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A NOTE ON A GENERALISATION OF A DEFINITE INTEGRAL INVOLVING THE BESSEL FUNCTION OF THE FIRST KIND

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Abstract. We consider a generalisation of a definite integral involving the Bessel function of the first kind. It is shown that this integral can be expressed in terms of the Fox-Wright function ${}_p\Psi_q(z)$ of one variable. Some consequences of this representation are explored by suitable choice of parameters. Further, we compte the range of numerical approximation values of the Ramanujan's cosine integral $\phi_C(m,n)$ and sine integral $\phi_S(m,n)$ for distinct values of m and n by Wolfram Mathematica software. In addition, two closed-form evaluations of infinite series of the Fox-Wright function are deduced and these sums have been verified numerically using Mathematica.

1. Introduction and preliminaries

The theory of Bessel function is intimately connected with the theory of certain types of differential equations. A detailed account of application of Bessel function is represented in Watson [9]. The applications of Bessel function are found in the field of applied sciences, engineering, biological, chemical and Physical sciences [9]. The Fox-Wright function is a modified version of the generalized hypergeometric function derived by putting arbitrary positive scaling factor into the parameters of the summand's gamma functions. Its significance arises mainly from its place in fractional calculus, though other fascinating applications exist. The series describing the fox-Wright function has a finite non-zero region of convergence if the sums of the scaling factors in the top and bottom parameters are equal. Braaksma demonstrated in 1964 that the Fox-Wright function can be generalized to a holomorphic function in the complex plane by cutting a ray from the positive point on the disk's boundary to the point at infinity. The Fox-Wright function ${}_p\Psi_q(z)$ of one variable [2, 3] is given by

$$
{}_p\Psi_q\left[\begin{matrix}(\alpha_1,A_1),\ldots,(\alpha_p,A_p)\\(\beta_1,B_1),\ldots,(\beta_q,B_q)\end{matrix}\middle|\, z\right] = {}_p\Psi_q\left[\begin{matrix}(\alpha_j,A_j)_{1,p}\\(\beta_j,B_j)_{1,q}\end{matrix}\middle|\, z\right]
$$

$$
=\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+kA_1)\ldots\Gamma(\alpha_p+kA_p)}{\Gamma(\beta_1+kB_1)\ldots\Gamma(\beta_q+kB_q)}\,\frac{z^k}{k!}, \tag{1.1}
$$

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where $z \in \mathbf{C}$, α_j, $\beta_j \in \mathbf{C}$ and the coefficients $A_j \geqslant 0$, $B_j \geqslant 0$, it being supposed throughout that the α_j, β_j and the A_j, B_j are such that the gamma functions are well defined. With

$$\Delta = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j, \quad \delta = \prod_{j=1}^{p} A_j^{-A_j} \prod_{j=1}^{q} B_j^{B_j}, \quad \mu^* = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j + \tfrac{1}{2}(p-q),$$

the series in (1.1) converges for $|z| < \infty$ when $\Delta > -1$, for $|z| < \delta$ when $\Delta = -1$ and for $|z| = \delta$ if, in addition, $\Re(\mu^*) > \frac{1}{2}$. Particular cases of (1.1) that we shall employ are the Wright function [3, p. 438(1.2)] is defined by

$$\Phi(\alpha,\beta,z) = {}_0\Psi_1\left[\begin{matrix} \text{---} \\ (\beta,\alpha) \end{matrix}\middle| z\right] = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\beta+\alpha k)}, \tag{1.2}$$

where $z,\beta \in \mathbf{C}$ and $\alpha > 0$ and the Wright generalised-Bessel function [3, p. 438(1.3)] defined by

$$J_\nu^\mu(-z) = \Phi(\mu,\nu+1,-z) = {}_0\Psi_1\left[\begin{matrix} \text{---} \\ (\nu+1,\mu) \end{matrix}\middle| -z\right] = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(\nu+1+\mu k)}, \tag{1.3}$$

where $z,\nu \in \mathbf{C}$ and $\mu > 0$. The Mittag-Leffler function [3, p. 450(6.1)] is given by

$$E_{\alpha,\beta}(z) = {}_1\Psi_1\left[\begin{matrix} (1,1) \\ (\beta,\alpha) \end{matrix}\middle| z\right] = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta+\alpha k)}, \tag{1.4}$$

where $z,\beta \in \mathbf{C}$ and $\alpha > 0$. The Bessel function of the first kind is defined by (see [5, p. 217])

$$J_\nu(z) = (\tfrac{1}{2}z)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{k!\Gamma(\nu+1+k)}.$$

Of special interest in this paper are the elementary functions corresponding to $\nu = \pm\frac{1}{2}$, namely

