Journal of Classical Analysis
Volume 24, Number 2 (2024), 161–177 doi:10.7153/jca-2024-24-09

$\mathscr{I}$–CONVERGENCE OF PARTIAL MAPS

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Abstract. By a partial function or a partial map from a metric space (X,d) to a metric space (Y,μ), we mean a pair (A,u), where A is a non-empty closed subset of X and $u : A \to Y$ is a function. In this paper, using the notion of an ideal $\mathscr{I}$ on a directed set, we generalize the notion of bornological convergence of nets to the notion of bornological $\mathscr{I}$-convergence of nets and the notion of convergence of nets of partial maps to the notion of $\mathscr{I}$-convergence of nets of partial maps. Some basic properties of this notions are investigated including their interrelationship. We also introduce the notion of bornological $\mathscr{I}^*$-convergence of nets as well as the notion of $\mathscr{I}^*$-convergence of nets of partial maps and study their relationship with bornological $\mathscr{I}$-convergence of nets and $\mathscr{I}$-convergence of nets of partial maps respectively.

1. Introduction

Partial maps play a central role in mathematical economics. A partial map or a partial function from a metric space (X,d) to a metric space (Y,μ), is a pair (F,γ), where F is a closed non-empty subset of X and γ is a function from F to Y. The space of all partial maps from X to Y is denoted by $\mathscr{P}[X,Y]$. Tastes of agents on a space X are usually represented by a preference relation on X, i.e., a subset A of $X \times X$, where $(x,y) \in A$ means that the agent prefers alternative x to y. In this scenario, utility functions are considered as more appropriate mathematical tools to represent agents' preferences and partial maps are considered as utility functions for agents. One may use the notion of convergence or topology on partial maps to describe similarities of agents (for more details see Debreu [4]).

In [1], Beer et al. introduced a new notion of convergence, namely $\mathscr{P}(\mathscr{B})$-convergence of nets of partial maps, in the metric setting, via the notion of bornology. They showed that their notion of convergence of nets of partial maps is bornological convergence of the associated nets of graphs. Here we extend this notion of $\mathscr{P}(\mathscr{B})$-convergence of nets of partial maps via the notion of ideals.

Using the concept of natural density, the usual notion of convergence of real sequences was first generalized to the notion of statistical convergence independently by Fast [5] and Schoenberg [12]. For more works in this line one can see ([2, 6, 11] etc.). Then using the notion of an ideal $\mathscr{I}$ of subsets of $\mathbb{N}$, the notion of statistical convergence of sequences was generalized to $\mathscr{I}$-convergence in a metric space by Kostyrko et al. [7]. The notion of $\mathscr{I}$ - convergence of sequences has further been extended from

Mathematics subject classification (2020): 40A30, 40A35, 54E40.
Keywords and phrases: Partial maps, bornological $\mathscr{I}$-convergence, bornological $\mathscr{I}^*$-convergence, $\mathscr{I}$-convergence of partial maps, $\mathscr{I}^*$-convergence of partial maps.
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a metric space to a topological space [8]. Recently in [9], Lahiri et al. have introduced the notion of $\mathscr{I}$-convergence for nets in a topological space by using the concept of an ideal $\mathscr{I}$ of a directed set.

To broaden the ideas of bornological convergence [10] and convergence of partial maps [1, 3], in section 3 of this paper using the notion of an ideal $\mathscr{I}$ on a directed set, we first, generalize the notion of bornological convergence of nets to bornological $\mathscr{I}$-convergence of nets and then we generalize the concept of convergence of nets of partial maps to the notion of $\mathscr{I}$-convergence of nets of partial maps. We examine some basic properties of the notion of $\mathscr{I}$-convergence of nets of partial maps and examine its relationship with bornological $\mathscr{I}$-convergence. Further, in section 4 of this paper, we introduce the concept of bornological $\mathscr{I}^*$-convergence of nets as well as the concept of $\mathscr{I}^*$-convergence of nets of partial maps and study their relationship with bornological $\mathscr{I}$-convergence of nets and $\mathscr{I}$-convergence of nets of partial maps respectively.

2. Basic definitions and notation

In this section, we discuss some basic definitions and ideas, which will be needed in the successive sections.

DEFINITION 1. [7] If X is a non-empty set, then a family $\mathscr{I} \subset 2^X$ is called an ideal of X, if

(i) $\emptyset \in \mathscr{I}$,

(ii) $A, B \in \mathscr{I}$ implies $A \cup B \in \mathscr{I}$, and

(iii) $A \in \mathscr{I}$; $B \subset A$ implies $B \in \mathscr{I}$.

The ideal $\mathscr{I}$ is called non-trivial, if $\mathscr{I} \neq \{\emptyset\}$ and $X \notin \mathscr{I}$.

DEFINITION 2. [7] A non empty family $\mathscr{F}$ of subsets of a non-empty set X is called a filter of X, if

(i) $\emptyset \notin \mathscr{F}$,

(ii) $A, B \in \mathscr{F}$ implies $A \cap B \in \mathscr{F}$ and

(iii) $A \in \mathscr{F}$; $A \subset B$ implies $B \in \mathscr{F}$.

Clearly, if $\mathscr{I} \subset 2^X$ is a non-trivial ideal of X, then

$$\mathscr{F}(\mathscr{I}) = \{A \subset X \ : \ X \setminus A \in \mathscr{I}\}$$

is a filter on X, called the filter associated with $\mathscr{I}$.

A non-trivial ideal $\mathscr{I}$ of $X (\neq \emptyset)$ is called admissible, if $\{y\} \in \mathscr{I}$ for each $y \in X$.

We now recall the definitions of directed sets and nets.

DEFINITION 3. Let $\mathscr{G}$ be a non-empty set and $\geqslant$ be a binary relation on $\mathscr{G}$ such that $\geqslant$ is reflexive, transitive and for any two elements $\alpha, \beta \in \mathscr{G}$, there is an element $\gamma \in \mathscr{G}$ such that $\gamma \geqslant \alpha$ and $\gamma \geqslant \beta$. Then the pair $(\mathscr{G}, \geqslant)$ is called a directed set.

DEFINITION 4. Let $(\mathscr{G}, \geqslant)$ be a directed set and let X be a non-void set. A mapping $f : \mathscr{G} \to X$ is called a net in X and is denoted by $\{f_\gamma : \gamma \in \mathscr{G}\}$ or $\{f_\gamma\}_{\gamma \in \mathscr{G}}$.

Throughout the paper, $X=(X,d)$, $Y=(Y,\mu)$ will denote metric spaces, $\mathscr{P}[X,Y]$ will denote the space of all partial maps from the metric space X to the metric space Y, $\mathscr{G}$ will denote a directed set $(\mathscr{G},\geqslant)$ and $\mathscr{I}$ will denote a non-trivial ideal of $\mathscr{G}$, unless otherwise mentioned.

An ideal $\mathscr{I}$ of $\mathscr{G}$ will be written sometimes as $\mathscr{I}_{\mathscr{G}}$ to indicate the directed set $\mathscr{G}$ of which $\mathscr{I}$ is an ideal.

Let $\mathscr{M}_\gamma=\{\alpha\in\mathscr{G}:\alpha\geqslant\gamma\}$, $\gamma\in\mathscr{G}$. Then the collection

$$\mathscr{F}_0=\left\{A\subset\mathscr{G}:A\supset\mathscr{M}_\gamma \text{ for some } \gamma\in\mathscr{G}\right\}$$

forms a filter in $\mathscr{G}$. Let $\mathscr{I}_0=\{A\subset\mathscr{G}:\mathscr{G}\setminus A\in\mathscr{F}_0\}$. Then $\mathscr{I}_0$ is a non-trivial ideal in $\mathscr{G}$.

DEFINITION 5. [9] A non-trivial ideal $\mathscr{I}$ of $\mathscr{G}$ is called $\mathscr{G}$-*admissible*, if $\mathscr{M}_n\in\mathscr{F}(\mathscr{I})$ for all $n\in\mathscr{G}$.

DEFINITION 6. [9] A net $\{x_\gamma\}_{\gamma\in\mathscr{G}}$ in a metric space (X,d) is said to be $\mathscr{I}$-convergent to $x\in X$, if for every $\varepsilon>0$, $\{\gamma\in\mathscr{G}:d(x_\gamma,x)\geqslant\varepsilon\}\in\mathscr{I}$. In this case, we write $\mathscr{I}-\lim x_\gamma=x$.

The above definition of $\mathscr{I}$-convergence can be written as follows: a net $\{x_\gamma\}_{\gamma\in\mathscr{G}}$ in a metric space (X,d) is said to be $\mathscr{I}$-convergent to $x\in X$, if for every $\varepsilon>0$,

$$\left\{\gamma\in\mathscr{G}:d(x_\gamma,x)<\varepsilon\right\}\in\mathscr{F}(\mathscr{I}).$$

We now discuss the concept of bornology and the notion of bornological convergence on a metric space (for more details see [1, 10]).

If $a\in X$ and $\varepsilon>0$, then $B(a,\varepsilon)$ denotes the open ε-ball with center at a and radius ε. If G is a nonempty subset of X, we write $d(a,G)$ to denote the distance from a to G and G^ε to denote the ε-enlargement of the set G, i.e.,

$$G^\varepsilon=\{x:d(x,G)<\varepsilon\}=\bigcup_{x\in G}B(x,\varepsilon).$$

DEFINITION 7. If X is a non-empty set, then a family $\mathscr{B}$ of non-empty subsets of X is called a bornology on X, if the following conditions are satisfied:

(i) $\mathscr{B}$ covers X,

(ii) $A,B\in\mathscr{B}$ implies $A\cup B\in\mathscr{B}$ and

(iii) $A\in\mathscr{B}$; $B\subset A$ implies $B\in\mathscr{B}$.

