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A NOTE ON CAUCHY–TYPE CLASS OF FUNCTIONS

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Abstract. In the paper of Jachymski et al., a general Cauchy-type class of functions, $\mathscr{A}_M(X,Y)$, has been defined. Assuming M to be non-decreasing, the class is in some sense equivalent to the class of continuous functions or Baire-1 functions. In this paper, we provide an equivalent definition of the class and use it to obtain several results without assuming M to be non-decreasing. In particular, by assuming the limit of M at $(0,0)$ exists, we show that the class contains all continuous functions or it can only contain uniformly locally constant functions.

1. Introduction

Continuous functions and Baire-1 functions are two highly significant types of functions when it comes to understanding the properties of functions and solving more complex problems across various fields, both in mathematics and its applications. In mathematics, a continuous function f from a metric space (X,d) to another metric space (Y,ρ) is defined in the form of $\varepsilon-\delta$ formulation as follows: for every $x\in X$ and $\varepsilon>0$, there exists $\delta>0$ such that for every $y\in X$ with $d(x,y)<\delta$, it holds that $\rho(f(x),f(y))<\varepsilon$. Baire-1 functions, on the other hand, are functions expressible as pointwise limits of sequences of continuous functions. The concept of Baire-1 functions was first introduced by René-Louis Baire in 1899 ([2]). To this day, research related to Baire-1 functions remains highly active, particularly concerning the characterization of these types of functions. One significant outcome related to the characterization of Baire-1 functions is the characterization into the $\varepsilon-\delta$ version obtained by Lee et al. ([5]) in 2000, as presented in Theorem 1 below. The characterization serves as the foundation for many developments in the realm of Baire functions, as evident in the works of Alikhani-Koopaei ([1]), Balcerzak, Karlova, and Szuca ([3]), and Orge and Benitez ([6]).

THEOREM 1. (Lee, Tang, and Zhao [5]) *Let (X,d) and (Y,ρ) be complete and separable metric spaces. A function $f:X\to Y$ is a Baire-1 function on X if and only if for every $\varepsilon>0$, there exists a function $\delta:X\to(0,\infty)$ such that for every $x,y\in X$ with $d(x,y)<\min\{\delta(x),\delta(y)\}$, we have $\rho(f(x),f(y))<\varepsilon$.*

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This characterization was motivated by the fact that continuous functions themselves can be characterized in a similar manner by replacing the function $\min\{\cdot,\cdot\}$ with the function $\max\{\cdot,\cdot\}$. Based on this fact, it is natural to ask the following question: what kind of class of functions can be obtained, which has a similar formulation as continuous or Baire-1 functions where the functions $\min\{\cdot,\cdot\}$ or $\max\{\cdot,\cdot\}$ are replaced by other functions? Addressing this question, in 2004, Jachymski et al. ([4]) defined the class of functions $\mathscr{A}_M(X,Y)$, which is a class of functions that share a formulation similar to that in Theorem 1, with the generalization of the function $\min\{\cdot,\cdot\}$ into a function $M:(0,\infty)^2 \to (0,\infty)$. Assuming M to be non-decreasing, they successfully obtained results indicating that this class would essentially be equivalent to the class of constant functions, continuous functions, or Baire-1 functions. So far, without assuming the monotonicity of M, it remains unknown whether there are other new classes that can be obtained, as explicitly stated in their paper.

In this paper, we provided an equivalent definition of the class and several results related to the class without assuming M to be non-decreasing. Particularly, under the assumption that $\lim\limits_{(s,t)\to(0,0)} M(s,t)$ exists, we show that either $\mathscr{A}_M(X,Y)$ contains all continuous functions or $\mathscr{A}_M(X,Y)$ can only contain uniformly locally constant functions.

2. Results

Throughout this section, unless stated otherwise, we always assume (X,d) and (Y,ρ) to be metric spaces. We start this section by providing an equivalent definition of continuous and Baire-1 functions that plays an important role in proving the main results, i.e., Theorem 5 and Theorem 6. The proof is obvious by changing δ into $\min\{\delta,r\}$.