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sin z, \qquad J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\cos z. \tag{1.5}$$

The infinite Fourier sine and cosine transforms of $g(x)$ over the interval $[0,\infty)$ are defined by [1, 4]

$$F_{S,C}(g(x);b) = \int_0^\infty g(x) \begin{matrix}\sin\\ \cos\end{matrix}(bx)\,dx = G_{S,C}(b) \qquad (b>0). \tag{1.6}$$

If $\Re(\mu) > 0$ and $\Re(b) > 0$, then Mellin transforms of $\cos(bx)/(e^{ax}-1)$ and $\sin(bx)/(e^{ax}-1)$ are given by [1, p. 15, 1.4(7)]

$$\int_0^\infty \frac{x^{\mu-1}\cos(bx)}{e^{ax}-1}\,dx = \frac{\Gamma(\mu)}{2a^\mu}\Big\{\zeta(\mu,1+ib/a) + \zeta(\mu,1-ib/a)\Big\}$$

and

$$\int_0^\infty \frac{x^{\mu-1}\sin(bx)}{e^{ax}-1}\,dx = \frac{i\Gamma(\mu)}{2a^\mu}\Big\{\zeta(\mu,1+ib/a)-\zeta(\mu,1-ib/a)\Big\},$$

where $\zeta(a,z)=\sum_{k\geqslant 0}(k+z)^{-a}$ is the Hurwitz zeta function. In the special case $\mu=2$ and $a=2\pi$, $b=\pi n$, we have

$$\int_0^\infty \frac{x\cos(\pi nx)}{e^{2\pi x}-1}dx = \frac{1}{2\pi^2 n^2}+\frac{1}{4(1-\cosh \pi n)} \tag{1.7}$$

and [5, (5.15.1)]

$$\int_0^\infty \frac{x\sin(\pi nx)}{e^{2\pi x}-1}dx = \frac{i}{8\pi^2}\{\psi'(1+\tfrac{1}{2}in)-\psi'(1-\tfrac{1}{2}in)\}, \tag{1.8}$$

where $\psi(z)$ is the logarithmic derivative of $\Gamma(z)$ and the prime denotes differentiation.

A natural generalisation of the Gauss hypergeometric function ${}_2F_1(z)$ is the generalised hypergeometric function ${}_pF_q(z)$, with p numerator parameters $\alpha_1,\ldots,\alpha_p$ and q denominator parameters $\beta_1,\ldots,\beta_q$, defined by

$${}_pF_q\left(\begin{matrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{matrix};z\right) = {}_pF_q\left(\begin{matrix}(\alpha_j)_{1,p}\\ (\beta_j)_{1,q}\end{matrix};z\right) = \sum_{k=0}^\infty \frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\frac{z^k}{k!}, \tag{1.9}$$

where $\alpha_j\in\mathbf{C}$ $(j=1,\ldots,p)$ and $\beta_j\in\mathbf{C}\backslash\mathbf{Z}_0^-$ $(j=1,\ldots,q)$, $\mathbf{Z}_0^-=\{0,-1,-2,\ldots\}$ and $(a)_k=\Gamma(a+k)/\Gamma(a)$ is the Pochhammer symbol. The series in (1.9) is convergent for $|z|<\infty$ if $p\leqslant q$ and for $|z|<1$ if $p=q+1$. If we define the parametric excess by $\omega=\sum_{j=1}^q\beta_j-\sum_{j=1}^p\alpha_j$, then it is known that the ${}_pF_q(z)$ series with $p=q+1$ is (i) absolutely convergent for $|z|=1$ if $\Re(\omega)>0$, (ii) is conditionally converegnt for $|z|=1$ if $-1<\Re(\omega)\leqslant 0$ and (iii) is divergent for $|z|=1$ if $\Re(\omega)\leqslant -1$.

The central aim in this paper is to give a generalisation and find some consequences of a definite integral involving the Bessel function of the first kind which we express in terms of the Fox-Wright Ψ function. The work is motivated by the papers by one of the present authors in [6, 7, 8]. In order to generalise the definite integrals introduced by Ramanujan $\phi_{S,C}(m,n)$ defined by

$$\phi_{S,C}(m,n)=\int_0^\infty \frac{x^m}{e^{2\pi x}-1}\,\begin{matrix}\sin\\ \cos\end{matrix}\,(\pi nx)\,dx, \tag{1.10}$$

we introduce the following integrals:

$$\mathfrak{F}_1(\mu,\xi,a,\nu,y)=\int_0^\infty x^\mu e^{-ax^\xi}\sqrt{xy}\,J_\nu(xy)\,dx, \tag{1.11}$$