The family $\mathscr{F}$ of all non-empty finite subsets of X is the smallest bornology on X and the family $\mathscr{P}_0(X)$ of all non-empty subsets of X is the largest bornology on X. Other important bornologies are: the family $\mathscr{B}_d$ of the nonempty d-bounded subsets, the family $\mathscr{B}_{tb}$ of the nonempty d-totally bounded subsets and the family $\mathscr{K}$ of nonempty subsets of X whose closures are compact.

DEFINITION 8. [10] Let $\mathscr{B}$ be a bornology on a metric space (X,d). A subfamily $\mathscr{B}'$ of $\mathscr{B}$, which is cofinal in $\mathscr{B}$ with respect to inclusion, is called a *base* for the bornology $\mathscr{B}$. A bornology is called *local*, if it contains a small ball around each point of X. Also, a bornology is said to be *stable under small enlargements*, if for every $B \in \mathscr{B}$ there is $\varepsilon > 0$ such that $B^{\varepsilon} \in \mathscr{B}$.

DEFINITION 9. [1] Let (X,d) be a metric space and $\mathscr{B}$ be a bornology on (X,d). A net $\{D_\gamma\}_{\gamma\in\mathscr{G}}$ in $\mathscr{P}_0(X)$ is called $\mathscr{B}^-$-convergent (lower bornological convergent) to $D \in \mathscr{P}_0(X)$ if for every $B \in \mathscr{B}$ and $\varepsilon > 0$, the following condition occurs eventually:

$$D \cap B \subset D_\gamma^{\varepsilon}.$$

In this case, we write $D \in \mathscr{B}^- - \lim D_\gamma$.

Similarly the net is called $\mathscr{B}^+$-convergent (upper bornological convergent) to $D \in \mathscr{P}_0(x)$ if for every $B \in \mathscr{B}$ and $\varepsilon > 0$, the following condition occurs eventually:

$$D_\gamma \cap B \subset D^{\varepsilon}.$$

In this case, we write $D \in \mathscr{B}^+ - \lim D_\gamma$.

When both the upper and lower bornological convergences occur, we say two-sided bornological convergence occurs and we write $D \in \mathscr{B} - \lim D_\gamma$.

DEFINITION 10. [1] Let (X,d), (Y,μ) be metric spaces, $\mathscr{B}$ be a bornology on X. A net $\{(D_\gamma,u_\gamma)\}_{\gamma\in\mathscr{G}}$ in $\mathscr{P}[X,Y]$ is said to be $\mathscr{P}^-(\mathscr{B})$-convergent to $(D,u) \in \mathscr{P}[X,Y]$, if for every $B \in \mathscr{B}$ and $\varepsilon > 0$, the following inclusion holds eventually:

$$u(D\cap B_1) \subset [u_\gamma(D_\gamma \cap B_1^{\varepsilon})]^{\varepsilon}, \text{ for all } B_1 \subset B.$$

In this case, we write $(D,u) \in \mathscr{P}^-(\mathscr{B}) - \lim(D_\gamma,u_\gamma)$.

Similarly, the net $\{(D_\gamma,u_\gamma)\}_{\gamma\in\mathscr{G}}$ in $\mathscr{P}[X,Y]$ is said to be $\mathscr{P}^+(\mathscr{B})$-convergent to $(D,u) \in \mathscr{P}[X,Y]$, if for every $B \in \mathscr{B}$ and $\varepsilon > 0$, the following inclusion holds eventually:

$$u_\gamma(D_\gamma\cap B_1) \subset [u(D \cap B_1^{\varepsilon})]^{\varepsilon}, \text{ for all } B_1 \subset B.$$

In this case, we write $(D,u) \in \mathscr{P}^+(\mathscr{B}) - \lim(D_\gamma,u_\gamma)$.

If the net $\{(D_\gamma,u_\gamma)\}_{\gamma\in\mathscr{G}}$ is both $\mathscr{P}^-(\mathscr{B})$-convergent and $\mathscr{P}^+(\mathscr{B})$-convergent to (D,u), then we say that $\{(D_\gamma,u_\gamma)\}_{\gamma\in\mathscr{G}}$ is $\mathscr{P}(\mathscr{B})$-convergent to (D,u) and in this case, we write $(D,u) \in \mathscr{P}(\mathscr{B}) - \lim(D_\gamma,u_\gamma)$.

REMARK 1. The graphical representation of the above $\mathscr{P}(\mathscr{B})$-convergence is the following: A net $\{(D_\gamma,u_\gamma)\}_{\gamma\in\mathscr{G}}$ in $\mathscr{P}[X,Y]$ is $\mathscr{P}(\mathscr{B})$-convergent to $(D,u) \in \mathscr{P}[X,Y]$, if for each $B \in \mathscr{B}$ and $\varepsilon > 0$, both the inclusions $Gr(u_\gamma) \cap (B\times Y) \subset Gr(u)^{\varepsilon}$ and $Gr(u) \cap (B\times Y) \subset Gr(u_\gamma)^{\varepsilon}$ hold eventually, where $Gr(u_\gamma)$ and $Gr(u)$ are the graphs of u_γ and u respectively. Here the enlargement is taken with respect to any metric compatible with the product uniformity. For definiteness we choose the box metric given by $(d\times\mu)((x_1,y_1),(x_2,y_2)) = \max\{d(x_1,x_2),\mu(y_1,y_2)\}$.

DEFINITION 11. [1] Let $\mathscr{B}$ be a bornology on (X,d) and $(E,v) \in \mathscr{P}[X,Y]$. We say that (E,v) is uniformly continuous relative to the bornology $\mathscr{B}$, if for every $B \in \mathscr{B}$ with $E \cap B \neq \phi$, the map

$$v : E \cap B \to Y$$

is uniformly continuous.

We say that (E,v) is strongly uniformly continuous relative to the bornology $\mathscr{B}$, if for every $B \in \mathscr{B}$ and for every $\varepsilon > 0$, there is $\eta > 0$ such that $\mu(v(x),v(w)) < \varepsilon$, whenever $d(x,w) < \eta$ and $x,w \in E \cap B^{\eta}$.

3. Bornological $\mathscr{I}$ convergence and $\mathscr{I}$-convergence of partial maps

In this section, we generalize the notion of bornological convergence and the notion of convergence of partial maps via ideals of directed sets and investigate their basic properties.

DEFINITION 12. Let $\mathscr{B}$ be a bornology on X. A net $\{D_\gamma\}_{\gamma\in\mathscr{G}}$ in $\mathscr{P}_0(X)$ is called $\mathscr{B}^-_{\mathscr{I}}$-convergent (lower bornological $\mathscr{I}$-convergent) to $D \in \mathscr{P}_0(X)$, if for every $B \in \mathscr{B}$ and $\varepsilon > 0$,

$$\{\gamma \in \mathscr{G} : \ D \cap B \subset D^{\varepsilon}_{\gamma}\} \in \mathscr{F}(\mathscr{I}).$$

In this case, we write $D \in \mathscr{B}^-_{\mathscr{I}} - \lim D_\gamma$.

Similarly, the net $\{D_\gamma\}_{\gamma\in\mathscr{G}}$ in $\mathscr{P}_0(X)$ is called $\mathscr{B}^+_{\mathscr{I}}$-convergent (upper bornological $\mathscr{I}$-convergent) to $D \in \mathscr{P}_0(X)$, if for every $B \in \mathscr{B}$ and $\varepsilon > 0$,

$$\{\gamma \in \mathscr{G} : \ D_\gamma \cap B \subset D^{\varepsilon}\} \in \mathscr{F}(\mathscr{I}).$$

In this case, we write $D \in \mathscr{B}^+_{\mathscr{I}} - \lim D_\gamma$.

If $D \in \mathscr{B}^-_{\mathscr{I}} - \lim D_\gamma$ as well as $D \in \mathscr{B}^+_{\mathscr{I}} - \lim D_\gamma$, then we say that the net $\{D_\gamma\}_{\gamma\in\mathscr{G}}$ is bornological $\mathscr{I}$-convergent to D and in this case, we write $D \in \mathscr{B}_{\mathscr{I}} - \lim D_\gamma$.

DEFINITION 13. Let $\mathscr{B}$ be a bornology on X. A net $\{(D_\gamma,u_\gamma)\}_{\gamma\in\mathscr{G}}$ in $\mathscr{P}[X,Y]$ is said to be $\mathscr{P}_{\mathscr{I}}{}^-(\mathscr{B})$-convergent to $(D,u) \in \mathscr{P}[X,Y]$, if for every $B \in \mathscr{B}$ and $\varepsilon > 0$,

$$\{\gamma \in \mathscr{G} : \forall B_1(\subset B),\ u(D \cap B_1) \subset [u_\gamma(D_\gamma \cap B_1^{\varepsilon})]^{\varepsilon}\} \in \mathscr{F}(\mathscr{I}).$$

In this case, we write $(D,u) \in \mathscr{P}_{\mathscr{I}}{}^-(\mathscr{B}) - \lim(D_\gamma,u_\gamma)$.

Similarly, the net $\{(D_\gamma,u_\gamma)\}_{\gamma\in\mathscr{G}}$ in $\mathscr{P}[X,Y]$ is said to be $\mathscr{P}_{\mathscr{I}}{}^+(\mathscr{B})$-convergent to $(D,u) \in \mathscr{P}[X,Y]$, if for every $B \in \mathscr{B}$ and $\varepsilon > 0$,

$$\{\gamma \in \mathscr{G} : \forall B_1(\subset B),\ u_\gamma(D_\gamma \cap B_1) \subset [u(D \cap B_1^{\varepsilon})]^{\varepsilon}\} \in \mathscr{F}(\mathscr{I}).$$

In this case, we write $(D,u) \in \mathscr{P}_{\mathscr{I}}{}^+(\mathscr{B}) - \lim(D_\gamma,u_\gamma)$.