LEMMA 1. *Let f be a function from X to Y.*

(a) f is continuous on X if and only if for every $\varepsilon>0$ and $r>0$, there exists a function $\delta:X\to(0,r)$ such that for every $x,y\in X$ with $d(x,y)<\max\{\delta(x),\delta(y)\}$, we have

$$\rho(f(x),f(y))<\varepsilon.$$

(b) f is a Baire-1 function if and only if for every $\varepsilon>0$ and $r>0$, there exists a function $\delta:X\to(0,r)$ such that for every $x,y\in X$ with $d(x,y)<\min\{\delta(x),\delta(y)\}$, we have

$$\rho(f(x),f(y))<\varepsilon.$$

Motivated by the above lemma, we provide an equivalent definition of the class of functions $\mathscr{A}_M(X,Y)$ given in [4].

DEFINITION 1. Let $M:(0,\infty)^2\to(0,\infty)$. A function $f:X\to Y$ is said to be of class $\mathscr{A}_M(X,Y)$ if for every $\varepsilon>0$ and $r>0$, there exists a function $\delta:X\to(0,r)$ such that for every $x,y\in X$ with $d(x,y)<M(\delta(x),\delta(y))$, we have

$$\rho(f(x),f(y))<\varepsilon.$$

REMARK 1. Below are several known or trivial results regarding the class $\mathscr{A}_M(X,Y)$.

- Any constant function is of class $\mathscr{A}_M(X,Y)$.
- If $M(s,t) = \max\{s,t\}$, then $\mathscr{A}_M(X,Y) = C(X,Y)$, the class of all continuous functions from X to Y.
- If $M(s,t) = \min\{s,t\}$, then $\mathscr{A}_M(X,Y) = B_1(X,Y)$, the class of all Baire-1 functions from X to Y.
- The characterizations of $\mathscr{A}_M(X,Y)$ when M is non-decreasing have been obtained in [4]. In particular, they showed that when M is the arithmetic mean or geometric mean, then $\mathscr{A}_M(X,Y) = C(X,Y)$ or $B_1(X,Y)$, respectively.

The following question was explicitly asked in [4] which is the main motivation of this paper.

QUESTION 1. What can we get if M is not always assumed to be non-decreasing?

First, we provide a slight modification of [4, Proposition 1] that can be proved similarly by making use of Definition 1.

PROPOSITION 1. *Let* $M,N : (0,\infty)^2 \to (0,\infty)$. *If for every* $r > 0$ *and* $\delta : X \to (0,r)$, *the functional inequality*

$$M(\mu(x),\mu(y)) \leqslant N(\delta(x),\delta(y))$$

has a solution $\mu_0 : X \to (0,r)$, *then* $\mathscr{A}_N(X,Y) \subseteq \mathscr{A}_M(X,Y)$.

Using Proposition 1, we can obtain a direct proof for the fact that $\mathscr{A}_M(X,Y) = B_1(X,Y)$, where M is the geometric mean.

THEOREM 2. *Let* $M : (0,\infty)^2 \to (0,\infty)$ *and both* X *and* Y *be complete and separable metric spaces. If*

$$\min\{s,t\} \leqslant M(s,t) \leqslant \sqrt{st}$$

for all $s,t \in (0,\infty)$, *then* $\mathscr{A}_M(X,Y) = B_1(X,Y)$.

Proof. Let $M_2(s,t) = \sqrt{st}$ for all $s,t \in (0,\infty)$. Clearly, by Proposition 1,

$$\mathscr{A}_{M_2}(X,Y) \subseteq \mathscr{A}_M(X,Y) \subseteq B_1(X,Y).$$

It remains to show $B_1(X,Y) \subseteq \mathscr{A}_{M_2}(X,Y)$. Fix $r > 0$. Observe that for every $s,t \in (0,r)$, we have