$$\mathfrak{F}_2(\mu,\xi,b,c,\nu,y)=\sum_{k=0}^\infty\frac{\Theta(k)}{k!}\int_0^\infty x^\mu e^{-(b+ck)x^\xi}\sqrt{xy}\,J_\nu(xy)\,dx, \tag{1.12}$$

$$\mathfrak{F}_3(\mu,\xi,b,c,\nu,y)=\int_0^\infty x^\mu e^{-bx^\xi}\,{}_r\Psi_s\left[\begin{matrix}(\alpha_j,A_j)_{1,r}\\(\beta_j,B_j)_{1,s}\end{matrix}\middle|e^{-cx^\xi}\right]\sqrt{xy}\,J_\nu(xy)\,dx \tag{1.13}$$

and

$$\mathfrak{F}_4(\mu,\xi,b,c,\nu,y)=\int_0^\infty x^\mu e^{-bx^\xi}\,{}_rF_s\left(\begin{matrix}(\alpha_j)_{1,r}\\(\beta_j)_{1,s}\end{matrix};e^{-cx^\xi}\right)\sqrt{xy}\,J_\nu(xy)\,dx. \tag{1.14}$$

Here $\xi>0$ and $\{\Theta(k)\}_{k=0}^\infty$ is a bounded sequence of real or complex quantities. We show how the main general theorem given in Section 3 is applicable for obtaining new and interesting results by suitable adjustment of the parameters and variables.

2. Evaluation of the definite integral $\mathfrak{F}_1(\mu,\xi,a,\nu,y)$

In this section, we evaluate the integral in (1.11) involving the Bessel function of the first kind in terms of the Fox-Wright function. We suppose throughout that the parameter $\xi>0$.

THEOREM 1. *With* $\sigma:=(2\mu+2\nu+3)/(2\xi)$, *we have*

$$\mathfrak{F}_1(\mu,\xi,a,\nu,y)=\int_0^\infty x^\mu e^{-ax^\xi}\sqrt{xy}\,J_\nu(xy)\,dx=\frac{y^{\nu+1/2}}{2^\nu\xi a^\sigma}\,{}_1\Psi_1\left[\begin{matrix}(\sigma,2/\xi)\\(\nu+1,1)\end{matrix}\middle|-\frac{y^2}{4a^{2/\xi}}\right], \tag{2.1}$$

where $a>0$, $y>0$ *and* $\Re(\mu+\nu)>-\frac{1}{2}$.

Proof. Expanding the Bessel function in its series form, followed by reversal of the order of summation and integration, we find

$$\begin{aligned}\mathfrak{F}_1(\mu,\xi,a,\nu,y)&=\frac{y^{\nu+1/2}}{2^\nu\xi}\sum_{k=0}^\infty\frac{(-1)^k(y^2/4)^k}{k!\Gamma(\nu+1+k)}\int_0^\infty t^{\sigma+2k/\xi-1}e^{-at}dt\\&=\frac{y^{\nu+1/2}}{2^\nu\xi a^\sigma}\sum_{k=0}^\infty\frac{\Gamma(\sigma+2k/\xi)}{k!\Gamma(\nu+1+k)}\left(\frac{-y^2}{4a^{2/\xi}}\right)^k\end{aligned}$$

upon evaluation of the integral as a gamma function, where $\sigma=(2\mu+2\nu+3)/(2\xi)$. If we now employ the definition of the Fox-Wright function in (1.1) in the above series, we obtain the right-hand side of (2.1).

If we set $\xi=1$ and $\nu=\pm\frac{1}{2}$ in (2.1), we obtain the Fourier sine and cosine transforms of $x^{\eta-1}e^{-ax}$ in the form

$$\begin{aligned}\int_0^\infty x^{\eta-1}e^{-ax}\cos(xy)\,dx&=\frac{\Gamma(\eta)}{a^\eta}\sum_{k=0}^\infty\frac{(\eta)_{2k}}{(\frac{1}{2})_k k!}\left(-\frac{y^2}{4a^2}\right)^k\\&=\frac{\Gamma(\eta)}{a^\eta}\,{}_2F_1\left(\begin{matrix}\frac{1}{2}\eta,\frac{1}{2}\eta+\frac{1}{2}\\\frac{1}{2}\end{matrix};-\frac{y^2}{a^2}\right)\end{aligned}$$

and

$$\int_0^\infty x^{\eta-1}e^{-ax}\sin(xy)\,dx = \frac{y\Gamma(\eta+1)}{a^{\eta+1}}\sum_{k=0}^\infty \frac{(\eta+1)_{2k}}{(\frac{3}{2})_k k!}\left(-\frac{y^2}{4a^2}\right)^k$$
$$= \frac{y\Gamma(\eta+1)}{a^{\eta+1}}\,{}_2F_1\left(\begin{matrix}\frac{1}{2}\eta+\frac{1}{2},\frac{1}{2}\eta+1\\ \frac{3}{2}\end{matrix};-\frac{y^2}{a^2}\right).$$

