr{U}(\mathfrak{q})=\lambda\langle\mathfrak{e}_n|\mathfrak{q}\rangle=\lambda\mathfrak{q}_n$, where $\mathfrak{q}=\{\mathfrak{q}_n\}_{n\in\mathbb{N}}\in\ell^2_{\mathbb{H}}$.

We now give a relation between two Riesz OV-frames.

COROLLARY 1. *For two given Riesz OV-frames* $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ *and* $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ *of* H *having range contained in* K*, there always exists a bounded invertible operator* T *on* H *such that* $\mathscr{F}_n=\mathscr{G}_nT$ *or* $\mathscr{G}_n=\mathscr{F}_nT$*.*

Proof. From Theorem 1 and 2, there exist two bounded invertible operators $\mathscr{U}_1$ and $\mathscr{U}_2$ on H and an orthonormal OV-frame $\{\mathscr{E}_n\}_{n\in\mathbb{N}}$ of H having range contained in K such that for each $n\in\mathbb{N}$, $\mathscr{F}_n=\mathscr{E}_n\mathscr{U}_1$ and $\mathscr{G}_n=\mathscr{E}_n\mathscr{U}_2$. Then we have

$$\mathscr{F}_n=\mathscr{E}_n\mathscr{U}_2\mathscr{U}_2^{-1}\mathscr{U}_1=\mathscr{G}_nT,\quad n\in\mathbb{N}$$

where $T:=\mathscr{U}_2^{-1}\mathscr{U}_1$. □

Following that, the next result demonstrates that a Riesz OV-frame composed with a bounded invertible operator again turns out to be a Riesz OV-frame.

THEOREM 3. *For a given Riesz OV-frame* $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ *of* H *having range contained in* K *and a bounded invertible operator* $\mathscr{U}$ *on* H*, the system* $\{\mathscr{F}_n\mathscr{U}\}_{n\in\mathbb{N}}$ *also forms a Riesz OV-frame of* H *having range contained in* K*.*

Proof. Let $\mathscr{S}_{\mathscr{F}}$ be the frame operator of $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$. For $n\in\mathbb{N}$, let $\mathscr{G}_n:=\mathscr{F}_n\mathscr{U}$. Then

$$\sum_{n\in\mathbb{N}}\mathscr{G}_n^*\mathscr{G}_n=\sum_{n\in\mathbb{N}}\mathscr{U}^*\mathscr{F}_n^*\mathscr{F}_n\mathscr{U}=\mathscr{U}^*\mathscr{S}_{\mathscr{F}}\mathscr{U}.$$

Therefore, $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ forms an OV-frame of H having range contained in K, since $\mathscr{U}^*\mathscr{S}_{\mathscr{F}}\mathscr{U}$ is a bounded invertible operator with frame operator $\mathscr{S}_{\mathscr{G}}:=\mathscr{U}^*\mathscr{S}_{\mathscr{F}}\mathscr{U}$. Then, for $n,m\in\mathbb{N}$, consider

$$\begin{aligned}\mathscr{G}_n\mathscr{S}_{\mathscr{G}}^{-1}\mathscr{G}_m^*&=\mathscr{F}_n\mathscr{U}\mathscr{S}_{\mathscr{G}}^{-1}\mathscr{U}^*\mathscr{F}_m^*\\&=\mathscr{F}_n\mathscr{U}\mathscr{U}^{-1}\mathscr{S}_{\mathscr{F}}^{-1}(\mathscr{U}^*)^{-1}\mathscr{U}^*\mathscr{F}_m^*\\&=\mathscr{F}_n\mathscr{S}_{\mathscr{F}}^{-1}\mathscr{F}_m^*\\&=\delta_{m,n}I_{\mathsf{K}}.\end{aligned}$$

Thus $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ forms a Riesz OV-frame of H having range contained in K. □

REMARK 3. Theorem 3 and Corollary 1 implies that given a Riesz OV-frame, one can obtain the whole class of Riesz OV-frames using bounded invertible operators.

In the forthcoming result, it is shown that an orthonormal OV-frame composed with a unitary operator again turns out to be an orthonormal OV-frame.

THEOREM 4. *For a given orthonormal OV-frame $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ of H having range contained in K and a unitary operator $\mathscr{U}$ on H, the system $\{\mathscr{F}_n\mathscr{U}\}_{n\in\mathbb{N}}$ also forms an orthonormal OV-frame of H having range contained in K.*

Proof. For $n\in\mathbb{N}$, let $\mathscr{G}_n := \mathscr{F}_n\mathscr{U}$. Then, we have

$$\sum_{n\in\mathbb{N}} \mathscr{G}_n^*\mathscr{G}_n = \mathscr{U}^*\Big(\sum_{n\in\mathbb{N}} \mathscr{F}_n^*\mathscr{F}_n\Big)\mathscr{U} = \mathscr{U}^*\mathscr{U} = I_{\mathsf{H}}.$$

Also for $m,\ n\in\mathbb{N}$,

$$\mathscr{G}_n\mathscr{G}_m^* = \mathscr{F}_n\mathscr{U}\mathscr{U}^*\mathscr{F}_m^* = \mathscr{F}_n\mathscr{F}_m^* = \delta_{m,n}I_{\mathsf{H}}.$$

Thus, $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ forms an orthonormal OV-frame of H having range contained in K. □

EXAMPLE 2. Let $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ be as defined in Example 1 and $\mathscr{U} : \ell^2_{\mathbb{H}} \to \ell^2_{\mathbb{H}}$ be such that

$$\mathscr{U}(\mathfrak{q}) = \bar{\mathfrak{q}}, \quad \mathfrak{q} = \{\mathfrak{q}_n\}_{n\in\mathbb{N}} \in \ell^2_{\mathbb{H}}$$

where, $\bar{\mathfrak{q}} = \{\bar{\mathfrak{q}_n}\}_n \in \mathbb{N}$. It is easy to verify that $\mathscr{U}$ is a unitary operator on $\ell^2_{\mathbb{H}}$. Note that

$$\mathscr{F}_n\mathscr{U}(\mathfrak{q}) = \langle \mathfrak{e}_n|\bar{\mathfrak{q}}\rangle = \bar{\mathfrak{q}_n}.$$

Thus by Theorem 4, $\{\mathscr{F}_n\mathscr{U}\}_{n\in\mathbb{N}}$ forms an orthonormal OV-frame for $\ell^2_{\mathbb{H}}$ having range contained in $\mathbb{H}$.

We now give a relation between two orthonormal OV-frames.

THEOREM 5. *For two given orthonormal OV-frames $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ and $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ of H having range contained in K, there always exists a unitary operator $\mathscr{U}$ on H such that $\mathscr{F}_n = \mathscr{G}_n\mathscr{U}$ and $\mathscr{T}_{\mathscr{F}} = \mathscr{T}_{\mathscr{G}}\mathscr{U}$ where $\mathscr{T}_{\mathscr{F}}$ and $\mathscr{T}_{\mathscr{G}}$ are analysis operators of $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ and $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ respectively.*

Proof. Since $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ and $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ are orthonormal OV-frames, we have

$$\mathscr{T}_{\mathscr{F}}^*\mathscr{T}_{\mathscr{F}} = I_{\mathsf{H}}, \qquad \mathscr{T}_{\mathscr{F}}\mathscr{T}_{\mathscr{F}}^* = I_{\ell^2_{\mathbb{H}}\otimes\mathsf{K}},$$
$$\mathscr{T}_{\mathscr{G}}^*\mathscr{T}_{\mathscr{G}} = I_{\mathsf{H}}, \qquad \mathscr{T}_{\mathscr{G}}\mathscr{T}_{\mathscr{G}}^* = I_{\ell^2_{\mathbb{H}}\otimes\mathsf{K}}.$$

Also for $n \in \mathbb{N}$,

$$\mathscr{F}_n = {}_{\ell^2_{\mathbb{H}}}\langle \mathfrak{e}_n | \mathscr{T}_{\mathscr{F}} = {}_{\ell^2_{\mathbb{H}}}\langle \mathfrak{e}_n | \mathscr{T}_{\mathscr{G}} \mathscr{T}_{\mathscr{G}}^* \mathscr{T}_{\mathscr{F}} = \mathscr{G}_n \mathscr{T}_{\mathscr{G}}^* \mathscr{T}_{\mathscr{F}}.$$

This implies that $\mathscr{F}_n = \mathscr{G}_n \mathscr{U}$, where $\mathscr{U} := \mathscr{T}_{\mathscr{G}}^* \mathscr{T}_{\mathscr{F}}$ is a unitary operator. Moreover, $\mathscr{T}_{\mathscr{F}} = \mathscr{T}_{\mathscr{G}} \mathscr{T}_{\mathscr{G}}^* \mathscr{T}_{\mathscr{F}} = \mathscr{T}_{\mathscr{G}} \mathscr{U}$. □

REMARK 4. Note that Theorem 4 and 5 together gives the precise form of the orthonormal OV-frames of H having range contained in K.