If $(D,u) \in \mathscr{P}_{\mathscr{I}}^{-}(\mathscr{B}) - \lim(D_\gamma, u_\gamma)$ as well as $(D,u) \in \mathscr{P}_{\mathscr{I}}^{+}(\mathscr{B}) - \lim(D_\gamma, u_\gamma)$, then we say that the net $\{(D_\gamma, u_\gamma)\}_{\gamma \in \mathscr{G}}$ is $\mathscr{P}_{\mathscr{I}}(\mathscr{B})$-convergent to (D,u) and we denote it by $(D,u) \in \mathscr{P}_{\mathscr{I}}(\mathscr{B}) - \lim(D_\gamma, u_\gamma)$.

REMARK 2. Note that if $\mathscr{I}$ is a $\mathscr{G}$-*admissible* ideal, then $\mathscr{P}^{-}(\mathscr{B})$-convergence, $\mathscr{P}^{+}(\mathscr{B})$-convergence and $\mathscr{P}(\mathscr{B})$-convergence imply $\mathscr{P}_{\mathscr{I}}^{-}(\mathscr{B})$-convergence, $\mathscr{P}_{\mathscr{I}}^{+}(\mathscr{B})$-convergence and $\mathscr{P}_{\mathscr{I}}(\mathscr{B})$-convergence respectively. On the other hand if $\mathscr{I} = \mathscr{I}_0$, then $\mathscr{P}_{\mathscr{I}}^{-}(\mathscr{B})$-convergence, $\mathscr{P}_{\mathscr{I}}^{+}(\mathscr{B})$-convergence and $\mathscr{P}_{\mathscr{I}}(\mathscr{B})$-convergence coincide with $\mathscr{P}^{-}(\mathscr{B})$-convergence, $\mathscr{P}^{+}(\mathscr{B})$-convergence and $\mathscr{P}(\mathscr{B})$-convergence respectively as introduced by Beer et al. in [1].

We now cite an example of a net of partial maps which is $\mathscr{P}_{\mathscr{I}}^{+}(\mathscr{B})$-convergent but not $\mathscr{P}^{+}(\mathscr{B})$-convergent.

EXAMPLE 1. Let $X = [-1,0]$ and $Y = \mathbb{R}$ be two metric spaces with usual metric. Let $\mathscr{I}_d = \{A \in \mathbb{N} : d(A) = 0\}$, where $d(A)$ is the asymptotic density of the set A, defined by $d(A) = \lim_{n\to\infty} \frac{|A(n)|}{n}$, where $A(n) = \{j \in A : j \leqslant n\}$ and $|A(n)|$ represents the number of elements in $A(n)$. Then $\mathscr{I}_d$ is a non-trivial ideal in $\mathbb{N}$. Let $A = \{k^3 : k \in \mathbb{N}\}$. Then A is an infinite subset of $\mathbb{N}$ and $A \in \mathscr{I}_d$. Let us consider the function $u : X \to Y$, defined by $u(x) = 0$, $\forall x \in X$. Now for every $n \in A$, let us define $u_n : X \to Y$ by

$$u_n(x) = \begin{cases} -nx - 1, & \text{if } -\frac{1}{n} \leqslant x \\ u(x), & \text{otherwise,} \end{cases}$$

and for every $n \notin A$, we define, $u_n : X \to Y$ by $u_n(x) = u(x)$, $\forall x \in X$. Then with respect to the bornology $\mathscr{B} = \mathscr{P}_0(X)$, the sequence of partial maps $\{(X,u_n)\}_{n\in\mathbb{N}}$ is not $\mathscr{P}^{+}(\mathscr{B})$-convergent to (X,u). To show this, we choose $B = [-\frac{5}{6},0] \in \mathscr{B}$ and $\varepsilon = \frac{1}{3}$. Then for $B_1 = [-\frac{1}{2},0]$, $B_1^{\varepsilon} \cap X = [-\frac{5}{6},0]$, $u(B_1^{\varepsilon} \cap X) = \{0\}$ and $[u(B_1^{\varepsilon} \cap X)]^{\varepsilon} = (-\frac{1}{3},\frac{1}{3})$. Now for $n > 1$ and $n \in A$, $u_n(B_1 \cap X) = [-1,0]$ and so $u_n(B_1 \cap X) \not\subset [u(B_1^{\varepsilon} \cap X)]^{\varepsilon}$. This shows that, with respect to the bornology $\mathscr{B} = P_0(X)$, the sequence of partial maps $\{(X,u_n)\}_{n\in\mathbb{N}}$ is not $\mathscr{P}^{+}(\mathscr{B})$-convergent to (X,u). Moreover, the sequence is not $\mathscr{P}^{+}(\mathscr{B})$-convergent to any partial map.

Now, let $B \in \mathscr{B} = \mathscr{P}_0(X)$ and $\varepsilon > 0$ be given. Then for all $B_1 \subset B$ and $n \notin A$, $u_n(B_1 \cap X) = \{0\}$ and $u(B_1^{\varepsilon} \cap X) = \{0\}$ and so $[u(B_1^{\varepsilon} \cap X)]^{\varepsilon} = (-\varepsilon, \varepsilon)$. Thus $u_n(B_1 \cap X) \subset [u(B_1^{\varepsilon} \cap X)]^{\varepsilon}$. Therefore,

$$(\mathbb{N} \setminus A) \subset \{n \in \mathbb{N} : \forall B_1 (\subset B), u_n(B_1 \cap X) \subset [u(B_1^{\varepsilon} \cap X)]^{\varepsilon}\}.$$

Since $(\mathbb{N} \setminus A) \in \mathscr{F}(\mathscr{I}_d)$, it follows that

$$\{n \in \mathbb{N} : \forall B_1 (\subset B), u_n(B_1 \cap X) \subset [u(B_1^{\varepsilon} \cap X)]^{\varepsilon}\} \in \mathscr{F}(\mathscr{I}_d).$$

Thus, with respect to the bornology $\mathscr{B} = \mathscr{P}_0(X)$, the sequence of partial maps $\{(X,u_n)\}_{n\in\mathbb{N}}$ is $\mathscr{P}_{\mathscr{I}}^{+}(\mathscr{B})$-convergent to (X,u).

From the above example, it is clear that $\mathscr{P}_{\mathscr{I}}(\mathscr{B})$-convergence is a generalization of $\mathscr{P}(\mathscr{B})$-convergence.

Now we characterize the notions of $\mathscr{P}_{\mathscr{I}}^{-}(\mathscr{B})$-convergence and $\mathscr{P}_{\mathscr{I}}^{+}(\mathscr{B})$-convergence.

THEOREM 1. *Let* $\{(D_\gamma, u_\gamma)\}_{\gamma\in\mathscr{G}}$ *be a net in* $\mathscr{P}[X,Y]$ *and* $\mathscr{B}$ *be a bornology on* X.

(*a*) *If* $(D,u) \in \mathscr{P}_{\mathscr{I}}^{-}(\mathscr{B}) - \lim\ (D_\gamma, u_\gamma)$, *then* $\forall B \in \mathscr{B}$ *and* $\forall \varepsilon > 0$,

$$\{\gamma \in \mathscr{G} :\ D \cap B \subset D_\gamma^\varepsilon\} \in \mathscr{F}(\mathscr{I}).$$

(*b*) *If* $(D,u) \in \mathscr{P}_{\mathscr{I}}^{+}(\mathscr{B}) - \lim\ (D_\gamma, u_\gamma)$, *then* $\forall B \in \mathscr{B}$ *and*

$$\forall \varepsilon > 0,\ \ \{\gamma \in \mathscr{G} :\ D_\gamma \cap B \subset D^\varepsilon\} \in \mathscr{F}(\mathscr{I}).$$

Proof. (*a*) Let $B \in \mathscr{B}$ and $\varepsilon > 0$ be given. By assumption, we have

$$A = \{\gamma \in \mathscr{G} :\ \forall B_1(\subset B), u(D \cap B_1) \subset [u_\gamma(D_\gamma \cap B_1^\varepsilon)]^\varepsilon\} \in \mathscr{F}(\mathscr{I}).$$

Since $\mathscr{I}$ is non-trivial, $A \neq \emptyset$. Choose $\gamma_1 \in A$. Let $x \in D \cap B$. Then considering $B_1 = \{x\}$, we have

$$u(x) \in [u_{\gamma_1}(D_{\gamma_1} \cap \{x\}^\varepsilon)]^\varepsilon.$$

Then, for some $v \in D_{\gamma_1} \cap B_d(x,\varepsilon)$, we have $\mu(u(x), u_{\gamma_1}(v)) < \varepsilon$. In particular, $x \in B_d(v,\varepsilon) \subset D_{\gamma_1}^\varepsilon$. Since $x \in D \cap B$ is arbitrary, $D \cap B \subset D_{\gamma_1}^\varepsilon$. Therefore $A \subset \{\gamma \in \mathscr{G} : D \cap B \subset D_\gamma^\varepsilon\}$. Since $A \in \mathscr{F}(\mathscr{I})$, we have $\{\gamma \in \mathscr{G} : D \cap B \subset D_\gamma^\varepsilon\} \in \mathscr{F}(\mathscr{I})$. This completes the proof.