Use of the standard evaluations of the hypergeometric function given in [5, (15.4.8), (15.4.10)] then yields the results stated in [4]

$$\int_0^\infty x^{\eta-1}e^{-ax}\begin{matrix}\sin\\ \cos\end{matrix}(xy)\,dx = \Gamma(\eta)\,(a^2+y^2)^{-\eta/2}\begin{matrix}\sin\\ \cos\end{matrix}(\eta\arctan(y/a)),$$

where $\Re(a)>0$, $y>0$ and $\Re(\eta)>-1$ and $\Re(\eta)>0$ for the sine and cosine integral, respectively. □

3. Evaluation of the definite integrals $\mathfrak{F}_j(\mu,\xi,a,\nu,y)$, $j=2,3,4$

Here we let $\sigma=(2\mu+2\nu+3)/(2\xi)$ throughout this section.

THEOREM 2. *Let $\{\Theta(k)\}_{k=0}^\infty$ be a bounded sequence of arbitrary real or complex numbers. Then when $\xi>0$ and $\Re(b)>0$ we have*

$$\mathfrak{F}_2(\mu,\xi,b,c,\nu,y) = \sum_{k=0}^\infty \frac{\Theta(k)}{k!}\int_0^\infty x^\mu e^{-(b+ck)x^\xi}\sqrt{xy}\,J_\nu(xy)\,dx$$
$$= \frac{y^{\nu+1/2}}{2^\nu b^\sigma \xi}\sum_{k=0}^\infty \frac{\Theta(k)}{k!}\left(\frac{(b/c)_k}{(1+b/c)_k}\right)^\sigma {}_1\Psi_1\left[\begin{matrix}(\sigma,2/\xi)\\ (\nu+1,1)\end{matrix}\middle| -\frac{y^2}{4(b+ck)^{2/\xi}}\right]. \tag{3.1}$$

Proof. The proof follows the same procedure as in Theorem 1 by expressing the Bessel function in series form and integrating term by term. We find

$$\mathfrak{F}_2(\mu,\xi,b,c,\nu,y) = \frac{y^{\nu+1/2}}{2^\nu\xi}\sum_{k=0}^\infty \frac{\Theta(k)}{k!(b+ck)^\sigma}\sum_{\ell=0}^\infty \frac{\Gamma(\sigma+2\ell/\xi)}{\ell!\Gamma(\nu+1+\ell)}\left(\frac{-y^2}{4(b+ck)^{2/\xi}}\right)^\ell.$$

Employing the definition of the Fox-Wright function function in (1.1) then yields the stated result in (3.1). □

COROLLARY 1. *For $\xi>0$, $\Re(b)>0$ and $\Re(c)>0$, we have*

$$\mathfrak{F}_3(\mu,\xi,b,c,\nu,y) = \int_0^\infty x^\mu e^{-bx^\xi}\,{}_r\Psi_s\left[\begin{matrix}(\alpha_j,A_j)_{1,r}\\ (\beta_j,B_j)_{1,s}\end{matrix}\middle| e^{-cx^\xi}\right]\sqrt{xy}\,J_\nu(xy)\,dx$$
$$= \frac{y^{\nu+1/2}}{2^\nu b^\sigma\xi}\sum_{k=0}^\infty \frac{\prod_{j=1}^r\Gamma(\alpha_j+kA_j)}{k!\prod_{j=1}^s\Gamma(\beta_j+kB_j)}\left(\frac{(b/c)_k}{(1+b/c)_k}\right)^\sigma$$
$$\times\,{}_1\Psi_1\left[\begin{matrix}(\sigma,2/\xi)\\ (\nu+1,1)\end{matrix}\middle| -\frac{y^2}{4(b+ck)^{2/\xi}}\right], \tag{3.2}$$

where the parameters $\alpha_j,\beta_j\in\mathbf{C}$ *and* $A_j\geqslant 0,\ B_j\geqslant 0$. *This result follows immediately from* (3.1) *by substituting*

$$\Theta(k)=\frac{\Gamma(\alpha_1+kA_1)\ldots\Gamma(\alpha_r+kA_r)}{\Gamma(\beta_1+kB_1)\ldots\Gamma(\beta_s+kB_s)}\qquad(k=0,1,2,\ldots).$$