In the next result, it is proved that an OV-frame composed with a bounded invertible operator again turns out to be an OV-frame.

THEOREM 6. *For a given OV-frame $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ of H having range contained in K and a bounded invertible operator $\mathscr{U}$ on H, the system $\{\mathscr{F}_n \mathscr{U}\}_{n\in\mathbb{N}}$ also forms an OV-frame of H having range contained in K.*

Proof. Let $\mathscr{S}_{\mathscr{F}}$ be the frame operator of $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$. For $n \in \mathbb{N}$, let $\mathscr{G}_n := \mathscr{F}_n \mathscr{U}$. Then, we have

$$\sum_{n\in\mathbb{N}} \mathscr{G}_n^* \mathscr{G}_n = \sum_{n\in\mathbb{N}} \mathscr{U}^* \mathscr{F}_n^* \mathscr{F}_n \mathscr{U} = \mathscr{U}^* \Big(\sum_{n\in\mathbb{N}} \mathscr{F}_n^* \mathscr{F}_n \Big) \mathscr{U} = \mathscr{U}^* \mathscr{S}_{\mathscr{F}} \mathscr{U}.$$

Since $\mathscr{U}^* \mathscr{S}_{\mathscr{F}} \mathscr{U}$ is a bounded invertible operator, thus $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ forms an OV-frame of H having range contained in K with frame operator $\mathscr{S}_{\mathscr{G}} := \mathscr{U}^* \mathscr{S}_{\mathscr{F}} \mathscr{U}$. □

EXAMPLE 3. Let $\mathsf{H} = \mathsf{K} = \ell^2_{\mathbb{H}}$. For each $n \in \mathbb{N}$, define $\mathscr{F}_n : \ell^2_{\mathbb{H}} \to \ell^2_{\mathbb{H}}$ as

$$\mathscr{F}_n(\mathfrak{q}) = \frac{\mathfrak{q}}{n}, \quad \mathfrak{q} = \{\mathfrak{q}_n\}_{n\in\mathbb{N}} \in \ell^2_{\mathbb{H}}.$$

One can easily verify that

$$\begin{aligned}
\sum_{n\in\mathbb{N}} \|\mathscr{F}_n(\mathfrak{q})\|^2 &= \sum_{n\in\mathbb{N}} \sum_{i\in\mathbb{N}} \left| \frac{\mathfrak{q}_i}{n} \right|^2 \\
&= \left(\sum_{n\in\mathbb{N}} \frac{1}{n^2} \right) \left(\sum_{i\in\mathbb{N}} |\mathfrak{q}_i|^2 \right) \\
&= \frac{\pi^2}{6} \|\mathfrak{q}\|^2.
\end{aligned}$$

Therefore, $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ forms an OV-frame for $\ell^2_{\mathbb{H}}$ having range contained in $\ell^2_{\mathbb{H}}$. Now for any bounded invertible operator $\mathscr{U}$ on $\ell^2_{\mathbb{H}}$, the sequence $\{\mathscr{F}_n \mathscr{U}\}_{n\in\mathbb{N}}$ also forms an OV-frame.

REMARK 5. Note that, if instead of OV-frames, we consider $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ as a Parseval OV-frame in Theorem 6, then also the sequence $\{\mathscr{F}_n\mathscr{U}\}_{n\in\mathbb{N}}$ forms an OV-frame.

Indeed, the frame operator $\mathscr{S}_{\mathscr{F}}$ of a Parseval OV-frame is an identity operator and as above we get that $\sum_{n\in\mathbb{N}} \mathscr{G}_n^*\mathscr{G}_n = \mathscr{U}^*\mathscr{U}$ is a bounded invertible operator which further gives that $\{\mathscr{F}_n\mathscr{U}\}_{n\in\mathbb{N}}$ forms an OV-frame for H with range in K with frame operator $\mathscr{U}^*\mathscr{U}$.

REMARK 6. In Theorem 6, let A and B be the lower and upper frame bounds for $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ respectively. In particular for $\mathscr{U} = \mathscr{S}_{\mathscr{F}}^{-1}$, $\{\mathscr{G}_n\}_{n\in\mathbb{N}} := \{\mathscr{F}_n\mathscr{S}_{\mathscr{F}}^{-1}\}_{n\in\mathbb{N}}$ forms an OV-frame of H having range contained in K with lower and upper frame bounds B^{-1} and A^{-1} respectively and frame operator $\mathscr{S}_{\mathscr{G}} := \mathscr{S}_{\mathscr{F}}^{-1}$. Indeed since $AI_{\mathsf{H}} \leqslant \mathscr{S}_{\mathscr{F}} \leqslant BI_{\mathsf{H}}$, we have

$$B^{-1}\|\mathsf{h}\|^2 \leqslant \langle \mathscr{S}_{\mathscr{F}}^{-1}(\mathsf{h})|\mathsf{h}\rangle \leqslant A^{-1}\|\mathsf{h}\|^2, \quad \mathsf{h}\in\mathsf{H}.$$

Also

$$\langle \mathscr{S}_{\mathscr{F}}^{-1}(\mathsf{h})|\mathsf{h}\rangle = \Big\langle \sum_{n\in\mathbb{N}} \mathscr{G}_n^*\mathscr{G}_n(\mathsf{h})\Big|\mathsf{h}\Big\rangle = \sum_{n\in\mathbb{N}} \|\mathscr{G}_n(\mathsf{h})\|^2, \quad \mathsf{h}\in\mathsf{H}.$$

Thus

$$B^{-1}\|\mathsf{h}\|^2 \leqslant \sum_{n\in\mathbb{N}} \|\mathscr{G}_n(\mathsf{h})\|^2 \leqslant A^{-1}\|\mathsf{h}\|^2, \quad \mathsf{h}\in\mathsf{H}.$$

3. Dual operator valued frames

In this section, we will explore the duals of a quaternionic OV-frame and examine their various properties. The definition of the dual of an quaternionic OV-frame is stated as follows:

DEFINITION 3. Let $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ be a quaternionic OV-frame of H having range contained in K with the frame operator $\mathscr{S}_{\mathscr{F}}$. Then,

(i) the sequence $\{\mathscr{F}_n\mathscr{S}_{\mathscr{F}}^{-1}\}_{n\in\mathbb{N}}$ is known as the canonical dual OV-frame of $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$.

(ii) a sequence $\{\mathscr{G}_n\}_{n\in\mathbb{N}} \subseteq \mathscr{B}(\mathsf{H},\mathsf{K})$ is known as an alternate of the OV-frame $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ if $\sum_{n\in\mathbb{N}} \mathscr{F}_n^*\mathscr{G}_n = I_{\mathsf{H}}$.

The dual of an OV-frame $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ that forms an OV-frame in itself is known as a dual OV-frame of $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$.

REMARK 7. Note that, the canonical dual of the OV-frame $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ is also an alternate dual OV-frame of $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ and it is already shown in Remark 6 that $\{\mathscr{F}_n\mathscr{S}_{\mathscr{F}}^{-1}\}_{n\in\mathbb{N}}$ forms an OV-frame of H having range contained in K.

EXAMPLE 4. The following examples give the existence of the alternate duals of some OV-frame as follows:

(i) Let $\{\mathfrak{e}_n\}_{n\in\mathbb{N}}$ be the orthonormal basis for $\mathsf{H} := \ell^2_{\mathbb{H}}$ and $\mathsf{K} := \mathbb{H}$. For $n \in \mathbb{N}$, define $\mathscr{F}_n : \ell^2_{\mathbb{H}} \to \mathbb{H}$ such that $\mathscr{F}_n(\mathfrak{q}) = \langle \mathfrak{e}_n | \mathfrak{q}\rangle$. This gives $\mathscr{F}_n^*(p) = \mathfrak{e}_n p$, $p \in \mathbb{H}$. Note that $\sum_{n\in\mathbb{N}} \mathscr{F}_n^* \mathscr{F}_n = I_{\ell^2_{\mathbb{H}}}$. Thus $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ forms an alternate dual of itself.