(*b*) The proof is similar to that of (*a*), so it is omitted. □

THEOREM 2. *Let* $\{(D_\gamma, u_\gamma)\}_{\gamma\in\mathscr{G}}$ *be a net in* $\mathscr{P}[X,Y]$ *and* $\mathscr{B}$ *be a bornology on* X. *Then*

(*a*) $(D,u) \in \mathscr{P}_{\mathscr{I}}^{+}(\mathscr{B}) - \lim\ (D_\gamma, u_\gamma)$ *if and only if for every* $B \in \mathscr{B}$ *and* $\varepsilon > 0$,

$$\{\gamma \in \mathscr{G} : \sup_{z \in D_\gamma \cap B}\ \inf_{x \in D \cap B_d(z,\varepsilon)} \mu(u(x), u_\gamma(z)) < \varepsilon\} \in \mathscr{F}(\mathscr{I}).$$

(*b*) $(D,u) \in \mathscr{P}_{\mathscr{I}}^{-}(\mathscr{B}) - \lim\ (D_\gamma, u_\gamma)$ *if and only if for every* $B \in \mathscr{B}$ *and* $\varepsilon > 0$,

$$\{\gamma \in \mathscr{G} : \sup_{z \in D \cap B}\ \inf_{x \in D_\gamma \cap B_d(z,\varepsilon)} \mu(u(z), u_\gamma(x)) < \varepsilon\} \in \mathscr{F}(\mathscr{I}).$$

Proof. (*a*) Let $(D,u) \in \mathscr{P}_{\mathscr{I}}^{+}(\mathscr{B}) - \lim\ (D_\gamma, u_\gamma)$. Let $B \in \mathscr{B}$ and $\varepsilon > 0$ be given. Then we have,

$$A = \{\gamma \in \mathscr{G} : \forall B_1(\subset B),\ u_\gamma(D_\gamma \cap B_1) \subset [u(D \cap B_1^{\frac{\varepsilon}{2}})]^{\frac{\varepsilon}{2}}\} \in \mathscr{F}(\mathscr{I}).$$

Let $\gamma \in A$ and $z \in D_\gamma \cap B$. We choose $B_1 = \{z\}$. Then $u_\gamma(z) \in [u(D \cap B_1^{\frac{\varepsilon}{2}})]^{\frac{\varepsilon}{2}}$. This means that, for some $x \in D \cap B_d(z, \frac{\varepsilon}{2})$, we have $\mu(u_\gamma(z), u(x)) < \frac{\varepsilon}{2}$. Therefore, $\inf_{x \in D \cap B_d(z,\frac{\varepsilon}{2})} \mu(u_\gamma(z), u(x)) < \frac{\varepsilon}{2}$. Since $z \in D_\gamma \cap B$ is arbitrary, we have

$$\sup_{z \in D_\gamma \cap B}\ \inf_{x \in D \cap B_d(z,\frac{\varepsilon}{2})} \mu(u_\gamma(z), u(x)) \leqslant \frac{\varepsilon}{2} < \varepsilon.$$

Then clearly,

$$\sup_{z\in D_\gamma\cap B}\ \inf_{x\in D\cap B_d(z,\varepsilon)}\mu(u_\gamma(z),u(x)) \leqslant \sup_{z\in D_\gamma\cap B}\ \inf_{x\in D\cap B_d(z,\frac{\varepsilon}{2})}\mu(u_\gamma(z),u(x)) < \varepsilon.$$

Thus $A\subset\{\gamma\in\mathscr{G}: \sup_{z\in D_\gamma\cap B}\ \inf_{x\in D\cap B_d(z,\varepsilon)}\mu(u_\gamma(z),u(x)) < \varepsilon\}$. Since $A\in\mathscr{F}(\mathscr{I})$, we have $\{\gamma\in\mathscr{G}: \sup_{z\in D_\gamma\cap B}\ \inf_{x\in D\cap B_d(z,\varepsilon)}\mu(u_\gamma(z),u(x)) < \varepsilon\}\in\mathscr{F}(\mathscr{I})$. Therefore, the condition is necessary.

Conversely, assume that the given condition holds. Let $B\in\mathscr{B}$ and $\varepsilon>0$. Then

$$A=\{\gamma\in\mathscr{G}: \sup_{z\in D_\gamma\cap B}\ \inf_{x\in D\cap B_d(z,\varepsilon)}\mu(u_\gamma(z),u(x)) < \varepsilon\}\in\mathscr{F}(\mathscr{I}).$$

Let $\gamma\in A$. If $B_1\subset B$, then

$$\sup_{z\in D_\gamma\cap B_1}\ \inf_{x\in D\cap B_d(z,\ \varepsilon)}\mu(u_\gamma(z),u(x)) < \varepsilon.$$

Therefore, for all $z\in D_\gamma\cap B_1$, there exists $x\in D\cap\{z\}^\varepsilon\subset D\cap B_1^\varepsilon$ such that $\mu(u_\gamma(z),u(x))<\varepsilon$. Thus

$$A\subset\left\{\gamma\in\mathscr{G}:\forall B_1(\subset B),\ u_\gamma(D_\gamma\cap B_1)\subset[u(D\cap B_1^\varepsilon)]^\varepsilon\right\}.$$

Since $A\in\mathscr{F}(\mathscr{I})$, we have

$$\left\{\gamma\in\mathscr{G}:\forall B_1(\subset B),\ u_\gamma(D_\gamma\cap B_1)\subset[u(D\cap B_1^\varepsilon)]^\varepsilon\right\}\in\mathscr{F}(\mathscr{I}).$$

This gives, $(D,u)\in\mathscr{P}_{\mathscr{I}}{}^{+}(\mathscr{B})-\lim\ (D_\gamma,u_\gamma)$.

(b) The proof is similar to that of (a), so it is omitted. □

Now if $\mathscr{B}$ is a bornology on X, then $\{B\times Y: B\in\mathscr{B}\}$ forms a base for some bornology $\mathscr{B}^*$ (say) on $X\times Y$. Using this bornology $\mathscr{B}^*$ on $X\times Y$, we now show that the $\mathscr{P}_{\mathscr{I}}{}^{-}(\mathscr{B})$ and $\mathscr{P}_{\mathscr{I}}{}^{+}(\mathscr{B})$ convergences in $\mathscr{P}[X,Y]$ are actually the lower and the upper bornological $\mathscr{I}$-convergences of graphs respectively in $X\times Y$, which extend the results of [10] as well as of [1].

THEOREM 3. *Let* $\{(D_\gamma,u_\gamma)\}_{\gamma\in\mathscr{G}}$ *be a net in* $\mathscr{P}[X,Y]$ *and* $\mathscr{B}$ *be a bornology on* X. *Then, for* $(D,u)\in\mathscr{P}[X,Y]$, *the following equivalences hold:*

(a) $Gr(u)\in(\mathscr{B}^*_{\mathscr{I}})^{-}-\lim Gr(u_\gamma)$ *if and only if* $(D,u)\in\mathscr{P}_{\mathscr{I}}{}^{-}(\mathscr{B})-\lim(D_\gamma,u_\gamma)$

(b) $Gr(u)\in(\mathscr{B}^*_{\mathscr{I}})^{+}-\lim Gr(u_\gamma)$ *if and only if* $(D,u)\in\mathscr{P}_{\mathscr{I}}{}^{+}(\mathscr{B})-\lim(D_\gamma,u_\gamma)$.

Proof. (a) Let $(D,u)\in\mathscr{P}_{\mathscr{I}}{}^{-}(\mathscr{B})-\lim(D_\gamma,u_\gamma)$. To verify bornological convergence of graphs, it is suffices to work with the basic sets in $\mathscr{B}^*$. Let $B\times Y\in\mathscr{B}^*$, where $B\in\mathscr{B}$. Let $\varepsilon>0$ be given. Then by Theorem 1 and Theorem 2, we have

$$A=\{\gamma\in\mathscr{G}: D\cap B\subset D_\gamma^\varepsilon\}\in\mathscr{F}(\mathscr{I})$$

and

$$C=\{\gamma\in\mathscr{G}: \sup_{z\in D\cap B}\ \inf_{x\in D_\gamma\cap B_d(z,\varepsilon)}\mu(u(z),u_\gamma(x)) < \varepsilon\}\in\mathscr{F}(\mathscr{I}).$$

Now $A,\ C \in \mathscr{F}(\mathscr{I}) \Rightarrow A \cap C \in \mathscr{F}(\mathscr{I})$. Let $\gamma \in A \cap C$. Then both

$$D \cap B \subset D_\gamma^\varepsilon \tag{1}$$

and

$$\sup_{z \in D \cap B} \inf_{x \in D_\gamma \cap B_d(z,\varepsilon)} \mu(u(z), u_\gamma(x)) < \varepsilon \tag{2}$$

hold. Now fix $(z, u(z)) \in (B \times Y) \cap Gr(u)$ so that $z \in D$. Then by (1), $B_d(z,\varepsilon) \cap D_\gamma \neq \phi$ and by (2), there exists some $x \in B_d(z,\varepsilon) \cap D_\gamma$, so that $\mu(u_\gamma(x), u(z)) < \varepsilon$. So we have, $(x, u_\gamma(x)) \in Gr(u_\gamma)$ and $(d \times \mu)((z,u(z)),(x,u_\gamma(x))) < \varepsilon$. This gives, $Gr(u) \cap (B \times Y) \subset Gr(u_\gamma)^\varepsilon$. Therefore,

$$A \cap C \subset \{\gamma \in \mathscr{G} :\ Gr(u) \cap (B \times Y) \subset Gr(u_\gamma)^\varepsilon\}.$$

Since $A \cap C \in \mathscr{F}(\mathscr{I})$, we have $\{\gamma \in \mathscr{G} :\ Gr(u) \cap (B \times Y) \subset Gr(u_\gamma)^\varepsilon\} \in \mathscr{F}(\mathscr{I})$. Hence $Gr(u) \in (\mathscr{B}^*_{\mathscr{I}})^- - \lim\ Gr(u_\gamma)$.