COROLLARY 2. *For* $\xi>0$, $\Re(b)>0$ *and* $\Re(c)>0$, *we have*

$$\begin{aligned}\mathfrak{F}_4(\mu,\xi,b,c,\nu,y)&=\int_0^\infty x^\mu e^{-bx^\xi}{}_rF_s\left(\begin{matrix}(\alpha_j)_{1,r}\\(\beta_j)_{1,s}\end{matrix};e^{-cx^\xi}\right)\sqrt{xy}\,J_\nu(xy)\,dx\\&=\frac{y^{\nu+1/2}}{2^\nu b^\sigma\xi}\sum_{k=0}^\infty\frac{\prod_{j=1}^r(\alpha_j)_k}{k!\prod_{j=1}^s(\beta_j)_k}\left(\frac{(b/c)_k}{(1+b/c)_k}\right)^\sigma\\&\quad\times{}_1\Psi_1\left[\begin{matrix}(\sigma,2/\xi)\\(\nu+1,1)\end{matrix}\middle|-\frac{y^2}{4(b+ck)^{2/\xi}}\right],\end{aligned}\tag{3.3}$$

where the parameters $\alpha_j\in\mathbf{C}$, $\beta_j\in\mathbf{C}\backslash\mathbf{Z}_0^-$ *and* $r\leqslant s+1$. *This result follows by setting* $A_1=\ldots=A_r=1$ *and* $B_1=\ldots=B_s=1$ *in* (3.2).

3.1. Special cases of the integral $\mathfrak{F}_3(\mu,\xi,b,c,\nu,y)$

Special cases of the integral in (3.2) are given by the following. When $r=0$, $s=1$ in (3.2), we obtain

$$\int_0^\infty x^\mu e^{-bx^\xi}\Phi(B_1,\beta_1,e^{-cx^\xi})\sqrt{xy}\,J_\nu(xy)\,dx$$
$$=\frac{y^{\nu+1/2}}{2^\nu b^\sigma\xi}\sum_{k=0}^\infty\frac{1}{k!\Gamma(\beta_1+kB_1)}\left(\frac{(b/c)_k}{(1+b/c)_k}\right)^\sigma{}_1\Psi_1\left[\begin{matrix}(\sigma,2/\xi)\\(\nu+1,1)\end{matrix}\middle|-\frac{y^2}{4(b+ck)^{2/\xi}}\right],\tag{3.4}$$

where $\beta_1\in\mathbf{C}$, $B_1>0$ and $\Phi(B_1,\beta_1,z)$ is the Wright function defined in (1.2).

When $r=0$, $s=1$, $B_1=\mu$, $\beta_1=\gamma+1$ and e^{-cx^ξ} is replaced by $-e^{-cx^\xi}$ in (3.2), we obtain

$$\int_0^\infty x^\mu e^{-bx^\xi}J_\gamma^\mu(-e^{-cx^\xi})\sqrt{xy}\,J_\nu(xy)\,dx$$
$$=\frac{y^{\nu+1/2}}{2^\nu b^\sigma\xi}\sum_{k=0}^\infty\frac{1}{k!\Gamma(\gamma+1+\mu k)}\left(\frac{(b/c)_k}{(1+b/c)_k}\right)^\sigma{}_1\Psi_1\left[\begin{matrix}(\sigma,2/\xi)\\(\nu+1,1)\end{matrix}\middle|-\frac{y^2}{4(b+ck)^{2/\xi}}\right],\tag{3.5}$$

where $\gamma\in\mathbf{C}$, $\mu>0$ and $J_\gamma^\mu(z)$ is the Wright generalised Bessel function defined in (1.3).

When $r=s=1$ and $\alpha_1=A_1=1$ in (3.2), we obtain

$$\int_0^\infty x^\mu e^{-bx^\xi}E_{B_1,\beta_1}(e^{-cx^\xi})\sqrt{xy}\,J_\nu(xy)\,dx$$
$$=\frac{y^{\nu+1/2}}{2^\nu b^\sigma\xi}\sum_{k=0}^\infty\frac{1}{k!\Gamma(\beta_1+kB_1)}\left(\frac{(b/c)_k}{(1+b/c)_k}\right)^\sigma{}_1\Psi_1\left[\begin{matrix}(\sigma,2/\xi)\\(\nu+1,1)\end{matrix}\middle|-\frac{y^2}{4(b+ck)^{2/\xi}}\right],\tag{3.6}$$

where $\beta_1\in\mathbf{C}$, $B_1>0$ and $E_{B_1,\beta_1}(z)$ is the Mittag-Leffler function defined in (1.4).