(ii) Define $\mathscr{F}_n : \ell^2_{\mathbb{H}} \to \ell^2_{\mathbb{H}}$ as $\mathscr{F}_n(\mathfrak{q}) = \frac{\mathfrak{q}}{n}$. Let $\mathscr{G}_n : \ell^2_{\mathbb{H}} \to \ell^2_{\mathbb{H}}$ be defined as $\mathscr{G}_n(\mathfrak{q}) = \frac{6\mathfrak{q}}{\pi^2 n}$, $\mathfrak{q} \in \ell^2_{\mathbb{H}}$. Then $\sum_{n\in\mathbb{N}} \mathscr{F}_n^* \mathscr{G}_n(\mathfrak{q}) = \sum_{n\in\mathbb{N}} \frac{\mathscr{G}_n(\mathfrak{q})}{n} = \mathfrak{q}$. Thus $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ forms an alternate dual of $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$.

LEMMA 1. *Let $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ be a dual OV-frame of the OV-frame $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ and $\mathscr{T}_{\mathscr{F}}$, $\mathscr{T}_{\mathscr{G}}$ be the analysis operators of $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ and $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ respectively. Then,*

(i) $\mathscr{T}_{\mathscr{G}}^* \mathscr{T}_{\mathscr{F}} = I_{\mathsf{H}}$ *if and only if* $\mathscr{T}_{\mathscr{F}}^* \mathscr{T}_{\mathscr{G}} = I_{\mathsf{H}}$.

(ii) $\sum_{n\in\mathbb{N}} \mathscr{G}_n^* \mathscr{F}_n = I_{\mathsf{H}}$ *if and only if* $\sum_{n\in\mathbb{N}} \mathscr{F}_n^* \mathscr{G}_n = I_{\mathsf{H}}$.

Proof. (i) $\mathscr{T}_{\mathscr{G}}^* \mathscr{T}_{\mathscr{F}} = I_{\mathsf{H}}$ if and only if $(\mathscr{T}_{\mathscr{G}}^* \mathscr{T}_{\mathscr{F}})^* = \mathscr{T}_{\mathscr{F}}^* \mathscr{T}_{\mathscr{G}} = I_{\mathsf{H}}$. (ii) Follows on the same lines. □

Now, we give a relation between the analysis operators of an OV-frame and its dual.

LEMMA 2. *Let $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ be a dual OV-frame of the OV-frame $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ and $\mathscr{T}_{\mathscr{F}}$, $\mathscr{T}_{\mathscr{G}}$ be the analysis operators of $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ and $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ respectively. Then, $\mathscr{T}_{\mathscr{G}}^* \mathscr{T}_{\mathscr{F}} = I_{\mathsf{H}}$.*

Proof. Since $\sum_{n\in\mathbb{N}} \mathscr{G}_n^* \mathscr{F}_n = I_{\mathsf{H}}$, we have

$$\mathscr{T}_{\mathscr{G}}^* \mathscr{T}_{\mathscr{F}}(\mathsf{h}) = \mathscr{T}_{\mathscr{G}}^* \Big(\sum_{n\in\mathbb{N}} \mathfrak{e}_n \otimes \mathscr{F}_n(\mathsf{h}) \Big) = \sum_{n\in\mathbb{N}} \mathscr{G}_n^* \mathscr{F}_n(\mathsf{h}) = \mathsf{h}, \quad \mathsf{h} \in \mathsf{H}.$$

Thus $\mathscr{T}_{\mathscr{G}}^* \mathscr{T}_{\mathscr{F}} = I_{\mathsf{H}}$. □

The following result gives the precise form of the family of duals of an OV-frame.

LEMMA 3. *Let $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ be an OV-frame of H having range contained in K with analysis operator $\mathscr{T}_{\mathscr{F}}$. Then, the dual OV-frames of $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ are precisely of the form $\{{}_{\ell^2_{\mathbb{H}}}\langle \mathfrak{e}_n | \mathscr{V}\}_{n\in\mathbb{N}}$, where $\mathscr{V} : \mathsf{H} \to \ell^2_{\mathbb{H}} \otimes \mathsf{K}$ is a bounded operator with $\mathscr{V}^* \mathscr{T}_{\mathscr{F}} = I_{\mathsf{H}}$.*

Proof. Let $\mathscr{V} : \mathsf{H} \to \ell^2_{\mathbb{H}} \otimes \mathsf{K}$ be a given bounded operator such that $\mathscr{V}^* \mathscr{T}_{\mathscr{F}} = I_{\mathsf{H}}$. For $n \in \mathbb{N}$, let $\mathscr{G}_n := {}_{\ell^2_{\mathbb{H}}}\langle \mathfrak{e}_n | \mathscr{V}$. Now for $\mathsf{h} \in \mathsf{H}$, we have

$$\sum_{n\in\mathbb{N}} \mathscr{G}_n^* \mathscr{F}_n(\mathsf{h}) = \sum_{n\in\mathbb{N}} \mathscr{V}^* |\mathfrak{e}_n\rangle_{\ell^2_{\mathbb{H}}} \mathscr{F}_n(\mathsf{h}) = \mathscr{V}^* \Big(\sum_{n\in\mathbb{N}} \mathfrak{e}_n \otimes \mathscr{F}_n(\mathsf{h}) \Big) = \mathscr{V}^* \mathscr{T}_{\mathscr{F}}(\mathsf{h}) = \mathsf{h}.$$

Thus, $\{{}_{\ell^2_{\mathbb{H}}}\langle \mathfrak{e}_n|\mathscr{V}\}_{n\in\mathbb{N}}$ forms a dual OV-frame of $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ for any bounded operator $\mathscr{V}$.

Conversely, let $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ be a dual OV-frame of $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ and $\mathscr{T}_{\mathscr{G}}$ be its analysis operator. Then $\mathscr{G}_n = {}_{\ell^2_{\mathbb{H}}}\langle \mathfrak{e}_n|\mathscr{T}_{\mathscr{G}}$ and Lemma 2 gives that $\mathscr{T}^*_{\mathscr{G}}\mathscr{T}_{\mathscr{F}} = I_{\mathsf{H}}$. □

Now, we give the precise form of the left-inverses of the analysis operator of an OV-frame, which will be used in the next result.

LEMMA 4. *Let $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ be an OV-frame of H having range contained in K with analysis and frame operators $\mathscr{T}_{\mathscr{F}}$ and $\mathscr{S}_{\mathscr{F}}$ respectively. Then, every left-inverse of $\mathscr{T}_{\mathscr{F}}$ is given to be of the form $\mathscr{S}^{-1}_{\mathscr{F}}\mathscr{T}^*_{\mathscr{F}} + \mathscr{V}(I_{\ell^2_{\mathbb{H}}\otimes\mathsf{K}} - \mathscr{T}_{\mathscr{F}}\mathscr{S}^{-1}_{\mathscr{F}}\mathscr{T}^*_{\mathscr{F}})$, where $\mathscr{V} : \ell^2_{\mathbb{H}}\otimes \mathsf{K}\to\mathsf{H}$ is a bounded operator.*

Proof. For a given bounded operator $\mathscr{V} : \ell^2_{\mathbb{H}}\otimes\mathsf{K}\to\mathsf{H}$ consider

$$\begin{aligned}\Big(\mathscr{S}^{-1}_{\mathscr{F}}\mathscr{T}^*_{\mathscr{F}} + \mathscr{V}(I_{\ell^2_{\mathbb{H}}\otimes\mathsf{K}} - \mathscr{T}_{\mathscr{F}}\mathscr{S}^{-1}_{\mathscr{F}}\mathscr{T}^*_{\mathscr{F}})\Big)\mathscr{T}_{\mathscr{F}}&\\ &= \mathscr{S}^{-1}_{\mathscr{F}}\mathscr{T}^*_{\mathscr{F}}\mathscr{T}_{\mathscr{F}} + \mathscr{V}\mathscr{T}_{\mathscr{F}} - \mathscr{V}\mathscr{T}_{\mathscr{F}}\mathscr{S}^{-1}_{\mathscr{F}}\mathscr{T}^*_{\mathscr{F}}\mathscr{T}_{\mathscr{F}}\\ &= I_{\mathsf{H}}.\end{aligned}$$

Conversely, let $\mathscr{U}$ be a left-inverse of $\mathscr{T}_{\mathscr{F}}$ i.e., $\mathscr{U}\mathscr{T}_{\mathscr{F}} = I_{\mathsf{H}}$. Then,

$$\mathscr{U} = \mathscr{S}^{-1}_{\mathscr{F}}\mathscr{T}^*_{\mathscr{F}} + \mathscr{U}(I_{\ell^2_{\mathbb{H}}\otimes\mathsf{K}} - \mathscr{T}_{\mathscr{F}}\mathscr{S}^{-1}_{\mathscr{F}}\mathscr{T}^*_{\mathscr{F}})$$

and we get the desired result. □

Next we give another form of the duals of an OV-frame in terms of a family of Bessel OV-sequences in the following result.