$$M_2(s,t) \leqslant \min\{\sqrt{sr}, \sqrt{tr}\}$$

and hence,

$$M_2\left(\frac{s^2}{r}, \frac{t^2}{r}\right) \leqslant \min\{s,t\}.$$

Again, by Proposition 1, $B_1(X,Y) \subseteq \mathscr{A}_{M_2}(X,Y)$. $\square$

EXAMPLE 1. Let $M:(0,\infty)^2 \to (0,\infty)$ be the harmonic mean function, i.e., $M(s,t) = \frac{2}{\frac{1}{s}+\frac{1}{t}}$ for all $s,t>0$. By GM-HM inequality, we have

$$\min\{s,t\} \leqslant M(s,t) \leqslant \sqrt{st}.$$

By Theorem 2, we deduce that $\mathscr{A}_M(X,Y) = B_1(X,Y)$ where X and Y are complete and separable metric spaces.

We may also use Proposition 1 to have an alternative proof for the fact that $\mathscr{A}_M(X,Y) = C(X,Y)$ when M is the arithmetic mean. First, we show that when M is a homogeneous function, both M and αM $(\alpha>0)$ give us the same class of functions. Recall that a function $M:(0,\infty)^2 \to (0,\infty)$ is homogeneous of degree p $(p>0)$ if $M(\alpha s, \alpha t) = \alpha^p M(s,t)$ for every $s,t>0$.

PROPOSITION 2. *Let* $M,N:(0,\infty)^2 \to (0,\infty)$ *and* $\alpha>1$.

(*a*) $\mathscr{A}_{\alpha M}(X,Y) \subseteq \mathscr{A}_M(X,Y) \subseteq \mathscr{A}_{\frac{1}{\alpha}M}(X,Y)$.
Furthermore, if M *is homogeneous, then all are equal.*

(*b*) *If both* M *and* N *are homogeneous, then* $\mathscr{A}_{N+M}(X,Y) = \mathscr{A}_{\max\{N,M\}}(X,Y)$.

Proof.

(*a*) The inclusions are clear by using Proposition 1 or [4, Proposition 1] and the fact that $\frac{1}{\alpha}M(s,t) \leqslant M(s,t) \leqslant \alpha M(s,t)$. Now, if M is homogeneous of degree p, we have

$$M(\alpha^{2/p}s, \alpha^{2/p}t) = \alpha^2 M(s,t), \text{ and hence } \frac{1}{\alpha}M(\alpha^{2/p}s, \alpha^{2/p}t) \geqslant \alpha M(s,t).$$

By Proposition 1, $\mathscr{A}_{\frac{1}{\alpha}M}(X,Y) \subseteq \mathscr{A}_{\alpha M}(X,Y)$, and hence $\mathscr{A}_{\frac{1}{\alpha}M}(X,Y) = \mathscr{A}_{\alpha M}(X,Y)$.

(*b*) As M and N are homogeneous, so is $\max\{M,N\}$. Furthermore, we have

$$\max\{M,N\} \leqslant M+N \leqslant 2\max\{M,N\}.$$

The conclusion follows from part (a). □

THEOREM 3. *Let* $M:(0,\infty)^2 \to (0,\infty)$. *If*

$$\frac{s+t}{2} \leqslant M(s,t) \leqslant \max\{s,t\}$$

for all $s,t \in (0,\infty)$, *then* $\mathscr{A}_M(X,Y) = C(X,Y)$.

Proof. Let $M_1(s,t) = s$ and $M_2(s,t) = t$ for all $s,t \in (0,\infty)$. Note that $\max\{M_1, M_2\}(s,t) = \max\{s,t\}$ and hence $\mathscr{A}_{\max\{M_1,M_2\}} = C(X,Y)$. The conclusion follows from Proposition 2 and the fact that both M_1 and M_2 are homogeneous. □

EXAMPLE 2. Let $M:(0,\infty)^2 \to (0,\infty)$ be the quadratic mean function, i.e., $M(s,t) = \sqrt{\frac{s^2+t^2}{2}}$ for all $s,t>0$. By QM-AM inequality, we have

$$\frac{s+t}{2} \leqslant M(s,t) \leqslant \max\{s,t\}.$$

By Theorem 3, we deduce that $\mathscr{A}_M(X,Y) = C(X,Y)$. Note that the result remains hold if we replace M by any power mean function.