4. Expressions for Fourier cosine and sine transforms

The expressions for the Fourier cosine and sine transforms of $x^{\eta-1}e^{-(b+ck)x}$, $x^{\eta-1}e^{-bx}{}_r\Psi_s(e^{-cx})$ and $x^{\eta-1}e^{-bx}{}_rF_s(e^{-cx})$ in terms of infinite series of the Fox-Wright function ${}_1\Psi_1(\cdot)$ will hold true for $y>0$, $\Re(\eta)>0$, $\Re(b)>0$ and $\Re(c)>0$. If we let $\mu=\eta-1$, $\xi=1$ and $\nu=\pm\frac{1}{2}$ in the main theorem in (3.1), we obtain after some simplification the following expressions:

$$\begin{aligned}&\mathfrak{F}_2(\eta-1,1,b,c,-\tfrac{1}{2},y)\equiv\mathbf{F}_C^{(1)}(\eta,b,c,y)\\&=\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}\frac{\Theta(k)}{k!}\int_0^{\infty}x^{\eta-1}e^{-(b+ck)x}\cos(xy)\,dx\\&=\sqrt{2}\sum_{k=0}^{\infty}\frac{\Theta(k)}{k!}\frac{1}{(b+ck)^{\eta}}\,{}_1\Psi_1\left[\begin{matrix}(\eta,2)\\(\frac{1}{2},1)\end{matrix}\middle|-\frac{y^2}{4(b+ck)^2}\right],\end{aligned}\tag{4.1}$$

$$\begin{aligned}&\mathfrak{F}_2(\eta-1,1,b,c,\tfrac{1}{2},y)\equiv\mathbf{F}_S^{(1)}(\eta,b,c,y)\\&=\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}\frac{\Theta(k)}{k!}\int_0^{\infty}x^{\eta-1}e^{-(b+ck)x}\sin(xy)\,dx\\&=\frac{y}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{\Theta(k)}{k!}\frac{1}{(b+ck)^{\eta+1}}\,{}_1\Psi_1\left[\begin{matrix}(\eta+1,2)\\(\frac{3}{2},1)\end{matrix}\middle|-\frac{y^2}{4(b+ck)^2}\right],\end{aligned}\tag{4.2}$$

$$\begin{aligned}&\mathfrak{F}_3(\eta-1,1,b,c,-\tfrac{1}{2},y)\equiv\mathbf{F}_C^{(2)}(\eta,b,c,y)\\&=\sqrt{\frac{2}{\pi}}\int_0^{\infty}x^{\eta-1}e^{-bx}{}_r\Psi_s\left[\begin{matrix}(\alpha_j,A_j)_{1,r}\\(\beta_j,B_j)_{1,s}\end{matrix}\middle|e^{-cx}\right]\cos(xy)\,dx\\&=\sqrt{2}\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{r}\Gamma(\alpha_j+kA_j)}{k!\prod_{j=1}^{s}\Gamma(\beta_j+kB_j)}\frac{1}{(b+ck)^{\eta}}\,{}_1\Psi_1\left[\begin{matrix}(\eta,2)\\(\frac{1}{2},1)\end{matrix}\middle|-\frac{y^2}{4(b+ck)^2}\right],\end{aligned}\tag{4.3}$$

$$\begin{aligned}&\mathfrak{F}_3(\eta-1,1,b,c,\tfrac{1}{2},y)\equiv\mathbf{F}_S^{(2)}(\eta,b,c,y)\\&=\sqrt{\frac{2}{\pi}}\int_0^{\infty}x^{\eta-1}e^{-bx}{}_r\Psi_s\left[\begin{matrix}(\alpha_j,A_j)_{1,r}\\(\beta_j,B_j)_{1,s}\end{matrix}\middle|e^{-cx}\right]\sin(xy)\,dx\\&=\frac{y}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{r}\Gamma(\alpha_j+kA_j)}{k!\prod_{j=1}^{s}\Gamma(\beta_j+kB_j)}\frac{1}{(b+ck)^{\eta+1}}\\&\quad\times{}_1\Psi_1\left[\begin{matrix}(\eta+1,2)\\(\frac{3}{2},1)\end{matrix}\middle|-\frac{y^2}{4(b+ck)^2}\right],\end{aligned}\tag{4.4}$$

$$\begin{aligned}&\mathfrak{F}_4(\eta-1,1,b,c,-\tfrac{1}{2},y)\equiv\mathbf{F}_C^{(3)}(\eta,b,c,y)\\&=\sqrt{\frac{2}{\pi}}\int_0^{\infty}x^{\eta-1}e^{-bx}{}_rF_s\left[\begin{matrix}(\alpha_j)_{1,r}\\(\beta_j)_{1,s}\end{matrix}\middle|e^{-cx}\right]\cos(xy)\,dx\\&=\sqrt{2}\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{r}\Gamma(\alpha_j)_k}{k!\prod_{j=1}^{s}\Gamma(\beta_j)_k}\frac{1}{(b+ck)^{\eta}}\,{}_1\Psi_1\left[\begin{matrix}(\eta,2)\\(\frac{1}{2},1)\end{matrix}\middle|-\frac{y^2}{4(b+ck)^2}\right],\end{aligned}\tag{4.5}$$