THEOREM 7. *Let $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ be an OV-frame of H having range contained in K with analysis and frame operators $\mathscr{T}_{\mathscr{F}}$ and $\mathscr{S}_{\mathscr{F}}$ respectively. Then, the dual OV-frames of $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ are precisely of the form $\{\mathscr{F}_n\mathscr{S}^{-1}_{\mathscr{F}} + \mathscr{E}_n - \sum\limits_{i\in\mathbb{N}}\mathscr{F}_n\mathscr{S}^{-1}_{\mathscr{F}}\mathscr{F}^*_i\mathscr{E}_i\}_{n\in\mathbb{N}}$ where $\{\mathscr{E}_n\}_{n\in\mathbb{N}}$ is a Bessel OV-sequence of H having range contained in K.*

Proof. Let $\{\mathscr{E}_n\}_{n\in\mathbb{N}}$ be a Bessel OV-sequence of H having range contained in K. For $n\in\mathbb{N}$, let $\mathscr{G}_n := \mathscr{F}_n\mathscr{S}^{-1}_{\mathscr{F}} + \mathscr{E}_n - \sum_{i\in\mathbb{N}}\mathscr{F}_n\mathscr{S}^{-1}_{\mathscr{F}}\mathscr{F}^*_i\mathscr{E}_i$. Now consider

$$\sum_{n\in\mathbb{N}}\mathscr{F}^*_n\mathscr{G}_n = \sum_{n\in\mathbb{N}}\Big(\mathscr{F}^*_n\mathscr{F}_n\mathscr{S}^{-1}_{\mathscr{F}} + \mathscr{F}^*_n\mathscr{E}_n - \sum_{i\in\mathbb{N}}\mathscr{F}^*_n\mathscr{F}_n\mathscr{S}^{-1}_{\mathscr{F}}\mathscr{F}^*_i\mathscr{E}_i\Big) = I_{\mathsf{H}}.$$

Conversely, let $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ be a dual OV-frame of $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$. Then Lemma 3 implies that, there exists some bounded operator $\mathscr{V} : \mathsf{H}\to\ell^2_{\mathbb{H}}\otimes\mathsf{K}$ such that for each $n\in\mathbb{N}$, $\mathscr{G}_n = {}_{\ell^2_{\mathbb{H}}}\langle\mathfrak{e}_n|\mathscr{V}$ and $\mathscr{V}^*\mathscr{T}_{\mathscr{F}} = I_{\mathsf{H}}$. This implies that $\mathscr{V}^*$ is a left-inverse of $\mathscr{T}_{\mathscr{F}}$ and hence by Lemma 4, there exists some bounded operator $\mathscr{W} : \ell^2_{\mathbb{H}}\otimes\mathsf{K}\to\mathsf{H}$ such that

$$\mathscr{V}^* = \mathscr{S}^{-1}_{\mathscr{F}}\mathscr{T}^*_{\mathscr{F}} + \mathscr{W} - \mathscr{W}\mathscr{T}_{\mathscr{F}}\mathscr{S}^{-1}_{\mathscr{F}}\mathscr{T}^*_{\mathscr{F}}.$$

For each $n \in \mathbb{N}$, let $\mathscr{E}_n = {}_{\ell^2_{\mathbb{H}}}\langle \mathfrak{e}_n | \mathscr{W}^*$. Then

$$\sum_{n \in \mathbb{N}} \mathscr{E}_n^* \mathscr{E}_n = \mathscr{W} \Big(\sum_{n \in \mathbb{N}} |\mathfrak{e}_n\rangle_{\ell^2_{\mathbb{H}}\, \ell^2_{\mathbb{H}}} \langle \mathfrak{e}_n| \Big) \mathscr{W}^* = \mathscr{W}\mathscr{W}^*.$$

Thus, $\{\mathscr{E}_n\}_{n\in\mathbb{N}}$ forms a Bessel OV-sequence of H having range contained in K. Now for $n \in \mathbb{N}$, consider

$$\begin{aligned} \mathscr{G}_n &= {}_{\ell^2_{\mathbb{H}}}\langle \mathfrak{e}_n | \mathscr{V} \\ &= {}_{\ell^2_{\mathbb{H}}}\langle \mathfrak{e}_n | (\mathscr{T}_{\mathscr{F}} \mathscr{S}_{\mathscr{F}}^{-1} + \mathscr{W}^* - \mathscr{T}_{\mathscr{F}} \mathscr{S}_{\mathscr{F}}^{-1} \mathscr{T}_{\mathscr{F}}^* \mathscr{W}^*) \\ &= \mathscr{F}_n \mathscr{S}_{\mathscr{F}}^{-1} + \mathscr{E}_n - \mathscr{F}_n \mathscr{S}_{\mathscr{F}}^{-1} \mathscr{T}_{\mathscr{F}}^* \mathscr{W}^*. \end{aligned} \tag{4}$$

Also

$$\mathscr{T}_{\mathscr{F}}^* \mathscr{W}^*(\mathsf{h}) = \mathscr{T}_{\mathscr{F}}^* \Big(\sum_{n \in \mathbb{N}} \mathfrak{e}_n \otimes \mathscr{E}_n(\mathsf{h}) \Big) = \sum_{n \in \mathbb{N}} \mathscr{F}_n^* \mathscr{E}_n(\mathsf{h}), \quad \mathsf{h} \in \mathsf{H}.$$

Thus, from (4), we get

$$\mathscr{G}_n = \mathscr{F}_n \mathscr{S}_{\mathscr{F}}^{-1} + \mathscr{E}_n - \sum_{i \in \mathbb{N}} \mathscr{F}_n \mathscr{S}_{\mathscr{F}}^{-1} \mathscr{F}_i^* \mathscr{E}_i, \quad n \in \mathbb{N}. \quad \square$$

We now prove that for every Riesz OV-frame, there exists a unique Riesz OV-frame as its alternate dual.

THEOREM 8. *Given a Riesz OV-frame $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ of H having range contained in K, there exists a unique Riesz OV-frame $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} \mathscr{G}_n^* \mathscr{F}_n = I_{\mathsf{H}}$.*

Proof. Since $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ is a Riesz OV-frame, there exists a bounded invertible operator $\mathscr{U}$ on H and an orthonormal OV-frame $\{\mathscr{E}_n\}_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$, $\mathscr{F}_n = \mathscr{E}_n \mathscr{U}$. Now consider,

$$I_{\mathsf{H}} = \mathscr{U}^{-1}\mathscr{U} = \mathscr{U}^{-1} \Big(\sum_{n \in \mathbb{N}} \mathscr{E}_n^* \mathscr{E}_n \Big) \mathscr{U} = \sum_{n \in \mathbb{N}} \mathscr{U}^{-1} \mathscr{E}_n^* \mathscr{F}_n = \sum_{n \in \mathbb{N}} (\mathscr{E}_n (\mathscr{U}^{-1})^*)^* \mathscr{F}_n.$$

For each $n \in \mathbb{N}$, let $\mathscr{G}_n := \mathscr{E}_n (\mathscr{U}^{-1})^*$ and consider

$$\sum_{n \in \mathbb{N}} \mathscr{G}_n^* \mathscr{G}_n = \mathscr{U}^{-1} \Big(\sum_{n \in \mathbb{N}} \mathscr{E}_n^* \mathscr{E}_n \Big) (\mathscr{U}^{-1})^* = \mathscr{U}^{-1} (\mathscr{U}^{-1})^*.$$

Then $\mathscr{S}_{\mathscr{G}} := \mathscr{U}^{-1}(\mathscr{U}^{-1})^*$ is a bounded invertible operator and for $m,\, n \in \mathbb{N}$, we have

$$\mathscr{G}_n \mathscr{S}_{\mathscr{G}}^{-1} \mathscr{G}_m^* = \mathscr{E}_n (\mathscr{U}^{-1})^* (\mathscr{U}^* \mathscr{U}) \mathscr{U}^{-1} \mathscr{E}_m^* = \mathscr{E}_n \mathscr{E}_m^* = \delta_{m,n} I_{\mathsf{K}}.$$

Hence $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ forms a Riesz OV-frame of H having range contained in K such that $\sum_{n\in\mathbb{N}} \mathscr{G}_n^* \mathscr{F}_n = I_{\mathsf{H}}$. $\square$

4. Construction of operator valued frames

This section deals with the construction of OV-frames with the help of the partial isometries $\{|\mathfrak{e}_n\rangle_{\ell^2_{\mathbb{H}}}\}_{n\in\mathbb{N}}$ and some given operators.