Conversely, let $Gr(u) \in (\mathscr{B}^*_{\mathscr{I}})^- - \lim\ Gr(u_\gamma)$. Let $B \in \mathscr{B}$ and $\varepsilon > 0$ be given. Choosing $0 < \eta < \varepsilon$, we have

$$A_1 = \{\gamma \in \mathscr{G} :\ Gr(u) \cap (B \times Y) \subset Gr(u_\gamma)^\eta\} \in \mathscr{F}(\mathscr{I}).$$

Let $\gamma \in A_1$ and $z \in D \cap B$ be arbitrary. Then $(z, u(z)) \in (B \times Y) \cap Gr(u)$. So, there exists $(y_0, u_\gamma(y_0)) \in Gr(u_\gamma)$ such that

$$(d \times \mu)((z,u(z)),(y_0,u_\gamma(y_0))) < \eta.$$

Thus we get, $y_0 \in D_\gamma$ such that $d(z, y_0) < \eta < \varepsilon$ as well as $\mu(u(z), u_\gamma(y_0)) < \eta$. Therefore

$$\inf_{x \in B_d(z,\varepsilon) \cap D_\gamma} \mu(u(z), u_\gamma(x)) < \eta.$$

Since $z \in D \cap B$ is arbitrary, we have

$$\sup_{z \in D \cap B} \inf_{x \in B_d(z,\varepsilon) \cap D_\gamma} \mu(u(z), u_\gamma(x)) \leqslant \eta < \varepsilon.$$

Thus,

$$A_1 \subset \{\gamma \in \mathscr{G} : \sup_{z \in D \cap B} \inf_{x \in B_d(z,\varepsilon) \cap D_\gamma} \mu(u(z), u_\gamma(x)) < \varepsilon\}.$$

Since $A_1 \in \mathscr{F}(\mathscr{I})$, we have $\{\gamma \in \mathscr{G} : \sup_{z \in D \cap B} \inf_{x \in B_d(z,\varepsilon) \cap D_\gamma} \mu(u(z), u_\gamma(x)) < \varepsilon\} \in \mathscr{F}(\mathscr{I})$. Then, by Theorem 2 (b), we have $(D,u) \in \mathscr{P}_{\mathscr{I}}{}^-(\mathscr{B}) - \lim(D_\gamma, u_\gamma)$. This completes the proof.

(b) The proof of (b) is similar to that of (a), so it is omitted. □

COROLLARY 1. *Let $\{(D_\gamma, u_\gamma)\}_{\gamma \in \mathscr{G}}$ be a net in $\mathscr{P}[X,Y]$ and $\mathscr{B}$ be a bornology on X. Then $\mathscr{P}_{\mathscr{I}}(\mathscr{B})$-convergence on $\mathscr{P}[X,Y]$ coincide with $\mathscr{B}^*_{\mathscr{I}}$-convergence of graphs of partial maps in $X \times Y$.*

THEOREM 4. *Let $\{(D_\gamma,u_\gamma)\}_{\gamma\in\mathscr{G}}$ be a net in $\mathscr{P}[X,Y]$ and $\mathscr{B}$ be a local bornology on X. If the net $\{(D_\gamma,u_\gamma)\}_{\gamma\in\mathscr{G}}$ is $\mathscr{P}_{\mathscr{I}}(\mathscr{B})$-convergent to both the partial maps (S,u) and (T,v), then $S=T$.*

Proof. Let the net $\{(D_\gamma,u_\gamma)\}_{\gamma\in\mathscr{G}}$ be $\mathscr{P}_{\mathscr{I}}(\mathscr{B})$-convergent to both the partial maps (S,u) and (T,v). If possible, let $S\neq T$. Then, without any loss of generality, we assume $x\in S\setminus T$. Since T is a closed subset of X, we have $\{x\}^\varepsilon\cap T=\emptyset$, for some $\varepsilon>0$. Choose $0<\delta<\frac{\varepsilon}{2}$, so that $\{x\}^\delta\in\mathscr{B}$. Let $B=\{x\}^\delta$. Since the net is $\mathscr{P}_{\mathscr{I}}{}^{-}(\mathscr{B})$-convergent to (S,u), we have

$$\{\gamma\in\mathscr{G}:\forall B_1(\subset B),u(S\cap B_1)\subset[u_\gamma(D_\gamma\cap B_1^\delta)]^\delta\}\in\mathscr{F}(\mathscr{I}).$$

Let us take $B_1=\{x\}$. Then we have

$$P=\{\gamma\in\mathscr{G}:u(S\cap\{x\})\subset[u_\gamma(D_\gamma\cap\{x\}^\delta)]^\delta\}\in\mathscr{F}(\mathscr{I}).$$

Since $u(S\cap\{x\})\neq\emptyset$, we have $u_\gamma(D\gamma\cap\{x\}^\delta)\neq\emptyset$, for all $\gamma\in P$. Now

$$v(T\cap B^{\frac{\varepsilon}{2}})\subset v(T\cap\{x\}^\varepsilon)=\emptyset.$$

Thus for all $\gamma\in P$, we have

$$u_\gamma(D_\gamma\cap B)\not\subset[v(T\cap B^{\frac{\varepsilon}{2}})]^{\frac{\varepsilon}{2}}.$$

Hence

$$P\subset\{\gamma\in\mathscr{G}:u_\gamma(D_\gamma\cap B)\not\subset[v(T\cap B^{\frac{\varepsilon}{2}})]^{\frac{\varepsilon}{2}}\}.$$

Since $P\in\mathscr{F}(\mathscr{I})$, we have

$$\{\gamma\in\mathscr{G}:u_\gamma(D_\gamma\cap B)\not\subset[v(T\cap B^{\frac{\varepsilon}{2}})]^{\frac{\varepsilon}{2}}\}\in\mathscr{F}(\mathscr{I}).$$

Thus

$$\{\gamma\in\mathscr{G}:u_\gamma(D_\gamma\cap B)\subset[v(T\cap B^{\frac{\varepsilon}{2}})]^{\frac{\varepsilon}{2}}\}\notin\mathscr{F}(\mathscr{I}),$$

which is a contradiction, since $\{(D_\gamma,u_\gamma)\}_{\gamma\in\mathscr{G}}$ is $\mathscr{P}_{\mathscr{I}}(\mathscr{B})$-convergent to (T,v). Therefore $S=T$. □

THEOREM 5. *Let $\mathscr{B}$ be a bornology on X. Then $\mathscr{P}_{\mathscr{I}_\mathscr{G}}(\mathscr{B})$ limits of each net in $\mathscr{C}[X,Y]$ are unique if and only if $\mathscr{B}$ is local, where $\mathscr{I}_\mathscr{G}$ is any $\mathscr{G}$-admissible ideal on $\mathscr{G}$.*

Proof. Let $\mathscr{B}$ be a local bornology on X. Let (S,v) and (T,u) be two partial maps in $\mathscr{C}[X,Y]$ to which a net $\{(D_\gamma,u_\gamma)\}_{\gamma\in\mathscr{G}}\in\mathscr{C}[X,Y]$ be $\mathscr{P}_{\mathscr{I}_\mathscr{G}}(\mathscr{B})$-convergent. Since $\mathscr{B}$ is local, then, by Theorem 4, $S=T$. Now suppose that there exists $x\in S$ so that $u(x)\neq v(x)$. By continuity of u, there exists $\varepsilon^1>0$ so small, that

$$(u(T\cap B_d(x,2\varepsilon^1)))^{\varepsilon^1}\cap B_\mu(v(x),\varepsilon^1)=\emptyset, \tag{3}$$

and $B_d(x,\varepsilon^1)\in\mathscr{B}$. Since the net is $\mathscr{P}_{\mathscr{I}_\mathscr{G}}(\mathscr{B})$-convergent to (T,u), the net is $\mathscr{P}_{\mathscr{I}_\mathscr{G}}{}^{+}(\mathscr{B})$-convergent to (T,u). Thus we have,

$$P = \{\gamma \in \mathscr{G} : u_\gamma(D_\gamma \cap B_d(x,\varepsilon^1)) \subset [u(T \cap (B_d(x,\varepsilon^1))^{\varepsilon^1})]^{\varepsilon^1}\} \in \mathscr{F}(\mathscr{I}_\mathscr{G}).$$

Again, since the net is $\mathscr{P}_{\mathscr{I}_\mathscr{G}}(\mathscr{B})$-convergent to (S,v), the net is $\mathscr{P}_{\mathscr{I}_\mathscr{G}}{}^{-}(\mathscr{B})$-convergent to (S,v). Thus we have,

$$Q = \{\gamma \in \mathscr{G} : \sup_{y \in S \cap B_d(x,\varepsilon^1)} \inf_{z \in B_d(y,\varepsilon^1) \cap D_\gamma} \mu(v(y), u_\gamma(z)) < \varepsilon^1\} \in \mathscr{F}(\mathscr{I}_\mathscr{G}).$$

Since $P, Q \in \mathscr{F}(\mathscr{I}_\mathscr{G})$, we have $P \cap Q \in \mathscr{F}(\mathscr{I}_\mathscr{G})$ and so $P \cap Q \neq \emptyset$. Let $\omega \in P \cap Q$. Then there exists $z_\omega \in B_d(x,\varepsilon^1) \cap D_\omega$ such that $u_\omega(z_\omega) \in B_\mu(v(x),\varepsilon^1)$ and

$$u_\omega(z_\omega) \in u_\omega(D_\omega \cap B_d(x,\varepsilon^1)) \subset [u(T \cap (B_d(x,\varepsilon^1))^{\varepsilon^1})]^{\varepsilon^1} \subset [u(T \cap B_d(x,2\varepsilon^1))]^{\varepsilon^1},$$

which contradicts (3). Hence $\mathscr{P}_{\mathscr{I}_\mathscr{G}}(\mathscr{B})$ limits of each net in $\mathscr{C}[X,Y]$ are unique.