$$
\begin{aligned}
&\mathfrak{F}_4(\eta-1,1,b,c,\tfrac{1}{2},y) \equiv \mathbf{F}_S^{(3)}(\eta,b,c,y)\\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty x^{\eta-1}e^{-bx}\,{}_rF_s\left[\begin{matrix}(\alpha_j)_{1,r}\\(\beta_j)_{1,s}\end{matrix}\middle| e^{-cx}\right]\sin(xy)\,dx\\
&= \frac{y}{\sqrt{\pi}}\sum_{k=0}^\infty \frac{\prod_{j=1}^r\Gamma(\alpha_j)_k}{k!\prod_{j=1}^s\Gamma(\beta_j)_k}\frac{1}{(b+ck)^{\eta+1}}\,{}_1\Psi_1\left[\begin{matrix}(\eta+1,2)\\(\frac{3}{2},1)\end{matrix}\middle| -\frac{y^2}{4(b+ck)^2}\right]. \qquad (4.6)
\end{aligned}
$$

5. Closed-form infinite summation formulas

If we choose $\Theta(k)=k!$, $\eta=m+1$, $y=\pi n$ and $b=c=2\pi$ in (4.1)–(4.6), we find the following new expressions in the closed form-infinite sums of the Fox-Wright Psi function:

$$
\begin{aligned}
\phi_C(m,n) &= \int_0^\infty \frac{x^m\cos(\pi nx)}{e^{2\pi x}-1}\,dx = \sum_{k=0}^\infty\int_0^\infty x^m e^{-2\pi x(1+k)}\cos(\pi nx)\,dx\\
&= \frac{\sqrt{\pi}}{(2\pi)^{m+1}}\sum_{k=0}^\infty\frac{1}{(k+1)^{m+1}}\,{}_1\Psi_1\left[\begin{matrix}(m+1,2)\\(\frac{1}{2},1)\end{matrix}\middle| -\frac{n^2}{16(1+k)^2}\right], \qquad (5.1)\\
\phi_S(m,n) &= \int_0^\infty \frac{x^m\sin(\pi nx)}{e^{2\pi x}-1}\,dx = \sum_{k=0}^\infty\int_0^\infty x^m e^{-2\pi x(1+k)}\sin(\pi nx)\,dx\\
&= \frac{\sqrt{\pi}\,n}{4(2\pi)^{m+1}}\sum_{k=0}^\infty\frac{1}{(k+1)^{m+2}}\,{}_1\Psi_1\left[\begin{matrix}(m+2,2)\\(\frac{3}{2},1)\end{matrix}\middle| -\frac{n^2}{16(1+k)^2}\right], \qquad (5.2)
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^\infty x^m e^{-2\pi x}\,{}_r\Psi_s\left[\begin{matrix}(\alpha_j,A_j)_{1,r}\\(\beta_j,B_j)_{1,s}\end{matrix}\middle| e^{-2\pi x}\right]\cos(\pi nx)\,dx\\
&= \frac{\sqrt{\pi}}{(2\pi)^{m+1}}\sum_{k=0}^\infty\frac{\prod_{j=1}^r\Gamma(\alpha_j+kA_j)}{\prod_{j=1}^s\Gamma(\beta_j+kB_j)}\frac{1}{(k+1)^{m+1}}\,{}_1\Psi_1\left[\begin{matrix}(m+1,2)\\(\frac{1}{2},1)\end{matrix}\middle| -\frac{n^2}{16(1+k)^2}\right], \qquad (5.3)\\
&\int_0^\infty x^m e^{-2\pi x}\,{}_r\Psi_s\left[\begin{matrix}(\alpha_j,A_j)_{1,r}\\(\beta_j,B_j)_{1,s}\end{matrix}\middle| e^{-2\pi x}\right]\sin(\pi nx)\,dx\\
&= \frac{\sqrt{\pi}\,n}{4(2\pi)^{m+1}}\sum_{k=0}^\infty\frac{\prod_{j=1}^r\Gamma(\alpha_j+kA_j)}{\prod_{j=1}^s\Gamma(\beta_j+kB_j)}\frac{1}{(k+1)^{m+2}}\,{}_1\Psi_1\left[\begin{matrix}(m+2,2)\\(\frac{3}{2},1)\end{matrix}\middle| -\frac{n^2}{16(1+k)^2}\right], \qquad (5.4)\\
&\int_0^\infty x^m e^{-2\pi x}\,{}_rF_s\left[\begin{matrix}(\alpha_j)_{1,r}\\(\beta_j)_{1,s}\end{matrix}\middle| e^{-2\pi x}\right]\cos(\pi nx)\,dx\\
&= \frac{\sqrt{\pi}}{(2\pi)^{m+1}}\sum_{k=0}^\infty\frac{\prod_{j=1}^r\Gamma(\alpha_j)_k}{\prod_{j=1}^s\Gamma(\beta_j)_k}\frac{1}{(k+1)^{m+1}}\,{}_1\Psi_1\left[\begin{matrix}(m+1,2)\\(\frac{1}{2},1)\end{matrix}\middle| -\frac{n^2}{16(1+k)^2}\right], \qquad (5.5)\\
&\int_0^\infty x^m e^{-2\pi x}\,{}_rF_s\left[\begin{matrix}(\alpha_j,A_j)_{1,r}\\(\beta_j,B_j)_{1,s}\end{matrix}\middle| e^{-2\pi x}\right]\sin(\pi nx)\,dx\\
&= \frac{\sqrt{\pi}\,n}{4(2\pi)^{m+1}}\sum_{k=0}^\infty\frac{\prod_{j=1}^r\Gamma(\alpha_j)_k}{\prod_{j=1}^s\Gamma(\beta_j)_k}\frac{1}{(k+1)^{m+2}}\,{}_1\Psi_1\left[\begin{matrix}(m+2,2)\\(\frac{3}{2},1)\end{matrix}\middle| -\frac{n^2}{16(1+k)^2}\right]. \qquad (5.6)
\end{aligned}
$$