THEOREM 9. *Let* $\mathscr{U}:\mathsf{H}\to\ell^2_{\mathbb{H}}\otimes\mathsf{K}$ *be a given operator. Then, the system* $\{{}_{\ell^2_{\mathbb{H}}}\langle\mathfrak{e}_n|\mathscr{U}\}_{n\in\mathbb{N}}$ *forms*

(i) an OV-Bessel sequence of H *having range contained in* K*, if* $\mathscr{U}$ *is a bounded operator.*

(ii) an OV-frame of H *having range contained in* K*, if* $\mathscr{U}$ *is a bounded injective operator with closed range.*

(iii) a Riesz OV-frame of H *having range contained in* K*, if* $\mathscr{U}$ *is a bounded invertible operator.*

(iv) an orthonormal OV-frame of H *having range contained in* K*, if* $\mathscr{U}$ *is a unitary operator.*

Proof. For $n\in\mathbb{N}$, let $\mathscr{F}_n := {}_{\ell^2_{\mathbb{H}}}\langle\mathfrak{e}_n|\mathscr{U}$. Then,

$$\sum_{n\in\mathbb{N}}\mathscr{F}_n^*\mathscr{F}_n=\mathscr{U}^*\Big(\sum_{n\in\mathbb{N}}|\mathfrak{e}_n\rangle_{\ell^2_{\mathbb{H}}\,\ell^2_{\mathbb{H}}}\langle\mathfrak{e}_n|\Big)\mathscr{U}=\mathscr{U}^*\mathscr{U}. \tag{5}$$

(i) Let $\mathscr{U}$ be a bounded operator with bound C (say). Using (5), we have

$$\sum_{n\in\mathbb{N}}\|\mathscr{F}_n(\mathsf{h})\|^2=\|\mathscr{U}(\mathsf{h})\|^2\leqslant C\|\mathsf{h}\|^2,\quad \mathsf{h}\in\mathsf{H}.$$

Thus, $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ forms an OV-Bessel sequence of H having range contained in K.

(ii) Since $\mathscr{U}$ is given to be a bounded injective operator with closed range, therefore there exists two positive constants C_1 and C_2 satisfying,

$$C_1\|\mathsf{h}\|^2\leqslant\sum_{n\in\mathbb{N}}\|\mathscr{F}_n(\mathsf{h})\|^2=\|\mathscr{U}(\mathsf{h})\|^2\leqslant C_2\|\mathsf{h}\|^2,\quad \mathsf{h}\in\mathsf{H}.$$

Thus, $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ forms an OV-frame of H having range contained in K.

(iii) From (ii), $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ forms an OV-frames for H having range contained in K. Moreover for n, $m\in\mathbb{N}$,

$$\mathscr{F}_n\mathscr{U}^*\mathscr{U}\mathscr{F}_m^*={}_{\ell^2_{\mathbb{H}}}\langle\mathfrak{e}_n|\mathscr{U}\mathscr{U}^{-1}(\mathscr{U}^*)^{-1}\mathscr{U}^*|\mathfrak{e}_m\rangle_{\ell^2_{\mathbb{H}}}={}_{\ell^2_{\mathbb{H}}}\langle\mathfrak{e}_n||\mathfrak{e}_m\rangle_{\ell^2_{\mathbb{H}}}=\delta_{m,n}I_{\mathsf{K}}.$$

Thus, $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ forms a Riesz OV-frame of H having range contained in K.

(iv) Since, $\mathscr{U}$ is unitary, $\mathscr{U}^*\mathscr{U}=I_{\mathsf{H}}$. For, m, $n\in\mathbb{N}$, we have

$$\sum_{n\in\mathbb{N}}\mathscr{F}_n^*\mathscr{F}_n=I_{\mathsf{H}}\quad\text{and}\quad \mathscr{F}_n\mathscr{F}_m^*={}_{\ell^2_{\mathbb{H}}}\langle\mathfrak{e}_n|\mathscr{U}\mathscr{U}^*|\mathfrak{e}_m\rangle_{\ell^2_{\mathbb{H}}}=\delta_{m,n}I_{\mathsf{K}}.$$

Thus, $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ forms an orthonormal OV-frame for H having range contained in K. □

EXAMPLE 5. Let $\mathsf{H} = \mathsf{K} := \ell^2_{\mathbb{H}}$ and $\mathscr{U} : \ell^2_{\mathbb{H}} \to \ell^2_{\mathbb{H}} \otimes \ell^2_{\mathbb{H}}$ be such that

$$\mathscr{U}(\mathfrak{q}) = \mathfrak{q} \otimes \mathfrak{q}, \quad \mathfrak{q} = \{\mathfrak{q}_n\}_{n\in\mathbb{N}} \in \ell^2_{\mathbb{H}}.$$

Then $\mathscr{U}$ is a bounded injective operator with closed range. Also, for each $n \in \mathbb{N}$

$$\begin{aligned}\left({}_{\ell^2_{\mathbb{H}}}\langle \mathfrak{e}_n|\mathscr{U}\right)(\mathfrak{q}) &= {}_{\ell^2_{\mathbb{H}}}\langle \mathfrak{e}_n|(\mathfrak{q}\otimes\mathfrak{q}) \\ &= \langle \mathfrak{e}_n|\mathfrak{q}\rangle\mathfrak{q} \\ &= \mathfrak{q}_n\mathfrak{q}, \quad \mathfrak{q} = \{\mathfrak{q}_n\}_{n\in\mathbb{N}} \in \ell^2_{\mathbb{H}}.\end{aligned}$$

Therefore by Theorem 9, $\{{}_{\ell^2_{\mathbb{H}}}\langle \mathfrak{e}_n|\mathscr{U}\}_{n\in\mathbb{N}}$ forms an OV-frame for $\ell^2_{\mathbb{H}}$ with range in $\ell^2_{\mathbb{H}}$.

5. Perturbation of operator valued frames

In this section, the stability of OV-frames under some perturbation conditions is established. First of all, we prove a result on perturbation of OV-frames, inspired by the most fundamental form of the Paley-Wiener perturbation theorem given by Casazza and Christensen [3], as follows:

THEOREM 10. *Let $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ be an OV-frame of H having range contained in K with lower and upper frame bounds A and B respectively. Let $\{\mathscr{G}_n\}_{n\in\mathbb{N}} \subseteq \mathscr{B}(\mathsf{H},\mathsf{K})$ be given and there exist α,β, and $\gamma > 0$ such that for all finite subset $\mathscr{I} \subset \mathbb{N}$ and for each $k \in \mathsf{K}$*

$$\begin{aligned}\Big\|\sum_{i\in\mathscr{I}}\Big(\mathscr{F}_i^*(c_ik) - \mathscr{G}_i^*(c_ik)\Big)\Big\| \leqslant \alpha\Big\|\sum_{i\in\mathscr{I}}\mathscr{F}_i^*(c_ik)\Big\| + \beta\Big\|\sum_{i\in\mathscr{I}}\mathscr{G}_i^*(c_ik)\Big\| \\ + \gamma\|k\|\Big(\sum_{i\in\mathscr{I}}|c_i|^2\Big)^{1/2}\end{aligned} \tag{6}$$

where, $c = \{c_n\}_{n\in\mathbb{N}} \in \ell^2_{\mathbb{H}}$ and $0 \leqslant \max\{\alpha + \gamma\sqrt{B}A^{-1}, \beta\} < 1$. Then, $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ also forms an OV-frame of H having range contained in K with lower and upper frame bounds $\frac{\left(1-(\alpha+\gamma\sqrt{B}A^{-1})\right)^2A^2}{(1+\beta)^2B}$ and $\frac{\left((1+\alpha)\sqrt{B}+\gamma\right)^2}{(1-\beta)^2}$ respectively.

Proof. Let $\mathscr{T}_{\mathscr{F}}$ and $\mathscr{S}_{\mathscr{F}}$ be the analysis and frame operators of $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ respectively. Then, from the given condition (6), for $k \in \mathsf{K}$ and $c = \{c_n\}_{n\in\mathbb{N}} \in \ell^2_{\mathbb{H}}$, we have

$$\begin{aligned}\Big\|\sum_{i\in\mathscr{I}}\mathscr{G}_i^*(c_ik)\Big\| &\leqslant \Big\|\sum_{i\in\mathscr{I}}\Big(\mathscr{F}_i^*(c_ik) - \mathscr{G}_i^*(c_ik)\Big)\Big\| + \Big\|\sum_{i\in\mathscr{I}}\mathscr{F}_i^*(c_ik)\Big\| \\ &\leqslant (1+\alpha)\Big\|\sum_{i\in\mathscr{I}}\mathscr{F}_i^*(c_ik)\Big\| + \beta\Big\|\sum_{i\in\mathscr{I}}\mathscr{G}_i^*(c_ik)\Big\| + \gamma\|k\|\|c\|,\end{aligned}$$

which implies that

$$\Big\|\sum_{i\in\mathscr{I}}\mathscr{G}_i^*(c_ik)\Big\| \leqslant \frac{(1+\alpha)\big\|\sum_{i\in\mathscr{I}}\mathscr{F}_i^*(c_ik)\big\|+\gamma\|k\|\|c\|}{1-\beta}$$
$$\leqslant \frac{(1+\alpha)\big\|\mathscr{T}_{\mathscr{F}}^*(c\otimes k)\big\|+\gamma\|c\otimes k\|}{1-\beta}$$
$$\leqslant \left(\frac{(1+\alpha)\big\|\mathscr{T}_{\mathscr{F}}^*\big\|+\gamma}{1-\beta}\right)\|c\otimes k\|$$
$$\leqslant \left(\frac{(1+\alpha)\sqrt{B}+\gamma}{1-\beta}\right)\|c\otimes k\|.$$