Conversely, let the bornology $\mathscr{B}$ be not local. We show that there exists a net of continuous partial maps which has different $\mathscr{P}_{\mathscr{I}_\mathscr{G}}(\mathscr{B})$ limits. Since $\mathscr{B}$ is not local, there exists $\alpha_0 \in X$, such that for each $n \in \mathbb{N}$ and $B \in \mathscr{B}$, we can choose $\alpha(n,B) \in B_d(\alpha_0, \frac{1}{n}) \setminus B$. Let us choose $\beta \neq \alpha_0$ in X and $\zeta \in Y$. We consider the directed set $\mathscr{G} = \mathbb{N} \times \mathscr{B}$ with the product direction, where $\mathbb{N}$ is directed in the usual way and $\mathscr{B}$ is directed by subset inclusion. We consider $D_{(n,B)} = \{\alpha(n,B), \beta\}$ and the continuous mapping $u_{(n,B)}$ that maps both $\alpha(n,B)$ and β to ζ, for every $n \in \mathbb{N}$ and $B \in \mathscr{B}$. Then the net of continuous partial maps $\{(D_{(n,B)}, u_{(n,B)})\}_{(n,B) \in \mathbb{N} \times \mathscr{B}}$ is $\mathscr{P}(\mathscr{B})$-convergent to the continuous partial maps (S,u) and (T,v), where $S = \{\alpha_0, \beta\}$, $T = \{\beta\}$, $u(S) = \{\zeta\}$ and $v(T) = \{\zeta\}$. So for any $\mathscr{G}$-*admissible* ideal $\mathscr{I}_\mathscr{G}$ of $\mathscr{G} = \mathbb{N} \times \mathscr{B}$, the net $\{(D_{(n,B)}, u_{(n,B)})\}_{(n,B) \in \mathbb{N} \times \mathscr{B}}$ of continuous partial maps is $\mathscr{P}_{\mathscr{I}_\mathscr{G}}(\mathscr{B})$-convergent to both (S,u) and (T,v). This completes the proof. □

THEOREM 6. *Let $\mathscr{B}$ be a bornology on X and $(D,u) \in \mathscr{P}[X,Y]$ be strongly uniformly continuous relative to $\mathscr{B}$. Then a net $\{(D_\gamma, u_\gamma)\}_{\gamma \in \mathscr{G}}$ in $\mathscr{P}[X,Y]$ is $\mathscr{P}_\mathscr{I}{}^{+}(\mathscr{B})$-convergent to (D,u) if and only if $\forall B \in \mathscr{B}$ and $\varepsilon > 0$, there exists $\zeta > 0$ such that*

$$\{\gamma \in \mathscr{G} : \sup_{z \in D_\gamma \cap B} \sup_{x \in B_d(z,\zeta) \cap D} \mu(u(x), u_\gamma(z)) < \varepsilon\} \in \mathscr{F}(\mathscr{I}).$$

Proof. Sufficient part of the statement follows from Theorem 2. We only show that the condition is necessary.

Let a net $\{(D_\gamma, u_\gamma)\}_{\gamma \in \mathscr{G}} \in \mathscr{P}[X,Y]$ be $\mathscr{P}_\mathscr{I}{}^{+}(\mathscr{B})$-convergent to (D,u). Let $B \in \mathscr{B}$ and $\varepsilon > 0$ be given. Let also $\eta > 0$ be such that $2\eta < \varepsilon$. By strong uniform continuity of (D,u) relative to $\mathscr{B}$, there exists δ, $0 < \delta < 2\eta$, such that $x, y \in D \cap B^\delta$ and $d(x,y) < \delta$ implies $\mu(u(x), u(y)) < \eta$. Now since $(D,u) \in \mathscr{P}_\mathscr{I}{}^{+}(\mathscr{B}) - \lim(D_\gamma, u_\gamma)$, by Theorem 1 and Theorem 2, we have

$$A = \{\gamma \in \mathscr{G} : D_\gamma \cap B \subset D^{\frac{\delta}{2}}\} \in \mathscr{F}(\mathscr{I})$$

and

$$C = \{\gamma \in \mathscr{G} : \sup_{z \in D_\gamma \cap B} \inf_{x \in B_d(z,\frac{\delta}{2}) \cap D} \mu(u(x), u_\gamma(z)) < \tfrac{\delta}{2}\} \in \mathscr{F}(\mathscr{I}).$$

Thus $A\cap C\in\mathscr{F}(\mathscr{I})$. Choose $\gamma\in A\cap C$. Then for every $z\in B\cap D_\gamma$, there exists $x_z\in B_d(z,\frac{\delta}{2})\cap D$ such that $\mu(u(x_z),u_\gamma(z))<\frac{\delta}{2}<\eta$. Since $B_d(z,\frac{\delta}{2})\subset B^\delta$, we have $B_d(z,\frac{\delta}{2})\cap D\subset B^\delta\cap D$. Let $x\in B_d(z,\frac{\delta}{2})\cap D\subset B^\delta\cap D$. Then $d(x,x_z)<\delta$. Then by strong uniform continuity of (D,u), we have $\mu(u(x),u(x_z))<\eta$. Thus for every $x\in B_d(z,\frac{\delta}{2})\cap D$, we have

$$\mu(u(x),u_\gamma(z))\leqslant\mu(u(x),u(x_z))+\mu(u(x_z),u_\gamma(z))<\eta+\eta=2\eta.$$

Hence

$$\sup_{z\in D_\gamma\cap B}\ \sup_{x\in B_d(z,\frac{\delta}{2})\cap D}\mu(u(x),u_\gamma(z))\leqslant 2\eta<\varepsilon.$$

Choose $\zeta=\frac{\delta}{2}$. Then we have,

$$A\cap C\subset\{\gamma\in\mathscr{G}:\sup_{z\in D_\gamma\cap B}\ \sup_{x\in B_d(z,\zeta)\cap D}\mu(u(x),u_\gamma(z))<\varepsilon\}.$$

Since $A\cap C\in\mathscr{F}(\mathscr{I})$, we have $\{\gamma\in\mathscr{G}:\sup_{z\in D_\gamma\cap B}\ \sup_{x\in B_d(z,\zeta)\cap D}\mu(u(x),u_\gamma(z))<\varepsilon\}\in\mathscr{F}(\mathscr{I})$. This completes the proof. □

THEOREM 7. *Let $\mathscr{B}$ be a bornology on X that is stable under small enlargements and $(D,u)\in\mathscr{P}[X,Y]$ be uniformly continuous relative to $\mathscr{B}$. Then a net $\{(D_\gamma,u_\gamma)\}_{\gamma\in\mathscr{G}}$ in $\mathscr{P}[X,Y]$ is $\mathscr{P}_{\mathscr{I}}(\mathscr{B})$-convergent to (D,u) if and only if both*

(a) for every $B\in\mathscr{B}$ and $\varepsilon>0$, there exists $\zeta>0$ such that

$$\{\gamma\in\mathscr{G}:\sup_{z\in D_\gamma\cap B}\ \sup_{x\in B_d(z,\zeta)\cap D}\mu(u(x),u_\gamma(z))<\varepsilon\}\in\mathscr{F}(\mathscr{I})\ ;$$

and

(b) for every $B\in\mathscr{B}$ and $\varepsilon>0$, $\{\gamma\in\mathscr{G}:D\cap B\subset D_\gamma^\varepsilon\}\in\mathscr{F}(\mathscr{I})$ hold.

Proof. Since (D,u) is uniformly continuous relative to $\mathscr{B}$ and $\mathscr{B}$ is stable under small enlargements, it follows that (D,u) is strongly uniformly continuous relative to $\mathscr{B}$. Then, by Theorem 1 and Theorem 6, we see that the conditions are necessary.

Conversely, let the conditions hold. Let $B\in\mathscr{B}$ and $\varepsilon>0$ be given. It is sufficient to show that

$$\{\gamma\in\mathscr{G}:\sup_{z\in D\cap B}\ \inf_{x\in B_d(z,\varepsilon)\cap D_\gamma}\mu(u(z),u_\gamma(x))<\varepsilon\}\in\mathscr{F}(\mathscr{I}).$$

Choose $\zeta<\varepsilon$ so small that $B^\zeta\in\mathscr{B}$,

$$C=\{\gamma\in\mathscr{G}:\sup_{z\in D_\gamma\cap B^\zeta}\ \sup_{x\in B_d(z,\zeta)\cap D}\mu(u(x),u_\gamma(z))<\tfrac{\varepsilon}{2}\}\in\mathscr{F}(\mathscr{I})$$

and

$$E=\{\gamma\in\mathscr{G}:D\cap B\subset D_\gamma^\zeta\}\in\mathscr{F}(\mathscr{I}).$$

Choose $\gamma\in C\cap E$. Let $z\in D\cap B$. Then there exists $x(z,\gamma)\in D_\gamma\cap B^\zeta$ with $d(z,x(z,\gamma))<\zeta$. Since $\gamma\in C$, $x(z,\gamma)\in D_\gamma\cap B^\zeta$ and $z\in B_d(x(z,\gamma),\zeta)\cap D$, we have $\mu(u(z),u_\gamma(x(z,\gamma))) < \frac{\varepsilon}{2}$. Thus $\inf\limits_{x\in B_d(z,\zeta)\cap D_\gamma}\mu(u(z),u_\gamma(x))<\frac{\varepsilon}{2}$. Hence

$$\sup_{z\in D\cap B}\ \inf_{x\in B_d(z,\varepsilon)\cap D_\gamma}\mu(u(z),u_\gamma(x))\leqslant \sup_{z\in D\cap B}\ \inf_{x\in B_d(z,\zeta)\cap D_\gamma}\mu(u(z),u_\gamma(x))\leqslant\frac{\varepsilon}{2}<\varepsilon.$$

Therefore, $C\cap E\subset\{\gamma\in\mathscr{G}:\ \sup\limits_{z\in D\cap B}\ \inf\limits_{x\in B_d(z,\varepsilon)\cap D_\gamma}\mu(u(z),u_\gamma(x))<\varepsilon\}$. Since $C\cap E\in\mathscr{F}(\mathscr{I})$, we have $\{\gamma\in\mathscr{G}:\ \sup\limits_{z\in D\cap B}\ \inf\limits_{x\in B_d(z,\varepsilon)\cap D_\gamma}\mu(u(z),u_\gamma(x))<\varepsilon\}\in\mathscr{F}(\mathscr{I})$. This completes the proof. □

4. $\mathscr{I}^*$-convergence of partial maps

In this section, we introduce the notions of bornological $\mathscr{I}^*$-convergence and $\mathscr{I}^*$-convergence of nets of partial maps and study their relationship with bornological $\mathscr{I}$-convergence and $\mathscr{I}$-convergence of nets of partial maps.