Finally, from (1.7), (1.8), (5.1) and (5.2), we calculate the non terminating summation formulas:

$$\sum_{k=0}^{\infty}\frac{1}{(1+k)^2}\,{}_1\Psi_1\left[\begin{matrix}(2,2)\\(\frac{1}{2},1)\end{matrix}\middle|-\frac{n^2}{16(1+k)^2}\right]=\sqrt{\pi}\left(\frac{2}{\pi n^2}+\frac{\pi}{1-\cosh\pi n}\right), \tag{5.7}$$

and

$$\sum_{k=0}^{\infty}\frac{1}{(1+k)^3}\,{}_1\Psi_1\left[\begin{matrix}(3,2)\\(\frac{3}{2},1)\end{matrix}\middle|-\frac{n^2}{16(1+k)^2}\right]=\frac{2i}{\sqrt{\pi}n}\left\{\psi'(1+\tfrac{1}{2}in)-\psi'(1-\tfrac{1}{2}in)\right\}, \tag{5.8}$$

where n is free to be chosen. These two sums have been verified numerically using *Mathematica*.

Case study. In Table-1 we compute numerical approximation values of the Ramanujan's integrals (5.1) and (5.2) for different values of m and n.

Table 1: *The numerical approximation table of cosine integral $\phi_C(m,n)$ and sine integral $\phi_S(m,n)$ in (5.1) and (5.2) for distinct values of m and n are calculated by Wolfram Mathematica software.*

m	n	$\phi_C(m,n)$	$\phi_S(m,n)$	m	n	$\phi_C(m,n)$	$\phi_S(m,n)$
1.00	1/2	0.0369893	0.0137208	1/7	1/8	0.0059067	0.0005401
1/2	1/3	0.0197389	0.0048411	1/8	1/9	0.0051767	0.0004206
1/3	1/4	0.0134708	0.0024705	1/9	1/10	0.0046068	0.0003368
1/4	1/5	0.0102142	0.0014965	1/10	1/11	0.0041497	0.0002758
1/5	1/6	0.0082203	0.0010029	1/11	1/12	0.0037749	0.0002299
1/6	1/7	0.0068750	0.0007186	1/12	1/13	0.0034621	0.0001947

6. Conclusion

In this note we have shown that a certain integral involving the Bessel function of the first kind can be expressed in terms of the Fox-Wright hypergeometric function of a single variable. Special cases of this integral lead to similar representations given in (5.1) and (5.2) for an integral considered by Ramanujan. Two infinite sums involving the Fox-Wright function are evaluated in closed form.

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2022. Prof R. B. Paris was a classical analyst, using tool from complex analysis and asymptotic. In the early years of his career he published two books, namely, *Asymptotics of Higher Order Differential equations* (with Prof A. D. Wood), and *Asymptotics and Mellin Barns Integrals*. He was also an author of the *DLMF* chapters in the *NIST Handbook of Mathematical functions*. He remained mathematically active until the end of his life.

Declarations. The authors declare that they have no conflict of interest.

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