Now define $\mathscr{U}_{\mathscr{G}}:\ell^2_{\mathbb{H}}\otimes\mathsf{K}\to\mathsf{H}$ as

$$\mathscr{U}_{\mathscr{G}}(c\otimes k)=\sum_{n\in\mathbb{N}}\mathscr{G}_n^*(c_nk),\quad k\in\mathsf{K},\ c=\{c_n\}\in\ell^2_{\mathbb{H}}.$$

It is easy to verify that $\mathscr{U}_{\mathscr{G}}$ is a well-defined bounded operator with $\|\mathscr{U}_{\mathscr{G}}\|\leqslant\frac{(1+\alpha)\sqrt{B}+\gamma}{(1-\beta)}$. Then, $\mathscr{T}_{\mathscr{G}}:=\mathscr{U}_{\mathscr{G}}^*:\mathsf{H}\to\ell^2_{\mathbb{H}}\otimes\mathsf{K}$ is defined as

$$\mathscr{T}_{\mathscr{G}}(\mathsf{h})=\sum_{n\in\mathbb{N}}\mathfrak{e}_n\otimes\mathscr{G}_n(\mathsf{h}),\quad \mathsf{h}\in\mathsf{H}.$$

Also for $\mathsf{h}\in\mathsf{H}$,

$$\sum_{n\in\mathbb{N}}\|\mathscr{G}_n(\mathsf{h})\|^2=\|\mathscr{T}_{\mathscr{G}}(\mathsf{h})\|^2\leqslant\left(\frac{(1+\alpha)\sqrt{B}+\gamma}{1-\beta}\right)^2\|\mathsf{h}\|^2. \tag{7}$$

Let $\mathscr{W}:=\mathscr{T}_{\mathscr{G}}^*\mathscr{T}_{\mathscr{F}}\mathscr{S}_{\mathscr{F}}^{-1}$. Then, we have

$$\begin{aligned}\|\mathsf{h}-\mathscr{W}(\mathsf{h})\| &= \|\mathscr{T}_{\mathscr{F}}^*\mathscr{T}_{\mathscr{F}}\mathscr{S}_{\mathscr{F}}^{-1}(\mathsf{h})-\mathscr{T}_{\mathscr{G}}^*\mathscr{T}_{\mathscr{F}}\mathscr{S}_{\mathscr{F}}^{-1}(\mathsf{h})\|\\ &\leqslant \alpha\|\mathsf{h}\|+\beta\|\mathscr{W}(\mathsf{h})\|+\gamma\|\mathscr{T}_{\mathscr{F}}\mathscr{S}_{\mathscr{F}}^{-1}(\mathsf{h})\|\\ &\leqslant \alpha\|\mathsf{h}\|+\beta\|\mathscr{W}(\mathsf{h})\|+\gamma\|\mathscr{T}_{\mathscr{F}}\|\|\mathscr{S}_{\mathscr{F}}^{-1}\|\|\mathsf{h}\|\\ &\leqslant (\alpha+\gamma\sqrt{B}A^{-1})\|\mathsf{h}\|+\beta\|\mathscr{W}(\mathsf{h})\|\\ &< \|\mathsf{h}\|+\|\mathscr{W}(\mathsf{h})\|,\quad \mathsf{h}\in\mathsf{H}.\end{aligned}$$

This further gives that $\mathscr{W}$ is an invertible operator. Moreover,

$$\begin{aligned}\|\mathscr{W}(\mathsf{h})\| &= \|\mathsf{h}-(\mathsf{h}-\mathscr{W}(\mathsf{h}))\|\\ &\geqslant \|\mathsf{h}\|-\|\mathsf{h}-\mathscr{W}(\mathsf{h})\|\\ &\geqslant \|\mathsf{h}\|-(\alpha+\gamma\sqrt{B}A^{-1})\|\mathsf{h}\|-\beta\|\mathscr{W}(\mathsf{h})\|.\end{aligned}$$

$$\implies \|\mathsf{h}\|\left(\frac{1-(\alpha+\gamma\sqrt{B}A^{-1})}{1+\beta}\right)\leqslant\|\mathscr{W}(\mathsf{h})\|.$$
$$\implies \|\mathscr{W}^{-1}\|\leqslant\frac{1+\beta}{1-(\alpha+\gamma\sqrt{B}A^{-1})}.$$

Also for $\mathsf{h} \in \mathsf{H}$, we have

$$\begin{aligned}\|\mathsf{h}\|^2 &= \|(\mathscr{W}^*)^{-1}\mathscr{W}^*(\mathsf{h})\|^2 \\ &\leqslant \|(\mathscr{W}^*)^{-1}\|^2\|\mathscr{S}_{\mathscr{F}}^{-1}\mathscr{T}_{\mathscr{F}}^*\mathscr{T}_{\mathscr{G}}(\mathsf{h})\|^2 \\ &\leqslant \|(\mathscr{W}^*)^{-1}\|^2\|\mathscr{S}_{\mathscr{F}}^{-1}\|^2\|\mathscr{T}_{\mathscr{F}}^*\|^2\|\mathscr{T}_{\mathscr{G}}(\mathsf{h})\|^2 \\ &\leqslant \left(\frac{1+\beta}{1-(\alpha+\gamma\sqrt{B}A^{-1})}\right)^2(\sqrt{B}A^{-1})^2\sum_{n\in\mathbb{N}}\|\mathscr{G}_n(\mathsf{h})\|^2.\end{aligned}$$

Thus

$$\frac{\left(1-(\alpha+\gamma\sqrt{B}A^{-1})\right)^2A^2}{(1+\beta)^2B}\|\mathsf{h}\|^2 \leqslant \sum_{n\in\mathbb{N}}\|\mathscr{G}_n(\mathsf{h})\|^2, \quad \mathsf{h}\in\mathsf{H}. \tag{8}$$

Thus from (7) and (8), we conclude that $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ forms an OV-frame of H having range contained in K with lower and upper frame bounds $\frac{\left(1-(\alpha+\gamma\sqrt{B}A^{-1})\right)^2A^2}{(1+\beta)^2B}$ and $\frac{\left((1+\alpha)\sqrt{B}+\gamma\right)^2}{(1-\beta)^2}$ respectively. □

THEOREM 11. *Let $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ be an OV-frame of H having range contained in K with lower and upper frame bounds A and B respectively. Let $\{\mathscr{G}_n\}_{n\in\mathbb{N}} \subseteq \mathscr{B}(\mathsf{H},\mathsf{K})$ be given and there exist α,β, and $\gamma>0$ such that for all finite subset $\mathscr{I}\subset\mathbb{N}$ and for each $\mathsf{h}\in\mathsf{H}$*

$$\Big\|\sum_{i\in\mathscr{I}}\mathfrak{e}_i\otimes\Big(\mathscr{F}_i(\mathsf{h})-\mathscr{G}_i(\mathsf{h})\Big)\Big\| \leqslant \alpha\Big\|\sum_{i\in\mathscr{I}}\mathfrak{e}_i\otimes\mathscr{F}_i(\mathsf{h})\Big\| + \beta\Big\|\sum_{i\in\mathscr{I}}\mathfrak{e}_i\otimes\mathscr{G}_i(\mathsf{h})\Big\| + \gamma\|\mathsf{h}\| \tag{9}$$

where, $\beta\neq 1$. Then, $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ also forms an OV-frame of H having range contained in K with lower and upper frame bounds $\frac{\left((1-\alpha)A(\sqrt{B})^{-1}-\gamma\right)^2}{(1+\beta)^2}$ and $\frac{\left((1+\alpha)\sqrt{B}+\gamma\right)^2}{(1-\beta)^2}$ respectively.