Note that by AM-GM inequality, the geometric mean is always less than or equal to the arithmetic mean, i.e.,

$$\sqrt{st} \leqslant \frac{s+t}{2} \quad \forall s,t>0.$$

It is natural to ask whether $\mathscr{A}_M(X,Y)$ will be equal to $C(X,Y)$ or $B_1(X,Y)$ if M lies between the geometric mean and arithmetic mean. The answer in general remains unknown. However, if M is a convex combination of the arithmetic mean and the geometric mean, we have $\mathscr{A}_M(X,Y) = C(X,Y)$. In fact, we have a more general result as follows.

THEOREM 4. *If $M:(0,\infty)^2 \to (0,\infty)$ is a positive linear combination of the arithmetic mean and the geometric mean, then $\mathscr{A}_M(X,Y) = C(X,Y)$.*

Proof. Let $M = aM_1 + bM_2$ where M_1 and M_2 are the arithmetic mean and the geometric mean, respectively, and $a,b>0$. Note that both M_1 and M_2 are homogenous. Since $\min\{a,b\}(M_1+M_2) \leqslant M \leqslant \max\{a,b\}(M_1+M_2)$, by Proposition 2, we have

$$\mathscr{A}_M(X,Y) = \mathscr{A}_{M_1+M_2}(X,Y) = \mathscr{A}_{\max\{M_1,M_2\}}(X,Y) = \mathscr{A}_{M_2}(X,Y) = C(X,Y). \quad \square$$

In the next theorem, we show that under the assumption that $\lim_{(s,t)\to(0,0)} M(s,t)$ exists, either $\mathscr{A}_M(X,Y)$ contains $C(X,Y)$ or $\mathscr{A}_M(X,Y)$ can only contain uniformly locally constant functions. We say a function $f:X\to Y$ is uniformly locally constant if there is $\lambda>0$ such that $f(x)=f(y)$ for every $x,y\in X$ with $d(x,y)<\lambda$. Note that any constant function is uniformly locally constant and any uniformly locally constant function is continuous.

THEOREM 5. *Let $M:(0,\infty)^2 \to (0,\infty)$.*

(1) *If $\lim_{(s,t)\to(0,0)} M(s,t) = 0$, then $C(X,Y) \subseteq \mathscr{A}_M(X,Y)$.*

(2) *If $\lim_{(s,t)\to(0,0)} M(s,t) > 0$, then $\mathscr{A}_M(X,Y)$ can only contain uniformly locally constant functions.*

Proof.

(1) Let $\delta : X \to (0,\infty)$. Since $\lim_{(s,t)\to(0,0)} M(s,t) = 0$, for a given $x \in X$, there exists $\lambda(x)$ such that $M(s,t) < \delta(x)$ for all $\|(s,t)\| < 2\lambda(x)$. Then for every $x,y \in X$, we have

$$M(\lambda(x),\lambda(y)) \leqslant \max\{\delta(x),\delta(y)\}.$$

By Proposition 1 or [4, Proposition 1], we deduce that $C(X,Y) \subseteq \mathscr{A}_M(X,Y)$.

(2) Since $2\lambda := \lim_{(s,t)\to(0,0)} M(s,t) > 0$, one can find $r_0 > 0$ such that $M(s,t) > \lambda$ for every $s,t < r_0$.

Let $f \in \mathscr{A}_M(X,Y)$. Then for every $\varepsilon > 0$ and the above r_0, there exists $\delta : X \to (0,r_0)$ such that $\rho(f(x),f(y)) < \varepsilon$ for every $x,y \in X$ with $d(x,y) \leqslant M(\delta(x),\delta(y))$. Note that for every $x,y \in X$ with $d(x,y) < \lambda$, we have

$$d(x,y) < \lambda < M(\delta(x),\delta(y))$$

and hence, $\rho(f(x),f(y)) < \varepsilon$. Since this holds for any $\varepsilon > 0$, we conclude that $f(x) = f(y)$ for any $x,y \in X$ with $d(x,y) < \lambda$. Thus, f is uniformly locally constant. □

Note that when X is connected, any uniformly locally constant function is a constant function. Consequently, by Theorem 5 (2), if $\lim_{(s,t)\to(0,0)} M(s,t) > 0$ and X is connected, then $\mathscr{A}_M(X,Y)$ contains only constant functions.