DEFINITION 14. Let (X,d) be a metric space and $\mathscr{B}$ be a bornology on X. A net $\{D_\gamma\}_{\gamma\in\mathscr{G}}\in\mathscr{P}_0(X)$ is said to be $\mathscr{B}_{\mathscr{I}^*}$-convergent (bornological $\mathscr{I}^*$ convergent) to $D\in\mathscr{P}_0(X)$, if there exists a set $\mathscr{G}'\in\mathscr{F}(\mathscr{I})$ such that $\mathscr{G}'$ itself is a directed set with respect to the binary relation induced from $(\mathscr{G},\geqslant)$ and the net $\{D_\gamma\}_{\gamma\in\mathscr{G}'}$ is $\mathscr{B}$-convergent to D.

In this case, we write $D\in\mathscr{B}_{\mathscr{I}^*}-\lim D_\gamma$.

THEOREM 8. *Let $\mathscr{I}$ be a $\mathscr{G}$-admissible ideal of a directed set $(\mathscr{G},\geqslant)$, $\{D_\gamma\}_{\gamma\in\mathscr{G}}$ be a net in $\mathscr{P}_0(X)$ and $D\in\mathscr{P}_0(X)$. Then $D\in\mathscr{B}_{\mathscr{I}^*}-\lim D_\gamma$ implies $D\in\mathscr{B}_{\mathscr{I}}-\lim D_\gamma$.*

Proof. Let $D\in\mathscr{B}_{\mathscr{I}^*}-\lim D_\gamma$. Then there exists $\mathscr{G}'\in\mathscr{F}(\mathscr{I})$ such that $\mathscr{G}'$ itself is a directed set with respect to the binary relation induced from $(\mathscr{G},\geqslant)$ and $\{D_\gamma\}_{\gamma\in\mathscr{G}'}$ is $\mathscr{B}$-convergent to D.

Let $B\in\mathscr{B}$ and $\varepsilon>0$ be given. Then there exists $\gamma_0\in\mathscr{G}'$, such that for all $\gamma\in\mathscr{G}'$ with $\gamma\geqslant\gamma_0$, we have

$$D\cap B\subset D_\gamma^\varepsilon \quad\text{and}\quad D_\gamma\cap D\subset D^\varepsilon.$$

Since $\mathscr{I}$ is $\mathscr{G}$-admissible, $\mathscr{M}_{\gamma_0}\in\mathscr{F}(\mathscr{I})$. Then $\mathscr{M}_{\gamma_0}\cap\mathscr{G}'\in\mathscr{F}(\mathscr{I})$. Now

$$\mathscr{M}_{\gamma_0}\cap\mathscr{G}'\subset\{\gamma\in\mathscr{G}:D\cap B\subset D_\gamma^\varepsilon\}\cap\{\gamma\in\mathscr{G}:D_\gamma\cap B\subset D^\varepsilon\}.$$

So $\{\gamma\in\mathscr{G}:D\cap B\subset D_\gamma^\varepsilon\}\in\mathscr{F}(\mathscr{I})$ as well as $\{\gamma\in\mathscr{G}:D_\gamma\cap B\subset D^\varepsilon\}\in\mathscr{F}(\mathscr{I})$. Hence $D\in\mathscr{B}_{\mathscr{I}}-\lim D_\gamma$. □

DEFINITION 15. [9] Let $\mathscr{I}$ be a $\mathscr{G}$-*admissible* ideal of a directed set $\mathscr{G}$. Then $\mathscr{I}$ is said to satisfy the condition (DP), if for every countable family of mutually disjoint sets $\{P_1, P_2, \ldots\}$ in $\mathscr{I}$, there exists a countable family of sets $\{Q_1, Q_2, \ldots\}$ in $\mathscr{G}$ such that for each $i \in \mathbb{N}$, $P_i \Delta Q_i \subset \mathscr{G} \setminus \mathscr{M}_{\gamma_i}$ for some $\gamma_i \in \mathscr{G}$ and $Q = \bigcup_{i=1}^{\infty} Q_i \in \mathscr{I}$, where Δ stands for the symmetric difference between two sets.

LEMMA 1. *Let $\mathscr{I}$ be an ideal of a directed set $(\mathscr{G}, \geqslant)$ satisfying the condition (DP). Then for any countable family of sets $\{E_1, E_2, \ldots\}$ in $\mathscr{F}(\mathscr{I})$ there exists a $E \in \mathscr{F}(\mathscr{I})$ such that E itself is a directed set with respect to the binary relation induced from $(\mathscr{G}, \geqslant)$ and for each $i \in \mathbb{N}$, $E \setminus E_i \subset \mathscr{G} \setminus \mathscr{M}_{\gamma^{(i)}}$ for some $\gamma^{(i)} \in \mathscr{G}$.*

Proof. Let $\{E_1, E_2, \ldots\}$ be a countable family of sets in $\mathscr{F}(\mathscr{I})$. Then $\{F_1, F_2, \ldots\}$ is a countable family of sets in $\mathscr{I}$, where $F_i = \mathscr{G} \setminus E_i, \forall i \in \mathbb{N}$. Now we construct a sequence of sets $\{P_i\}_{i \in \mathbb{N}}$ as follows:

$$P_1 = F_1,\ P_2 = F_2 \setminus F_1, \ldots,\ P_i = F_i \setminus (F_1 \cup F_2 \cup \ldots \cup F_{i-1}), \ldots .$$

Clearly, $P_i \in \mathscr{I}$, for all $i \in \mathbb{N}$ and $P_i \cap P_j = \emptyset$ for $i \neq j$. Since $\mathscr{I}$ satisfies the condition (DP), there exists a countable family of sets $\{Q_1, Q_2, \ldots\}$ in $\mathscr{G}$ such that for each $i \in \mathbb{N}$, $P_i \Delta Q_i \subset \mathscr{G} \setminus M_{\gamma_i}$ for some $\gamma_i \in \mathscr{G}$ and $Q = \bigcup_{i=1}^{\infty} Q_i \in \mathscr{I}$. Now, fix $i \in \mathbb{N}$. Then

$$F_i \setminus Q \subset (\bigcup_{j=1}^{i} F_j) \setminus Q = (\bigcup_{j=1}^{i} P_j) \setminus Q = \bigcup_{j=1}^{i} (P_j \setminus Q) \subset \bigcup_{j=1}^{i} (P_j \setminus Q_j) \subset \bigcup_{j=1}^{i} (P_j \Delta Q_j)$$
$$\subset \bigcup_{j=1}^{i} (\mathscr{G} \setminus \mathscr{M}_{\gamma_j}).$$

Now, for $\gamma_1, \gamma_2, \ldots, \gamma_i$, there exists $\gamma^{(i)} \in \mathscr{G}$, such that $\gamma^{(i)} \geqslant \gamma_j$, $\forall j = 1, 2, \ldots, i$. Then

$$F_i \setminus Q \subset \bigcup_{j=1}^{i} (\mathscr{G} \setminus \mathscr{M}_{\gamma_j}) \subset \mathscr{G} \setminus \mathscr{M}_{\gamma^{(i)}}.$$

Let $E = \mathscr{G} \setminus Q$. Then $E \in \mathscr{F}(\mathscr{I})$ and $E \setminus E_i = F_i \setminus Q \subset G \setminus \mathscr{M}_{\gamma^{(i)}}$.

We now show that E itself is a directed set with respect to the binary relation induced from $(\mathscr{G}, \geqslant)$. It is clear that $\geqslant$ is reflexive and transitive on E. Now, let $e_1, e_2 \in E$. Since e_1, e_2 are two elements of $\mathscr{G}$, there exists $e \in \mathscr{G}$ such that $e \geqslant e_1$ and $e \geqslant e_2$. Now $E \cap \mathscr{M}_e \in \mathscr{F}(\mathscr{I})$. Let $e' \in E \cap \mathscr{M}_e$. Then $e' \geqslant e$ and so $e' \geqslant e_1$ and $e' \geqslant e_2$. Therefore, E is a directed set with respect to the binary relation induced from $(\mathscr{G}, \geqslant)$. □

THEOREM 9. *Let $\mathscr{I}$ be an ideal on a directed set $(\mathscr{G}, \geqslant)$ satisfying the condition (DP), (X, d) be a metric space and $\mathscr{B}$ be a bornology on X. Then for any net $\{D_\gamma\}_{\gamma \in \mathscr{G}}$ in $\mathscr{P}_0(X)$ and $D \in \mathscr{P}_0(X)$, $D \in \mathscr{B}_{\mathscr{I}} - \lim D_\gamma$ implies $D \in \mathscr{B}_{\mathscr{I}^*} - \lim D_\gamma$.*

Proof. Let $D \in \mathscr{B}_{\mathscr{I}} - \lim D_\gamma$. Let $B \in \mathscr{B}$ and $\varepsilon > 0$ be given. Since $D \in \mathscr{B}_{\mathscr{I}} - \lim D_\gamma$, for each $j \in \mathbb{N}$,

$$A_j = \{\gamma \in \mathscr{G} : D \cap B \subset D_\gamma^{\frac{1}{j}}\} \in \mathscr{F}(\mathscr{I})$$

and

$$B_j = \{\gamma \in \mathscr{G} : D_\gamma \cap B \subset D^{\frac{1}{j}}\} \in \mathscr{F}(\mathscr{I}).$$