Proof. Let $\mathscr{T}_{\mathscr{F}}$ and $\mathscr{S}_{\mathscr{F}}$ be the analysis and frame operators of $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ respectively. From the given condition (9), for $\mathsf{h}\in\mathsf{H}$,

$$\begin{aligned}\Big\|\sum_{i\in\mathscr{I}}\mathfrak{e}_i\otimes\mathscr{G}_i(\mathsf{h})\Big\| &\leqslant \Big\|\sum_{i\in\mathscr{I}}\mathfrak{e}_i\otimes\Big(\mathscr{F}_i(\mathsf{h})-\mathscr{G}_i(\mathsf{h})\Big)\Big\| + \Big\|\sum_{i\in\mathscr{I}}\mathfrak{e}_i\otimes\mathscr{F}_i(\mathsf{h})\Big\| \\ &\leqslant (1+\alpha)\Big\|\sum_{i\in\mathscr{I}}\mathfrak{e}_i\otimes\mathscr{F}_i(\mathsf{h})\Big\| + \beta\Big\|\sum_{i\in\mathscr{I}}\mathfrak{e}_i\otimes\mathscr{G}_i(\mathsf{h})\Big\| + \gamma\|\mathsf{h}\|,\end{aligned}$$

which implies that

$$\begin{aligned}\Big\|\sum_{i\in\mathscr{I}}\mathfrak{e}_i\otimes\mathscr{G}_i(\mathsf{h})\Big\| &\leqslant \frac{(1+\alpha)\|\mathscr{T}_{\mathscr{F}}(\mathsf{h})\|+\gamma\|\mathsf{h}\|}{1-\beta} \\ &\leqslant \left(\frac{(1+\alpha)\sqrt{B}+\gamma}{1-\beta}\right)\|\mathsf{h}\|.\end{aligned}$$

Now, define $\mathscr{T}_{\mathscr{G}} : \mathsf{H} \to \ell^2_{\mathbb{H}} \otimes \mathsf{K}$ as

$$\mathscr{T}_{\mathscr{G}}(\mathsf{h}) = \sum_{n\in\mathbb{N}} \mathfrak{e}_n \otimes \mathscr{G}_n(\mathsf{h}), \quad \mathsf{h} \in \mathsf{H}.$$

Clearly, $\mathscr{T}_{\mathscr{G}}$ is a well-defined bounded operator with $\|\mathscr{T}_{\mathscr{G}}\| \leqslant \frac{(1+\alpha)\sqrt{B}+\gamma}{1-\beta}$. Also,

$$\sum_{n\in\mathbb{N}} \|\mathscr{G}_n(\mathsf{h})\|^2 = \|\mathscr{T}_{\mathscr{G}}(\mathsf{h})\|^2 \leqslant \left(\frac{(1+\alpha)\sqrt{B}+\gamma}{1-\beta}\right)^2 \|\mathsf{h}\|^2, \quad \mathsf{h} \in \mathsf{H}. \tag{10}$$

Moreover, by the given condition,

$$\|\mathscr{T}_{\mathscr{F}}(\mathsf{h}) - \mathscr{T}_{\mathscr{G}}(\mathsf{h})\| \leqslant \alpha\|\mathscr{T}_{\mathscr{F}}(\mathsf{h})\| + \beta\|\mathscr{T}_{\mathscr{G}}(\mathsf{h})\| + \gamma\|\mathsf{h}\|,$$

which gives that

$$\|\mathscr{T}_{\mathscr{G}}(\mathsf{h})\| \geqslant \frac{(1-\alpha)\|\mathscr{T}_{\mathscr{F}}(\mathsf{h})\| - \gamma\|\mathsf{h}\|}{1+\beta}, \quad \mathsf{h} \in \mathsf{H}. \tag{11}$$

Also for $\mathsf{h} \in \mathsf{H}$,

$$\begin{aligned} \|\mathsf{h}\| &= \|\mathscr{S}_{\mathscr{F}}^{-1}\mathscr{S}_{\mathscr{F}}(\mathsf{h})\| \\ &\leqslant \|\mathscr{S}_{\mathscr{F}}^{-1}\|\|\mathscr{T}_{\mathscr{F}}^{*}\|\|\mathscr{T}_{\mathscr{F}}(\mathsf{h})\| \\ &\leqslant A^{-1}\sqrt{B}\|\mathscr{T}_{\mathscr{F}}(\mathsf{h})\|. \end{aligned} \tag{12}$$

From (11) and (12), we get

$$\sum_{n\in\mathbb{N}} \|\mathscr{G}_n(\mathsf{h})\|^2 = \|\mathscr{T}_{\mathscr{G}}(\mathsf{h})\|^2 \geqslant \left(\frac{(1-\alpha)A(\sqrt{B})^{-1}-\gamma}{1+\beta}\right)^2 \|\mathsf{h}\|^2, \quad \mathsf{h} \in \mathsf{H}. \tag{13}$$

Thus, from (10) and (13), we conclude that $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ forms an OV-frame of H having range contained in K with lower and upper frame bounds $\frac{\left((1-\alpha)A(\sqrt{B})^{-1}-\gamma\right)^2}{(1+\beta)^2}$ and $\frac{\left((1+\alpha)\sqrt{B}+\gamma\right)^2}{(1-\beta)^2}$ respectively. □

THEOREM 12. *Let $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ be an OV-frame of H having range contained in K with lower and upper frame bounds A and B respectively. Let $\{\mathscr{G}_n\}_{n\in\mathbb{N}} \subseteq \mathscr{B}(\mathsf{H},\mathsf{K})$ be given and there exist α, β, and $\gamma > 0$ such that for all finite subset $\mathscr{I} \subset \mathbb{N}$ and for each $\mathsf{h} \in \mathsf{H}$*

$$\begin{aligned} \Big\|\sum_{i\in\mathscr{I}} \Big(\mathscr{F}_i^*\mathscr{F}_i(\mathsf{h}) - \mathscr{G}_i^*\mathscr{G}_i(\mathsf{h})\Big)\Big\| &\leqslant \alpha\Big\|\sum_{i\in\mathscr{I}} \mathscr{F}_i^*\mathscr{F}_i(\mathsf{h})\Big\| + \beta\Big\|\sum_{i\in\mathscr{I}} \mathscr{G}_i^*\mathscr{G}_i(\mathsf{h})\Big\| \\ &\quad + \gamma\Big(\sum_{i\in\mathscr{I}} \|\mathscr{F}_i(\mathsf{h})\|^2\Big)^{1/2} \end{aligned} \tag{14}$$

where, $0 \leqslant \max\{\alpha+\gamma\sqrt{B}A^{-1}, \beta\} < 1$. *Then,* $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ *also forms an OV-frame of* H *having range contained in* K *with lower and upper frame bounds* $\frac{\left(1-(\alpha+\gamma\sqrt{B}A^{-1})\right)A}{1+\beta}$ *and* $\frac{(1+\alpha)B+\gamma\sqrt{B}}{1-\beta}$ *respectively.*

Proof. Let $\mathscr{S}_{\mathscr{F}}$ be the frame operator of $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$. From the given condition (14), for $\mathsf{h}\in\mathsf{H}$, we have

$$\Big\|\sum_{i\in\mathscr{I}}\mathscr{G}_i^*\mathscr{G}_i(\mathsf{h})\Big\| \leqslant \Big\|\sum_{i\in\mathscr{I}}\Big(\mathscr{F}_i^*\mathscr{F}_i(\mathsf{h})-\mathscr{G}_i^*\mathscr{G}_i(\mathsf{h})\Big)\Big\| + \Big\|\sum_{i\in\mathscr{I}}\mathscr{F}_i^*\mathscr{F}_i(\mathsf{h})\Big\|$$
$$\leqslant (1+\alpha)\Big\|\sum_{i\in\mathscr{I}}\mathscr{F}_i^*\mathscr{F}_i(\mathsf{h})\Big\| + \beta\Big\|\sum_{i\in\mathscr{I}}\mathscr{G}_i^*\mathscr{G}_i(\mathsf{h})\Big\| + \gamma\Big(\sum_{i\in\mathscr{I}}\|\mathscr{F}_i(\mathsf{h})\|^2\Big)^{1/2},$$

that gives

$$\Big\|\sum_{i\in\mathscr{I}}\mathscr{G}_i^*\mathscr{G}_i(\mathsf{h})\Big\| \leqslant \frac{(1+\alpha)\Big\|\sum_{i\in\mathscr{I}}\mathscr{F}_i^*\mathscr{F}_i(\mathsf{h})\Big\| + \gamma\Big(\sum_{i\in\mathscr{I}}\|\mathscr{F}_i(\mathsf{h})\|^2\Big)^{1/2}}{1-\beta}.$$