Recall that any constant function is contained in any class of $\mathscr{A}_M(X,Y)$. Hence, it is natural to ask whether the same is true for uniformly locally constant functions. The question remains open in general, but under the assumption of M to be homogeneous or $\lim_{(s,t)\to(0,0)} M(s,t) = 0$, we obtain an affirmative answer.

PROPOSITION 3. *Let $M : (0,\infty)^2 \to (0,\infty)$. If M is homogeneous or $\lim_{(s,t)\to(0,0)} M(s,t) = 0$, then any uniformly locally constant function on X is contained in $\mathscr{A}_M(X,Y)$.*

Proof. We only need to prove for the case M is homogeneous, the other case is a consequence of Theorem 5 part (1).

Let f be a uniformly locally constant function. Then there is $\lambda > 0$ such that $f(x) = f(y)$ for every $x,y \in X$ with $d(x,y) < \lambda$.

Let $\varepsilon > 0$ and $r > 0$ be arbitrary. Since M is homogeneous, by Proposition 2, $\mathscr{A}_M = \mathscr{A}_{\alpha M}$ for any $\alpha > 0$. Therefore, without loss of generality, we may assume that $M\left(\frac{r}{2},\frac{r}{2}\right) = \lambda$. Now, define $\delta : X \to (0,r)$ by $\delta(x) = \frac{r}{2}$ for every $x \in X$. Then for every $x,y \in X$ with $d(x,y) < M(\delta(x),\delta(y))$, we have $d(x,y) < \lambda$. Hence, $f(x) = f(y)$ and therefore, $\rho(f(x),f(y)) = 0 < \varepsilon$. Thus, $f \in \mathscr{A}_M(X,Y)$. □

We end our paper by providing a general result of Theorem 5 (1) as follows.

THEOREM 6. *Let* $M:(0,\infty)^2\to(0,\infty)$. *Suppose that* $\lim_{(s,t)\to(0,0)} M(s,t)$ *exists. The following are equivalent:*

(1) $\lim_{(s,t)\to(0,0)} M(s,t)=0$.

(2) $C(X,Y)\subseteq \mathscr{A}_M(X,Y)$ *for all metric spaces* X *and* Y.

Proof. $(1)\Rightarrow(2)$ is clear from Theorem 5.

$(2)\Rightarrow(1)$ By taking $X=Y=\mathbb{R}$ in (2), we have $C(\mathbb{R},\mathbb{R})\subseteq \mathscr{A}_M(\mathbb{R},\mathbb{R})$. Suppose that $\alpha:=\lim_{(s,t)\to(0,0)} M(s,t)>0$. Then there is $\lambda>0$ such that

$$\frac{\alpha}{2}\leqslant M(s,t)$$

for all $s,t<\lambda$.

Take $a,b\in\mathbb{R}$ with $0<|a-b|<\frac{\alpha}{2}$. Since $\mathbb{R}$ is normal, by Urysohn's lemma there exists $f\in C(\mathbb{R},\mathbb{R})$ such that $f(a)=0$ and $f(b)=1$. We show that $f\notin \mathscr{A}_M(\mathbb{R},\mathbb{R})$ and hence we get a contradiction.

Let $\varepsilon=1$ and $r=\frac{\lambda}{2}$. For every $\delta:\mathbb{R}\to(0,r)$, we have

$$|a-b|<\frac{\alpha}{2}\leqslant M(\delta(a),\delta(b)).$$

However,

$$|f(a)-f(b)|=1\geqslant\varepsilon. \quad\square$$

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