Let $E_j = A_j \cap B_j, j \in \mathbb{N}$. Then $E_j \in \mathscr{F}(\mathscr{I}), \forall j \in \mathbb{N}$. Since $\mathscr{I}$ satisfies the condition (DP), by Lemma 1, there exists $E \in \mathscr{F}(\mathscr{I})$ such that E itself is a directed set with respect to the binary relation induced from $(\mathscr{G}, \geqslant)$ and for each $j \in \mathbb{N}$, $E \setminus E_j \subset \mathscr{G} \setminus \mathscr{M}_{\gamma^{(j)}}$ for some $\gamma^{(j)} \in \mathscr{G}$. Now for the above $\varepsilon > 0$, there exists $j \in \mathbb{N}$ such that $\frac{1}{j} < \varepsilon$. Then for that j, $E \setminus E_j \subset \mathscr{G} \setminus \mathscr{M}_{\gamma^{(j)}}$ for some $\gamma^{(j)} \in \mathscr{G}$. Then for all $\gamma \in E$ and $\gamma \geqslant \gamma^{(j)}$ we have $\gamma \in E_j = A_j \cap B_j$ and so

$$D \cap B \subset D_\gamma^{\frac{1}{j}} \subset D_\gamma^\varepsilon$$

and

$$D_\gamma \cap B \subset D^{\frac{1}{j}} \subset D^\varepsilon.$$

Therefore, $D \in \mathscr{B}_{\mathscr{I}^*} - \lim D_\gamma$. □

We now introduce the notion of $\mathscr{I}^*$-convergence of nets of partial maps.

DEFINITION 16. Let $(X,d), (Y,\mu)$ be two metric spaces, $\mathscr{B}$ be a bornology on X. A net $\{(D_\gamma, u_\gamma)\}_{\gamma \in \mathscr{G}} \in \mathscr{P}[X,Y]$ is said to be $\mathscr{P}_{\mathscr{I}^*}(\mathscr{B})$-convergent to $(D,u) \in \mathscr{P}[X,Y]$, if there exists a set $\mathscr{G}' \in \mathscr{F}(\mathscr{I})$ such that $\mathscr{G}'$ itself is a directed set with respect to the binary relation induced from $(\mathscr{G}, \geqslant)$ and the net $\{(D_\gamma, u_\gamma)\}_{\gamma \in \mathscr{G}'}$ is $\mathscr{P}(\mathscr{B})$-convergent to (D,u).

In this case, we write $(D,u) \in \mathscr{P}_{\mathscr{I}^*}(\mathscr{B}) - \lim(D_\gamma, u_\gamma)$.

THEOREM 10. *Let $\mathscr{I}$ be a $\mathscr{G}$-admissible ideal of a directed set $(\mathscr{G}, \geqslant)$, $\{(D_\gamma, u_\gamma)\}_{\gamma \in \mathscr{G}}$ be a net in $\mathscr{P}[X,Y]$ and $(D,u) \in \mathscr{P}[X,Y]$. Then $(D,u) \in \mathscr{P}_{\mathscr{I}^*}(\mathscr{B}) - \lim(D_\gamma, u_\gamma)$ implies $(D,u) \in \mathscr{P}_{\mathscr{I}}(\mathscr{B}) - \lim(D_\gamma, u_\gamma)$.*

Proof. Let $(D,u) \in \mathscr{P}_{\mathscr{I}^*}(\mathscr{B}) - \lim(D_\gamma, u_\gamma)$. Then there exists $\mathscr{G}' \in \mathscr{F}(\mathscr{I})$ such that $\mathscr{G}'$ itself is a directed set with respect to the binary relation induced from $(\mathscr{G}, \geqslant)$ and $\{(D_\gamma, u_\gamma)\}_{\gamma \in \mathscr{G}'}$ is $\mathscr{P}(\mathscr{B})$-convergent to (D,u).

Let $B \in \mathscr{B}$ and $\varepsilon > 0$ be given. Then there exists $\gamma_0 \in \mathscr{G}'$, such that for all $B_1 \subset B$ and for all $\gamma \in \mathscr{G}'$ with $\gamma \geqslant \gamma_0$, we have

$$u(D \cap B_1) \subset [u_\gamma(D_\gamma \cap B_1^\varepsilon)]^\varepsilon \text{ and } u_\gamma(D_\gamma \cap B_1) \subset [u(D \cap B_1^\varepsilon)]^\varepsilon.$$

Since $\mathscr{I}$ is $\mathscr{G}$-admissible, $\mathscr{M}_{\gamma_0} \in \mathscr{F}(\mathscr{I})$ and so $\mathscr{M}_{\gamma_0} \cap \mathscr{G}' \in \mathscr{F}(\mathscr{I})$. Then

$$\mathscr{M}_{\gamma_0} \cap \mathscr{G}' \subset \{\gamma \in \mathscr{G} : \forall B_1(\subset B), u(D \cap B_1) \subset [u_\gamma(D_\gamma \cap B_1^\varepsilon)]^\varepsilon\}$$
$$\cap \{\gamma \in \mathscr{G} : \forall B_1(\subset B), u_\gamma(D_\gamma \cap B_1) \subset [u(D \cap B_1^\varepsilon)]^\varepsilon\}.$$

So $\{\gamma \in \mathscr{G} : \forall B_1(\subset B), u(D \cap B_1) \subset [u_\gamma(D_\gamma \cap B_1^\varepsilon)]^\varepsilon\} \in \mathscr{F}(\mathscr{I})$ as well as $\{\gamma \in \mathscr{G} : \forall B_1(\subset B), u_\gamma(D_\gamma \cap B_1) \subset [u(D \cap B_1^\varepsilon)]^\varepsilon\} \in \mathscr{F}(\mathscr{I})$. Hence, $(D,u) \in \mathscr{P}_{\mathscr{I}}(\mathscr{B}) - \lim(D_\gamma, u_\gamma)$. □

THEOREM 11. *Let $\mathscr{I}$ be an ideal on a directed set $(\mathscr{G},\geqslant)$ satisfying the condition (DP), $(X,d),(Y,\mu)$ be metric spaces and $\mathscr{B}$ be a bornology on X. Then for any net $\{(D_\gamma,u_\gamma)\}_{\gamma\in\mathscr{G}}$ in $\mathscr{P}[X,Y]$ and $(D,u)\in\mathscr{P}[X,Y]$, $(D,u)\in\mathscr{P}_{\mathscr{I}}(\mathscr{B})-\lim(D_\gamma,u_\gamma)$ implies $(D,u)\in\mathscr{P}_{\mathscr{I}^*}(\mathscr{B})-\lim(D_\gamma,u_\gamma)$.*

Proof. Let $(D,u)\in\mathscr{P}_{\mathscr{I}}(\mathscr{B})-\lim(D_\gamma,u_\gamma)$. Let $B\in\mathscr{B}$ and $\varepsilon>0$ be given. Since $(D,u)\in\mathscr{P}_{\mathscr{I}}(\mathscr{B})-\lim(D_\gamma,u_\gamma)$, for each $j\in\mathbb{N}$,

$$A_j=\{\gamma\in\mathscr{G}:\forall B_1(\subset B),u(D\cap B_1)\subset[u_\gamma(D_\gamma\cap B_1^{\frac{1}{j}})]^{\frac{1}{j}}\}\in\mathscr{F}(\mathscr{I})$$

and

$$C_j=\{\gamma\in\mathscr{G}:\forall B_1(\subset B),u_\gamma(D_\gamma\cap B_1)\subset[u(D\cap B_1^{\frac{1}{j}})]^{\frac{1}{j}}\}\in\mathscr{F}(\mathscr{I}).$$

Let $E_j=A_j\cap C_j, j\in\mathbb{N}$. Then $E_j\in\mathscr{F}(\mathscr{I}),\forall j\in\mathbb{N}$. Since $\mathscr{I}$ satisfies the condition (DP), by Lemma 1, there exists $E\in\mathscr{F}(\mathscr{I})$ such that E itself is a directed set with respect to the binary relation induced from $(\mathscr{G},\geqslant)$ and for each $j\in\mathbb{N}$, $E\setminus E_j\subset\mathscr{G}\setminus\mathscr{M}_{\gamma^{(j)}}$ for some $\gamma^{(j)}\in\mathscr{G}$. Now for the above $\varepsilon>0$, there exists $j\in\mathbb{N}$ such that $\frac{1}{j}<\varepsilon$. Then for that j, $E\setminus E_j\subset\mathscr{G}\setminus\mathscr{M}_{\gamma^{(j)}}$, for some $\gamma^{(j)}\in\mathscr{G}$. Then for all $\gamma\in E$ and $\gamma\geqslant\gamma^{(j)}$, we have $\gamma\in E_j=A_j\cap C_j$ and so

$$\forall B_1(\subset B),u(D\cap B_1)\subset[u_\gamma(D_\gamma\cap B_1^{\frac{1}{j}})]^{\frac{1}{j}}\subset[u_\gamma(D_\gamma\cap B_1^{\varepsilon})]^{\varepsilon}$$

and

$$\forall B_1(\subset B),u_\gamma(D_\gamma\cap B_1)\subset[u(D\cap B_1^{\frac{1}{j}})]^{\frac{1}{j}}\subset[u(D\cap B_1^{\varepsilon})]^{\varepsilon}.$$

Therefore, $(D,u)\in\mathscr{P}_{\mathscr{I}^*}(\mathscr{B})-\lim(D_\gamma,u_\gamma)$. □

REMARK 3. It is not clear, whether the (DP)-condition is necessary for Theorem 9 and Theorem 11, so examination of necessity of (DP)-condition for that two theorems remains open.

Acknowledgement. The second author is grateful to the University Grants Commission, India for fellowship funding under UGC-JRF (SRF fellowship) scheme during the preparation of this paper.

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(Received June 4, 2024)

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