Also, we have

$$\Big\|\sum_{i\in\mathscr{I}}\mathscr{F}_i^*\mathscr{F}_i(\mathsf{h})\Big\| = \sup_{\|\mathfrak{f}\|=1}\Big|\Big\langle\sum_{i\in\mathscr{I}}\mathscr{F}_i^*\mathscr{F}_i(\mathsf{h})\Big|\mathfrak{f}\Big\rangle\Big|$$
$$\leqslant \Big(\sum_{i\in\mathscr{I}}\|\mathscr{F}_i(\mathsf{h})\|^2\Big)^{1/2}\sup_{\|\mathfrak{f}\|=1}\Big(\sum_{i\in\mathscr{I}}\|\mathscr{F}_i(\mathfrak{f})\|^2\Big)^{1/2}$$
$$\leqslant \sqrt{B}\Big(\sum_{i\in\mathscr{I}}\|\mathscr{F}_i(\mathsf{h})\|^2\Big)^{1/2}.$$

This further gives

$$\Big\|\sum_{i\in\mathscr{I}}\mathscr{G}_i^*\mathscr{G}_i(\mathsf{h})\Big\| \leqslant \left(\frac{(1+\alpha)\sqrt{B}+\gamma}{1-\beta}\right)\Big(\sum_{i\in\mathscr{I}}\|\mathscr{F}_i(\mathsf{h})\|^2\Big)^{1/2}$$
$$\leqslant \left(\frac{(1+\alpha)\sqrt{B}+\gamma}{1-\beta}\right)\sqrt{B}\|\mathsf{h}\|$$
$$\leqslant \left(\frac{(1+\alpha)B+\gamma\sqrt{B}}{1-\beta}\right)\|\mathsf{h}\|.$$

Now, define $\mathscr{S}_{\mathscr{G}}:\mathsf{H}\to\mathsf{H}$ as

$$\mathscr{S}_{\mathscr{G}}(\mathsf{h}) = \sum_{n\in\mathbb{N}}\mathscr{G}_n^*\mathscr{G}_n(\mathsf{h}), \quad \mathsf{h}\in\mathsf{H}.$$

One may easily observe that $\mathscr{S}_{\mathscr{G}}$ is a well-defined bounded operator with $\|\mathscr{S}_{\mathscr{G}}\| \leqslant \frac{(1+\alpha)B+\gamma\sqrt{B}}{1-\beta}$. Moreover, for $\mathsf{h}\in\mathsf{H}$

$$
\begin{aligned}
\sum_{n\in\mathbb{N}} \|\mathscr{G}_n(\mathsf{h})\|^2 &= \langle \mathscr{S}_{\mathscr{G}}(\mathsf{h})|\mathsf{h}\rangle \\
&\leqslant \|\mathscr{S}_{\mathscr{G}}\|\|\mathsf{h}\|^2 \\
&\leqslant \left(\frac{(1+\alpha)B+\gamma\sqrt{B}}{1-\beta}\right)\|\mathsf{h}\|^2. \qquad (15)
\end{aligned}
$$

Also by (14), for $\mathsf{h}\in\mathsf{H}$, we have

$$
\|\mathscr{S}_{\mathscr{F}}(\mathsf{h})-\mathscr{S}_{\mathscr{G}}(\mathsf{h})\| \leqslant \alpha\|\mathscr{S}_{\mathscr{F}}(\mathsf{h})\|+\beta\|\mathscr{S}_{\mathscr{G}}(\mathsf{h})\|+\gamma\Big(\sum_{n\in\mathbb{N}}\|\mathscr{F}_n(\mathsf{h})\|^2\Big)^{1/2},
$$

which further gives

$$
\begin{aligned}
\|\mathsf{h}-\mathscr{S}_{\mathscr{G}}\mathscr{S}_{\mathscr{F}}^{-1}(\mathsf{h})\| &\leqslant \alpha\|\mathsf{h}\|+\beta\|\mathscr{S}_{\mathscr{G}}\mathscr{S}_{\mathscr{F}}^{-1}(\mathsf{h})\|+\gamma\Big(\sum_{n\in\mathbb{N}}\|\mathscr{F}_n\mathscr{S}_{\mathscr{F}}^{-1}(\mathsf{h})\|^2\Big)^{1/2} \\
&\leqslant \alpha\|\mathsf{h}\|+\beta\|\mathscr{S}_{\mathscr{G}}\mathscr{S}_{\mathscr{F}}^{-1}(\mathsf{h})\|+\gamma\Big(B\|\mathscr{S}_{\mathscr{F}}^{-1}(\mathsf{h})\|^2\Big)^{1/2} \\
&\leqslant \alpha\|\mathsf{h}\|+\beta\|\mathscr{S}_{\mathscr{G}}\mathscr{S}_{\mathscr{F}}^{-1}(\mathsf{h})\|+\gamma\Big(B\|\mathscr{S}_{\mathscr{F}}^{-1}\|^2\|\mathsf{h}\|^2\Big)^{1/2} \\
&\leqslant \alpha\|\mathsf{h}\|+\beta\|\mathscr{S}_{\mathscr{G}}\mathscr{S}_{\mathscr{F}}^{-1}(\mathsf{h})\|+\gamma\Big(BA^{-2}\|\mathsf{h}\|^2\Big)^{1/2} \\
&\leqslant (\alpha+\gamma\sqrt{B}A^{-1})\|\mathsf{h}\|+\beta\|\mathscr{S}_{\mathscr{G}}\mathscr{S}_{\mathscr{F}}^{-1}(\mathsf{h})\| \\
&< \|\mathsf{h}\|+\|\mathscr{S}_{\mathscr{G}}\mathscr{S}_{\mathscr{F}}^{-1}(\mathsf{h})\|.
\end{aligned}
$$

Thus, $\mathscr{S}_{\mathscr{G}}\mathscr{S}_{\mathscr{F}}^{-1}$ is invertible and hence $\mathscr{S}_{\mathscr{G}}$ is invertible. Moreover,

$$
\begin{aligned}
\|\mathscr{S}_{\mathscr{G}}\mathscr{S}_{\mathscr{F}}^{-1}(\mathsf{h})\| &= \|\mathsf{h}-(\mathsf{h}-\mathscr{S}_{\mathscr{G}}\mathscr{S}_{\mathscr{F}}^{-1}(\mathsf{h}))\| \\
&\geqslant \|\mathsf{h}\|-\|\mathsf{h}-\mathscr{S}_{\mathscr{G}}\mathscr{S}_{\mathscr{F}}^{-1}(\mathsf{h})\| \\
&\geqslant \|\mathsf{h}\|-(\alpha+\gamma\sqrt{B}A^{-1})\|\mathsf{h}\|-\beta\|\mathscr{S}_{\mathscr{G}}\mathscr{S}_{\mathscr{F}}^{-1}(\mathsf{h})\|.
\end{aligned}
$$

$$
\begin{aligned}
&\implies \|\mathscr{S}_{\mathscr{G}}\mathscr{S}_{\mathscr{F}}^{-1}(\mathsf{h})\| \geqslant \left(\frac{1-(\alpha+\gamma\sqrt{B}A^{-1})}{1+\beta}\right)\|\mathsf{h}\|. \\
&\implies \|\mathscr{S}_{\mathscr{F}}\mathscr{S}_{\mathscr{G}}^{-1}\| \leqslant \frac{1+\beta}{1-(\alpha+\gamma\sqrt{B}A^{-1})}.
\end{aligned}
$$

Thus

$$
\|\mathscr{S}_{\mathscr{G}}^{-1}\| = \|\mathscr{S}_{\mathscr{F}}^{-1}\mathscr{S}_{\mathscr{F}}\mathscr{S}_{\mathscr{G}}^{-1}\| \leqslant \frac{1+\beta}{A(1-(\alpha+\gamma\sqrt{B}A^{-1}))}.
$$

$$
\begin{aligned}
&\implies \mathscr{S}_{\mathscr{G}}^{-1} \leqslant \left(\frac{1+\beta}{(1-(\alpha+\gamma\sqrt{B}A^{-1}))A}\right)I_{\mathsf{H}}. \\
&\implies \left(\frac{(1-(\alpha+\gamma\sqrt{B}A^{-1}))A}{1+\beta}\right)I_{\mathsf{H}} \leqslant \mathscr{S}_{\mathscr{G}}.
\end{aligned}
$$

Therefore

$$\left(\frac{(1-(\alpha+\gamma\sqrt{B}A^{-1}))A}{1+\beta}\right)\|\mathsf{h}\|^2 \leqslant \sum_{n\in\mathbb{N}} \|\mathscr{G}_n(\mathsf{h})\|^2, \quad \mathsf{h}\in\mathsf{H}. \tag{16}$$

Thus, from (15) and (16), we get that $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ forms an OV-frame of H having range contained in K with lower and upper frame bounds $\frac{\left(1-(\alpha+\gamma\sqrt{B}A^{-1})\right)A}{1+\beta}$ and $\frac{(1+\alpha)B+\gamma\sqrt{B}}{1-\beta}$ respectively. □

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