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ON SOME IMPROPER INTEGRALS INVOLVING THE CUBE OF THE TAILS OF TWO MACLAURIN SERIES

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Abstract. Using a reformulation of a recently devised method for evaluating definite integrals known as *integration by differentiation* [*J. Phys. A: Math. Theor.* **50** (2017) 235201], a family of ten improper integrals containing the cube of the tails of the Maclaurin series for the sine and cosine functions are found. Contributions to the value of the improper integral from various terms that repeatedly appear in the integrands for the improper integrals to be evaluated when applying the method are explicitly found, thereby greatly helping to streamline the computational aspects of the process. A number of inter-relations between six of the ten improper integrals are established, leading to some intriguing binomial identities.

1. Introduction

The problem of evaluating improper integrals that contain the tails of the Maclaurin series for the sine and cosine functions dates back well over a century where it was posed as a problem [13, Problem 1914, p. 329]. Since then it has appeared in various guises [12, 8], [9, Ex. 2.68 and 2.69, pp. 46–47], [10, 7] and is most easily evaluated using elementary integration. More recently, the first two of the present authors gave evaluations for a family of improper integrals containing the square of the tails of the Maclaurin series for the sine and cosine functions [11, 2, 1]. More specifically, for each $n \in \mathbb{Z}_{\geqslant 0} = \{0,1,2,\ldots\}$, let

$$C_n(x) = \cos x - \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} \quad \text{and} \quad S_n(x) = \sin x - \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}.$$

For $a \in \mathbb{R}$ it can easily be shown that the improper integral $\int_0^\infty C_n^2(x)/x^a\,dx$ converges for $4n+1 < a < 4n+5$ while the improper integral $\int_0^\infty S_n^2(x)/x^a\,dx$ converges for $4n+3 < a < 4n+7$. So for $a \in \mathbb{Z}_{\geqslant 0}$, there are three convergent improper integrals in each of the families

$$\int_0^\infty \frac{C_n^2(x)}{x^a}\,dx,\ a = \begin{cases} 4n+2 \\ 4n+3 \\ 4n+4, \end{cases} \qquad \text{and} \qquad \int_0^\infty \frac{S_n^2(x)}{x^a}\,dx,\ a = \begin{cases} 4n+4 \\ 4n+5 \\ 4n+6 \end{cases} \tag{1}$$

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where $n \in \mathbb{Z}_{\geqslant 0}$. Four of the improper integrals in these families have integrands that are even functions. Their values were found using Fourier transform [11] and contour integration [2] methods, and an elementary method involving integration by parts and induction [1]. The values are:

$$I_n = \int_0^\infty \frac{C_n^2(x)}{x^{4n+2}}\,dx = \frac{1}{(4n+1)!}\binom{4n}{2n}\cdot\frac{\pi}{2}; \tag{2}$$

$$J_n = \int_0^\infty \frac{C_n^2(x)}{x^{4n+4}}\,dx = \frac{1}{(4n+3)!}\binom{4n+2}{2n+1}\cdot\frac{\pi}{2}; \tag{3}$$

$$\Lambda_n = \int_0^\infty \frac{S_n^2(x)}{x^{4n+4}}\,dx = \frac{1}{(4n+3)!}\binom{4n+2}{2n+1}\cdot\frac{\pi}{2}; \tag{4}$$

$$\Pi_n = \int_0^\infty \frac{S_n^2(x)}{x^{4n+6}}\,dx = \frac{1}{(4n+5)!}\binom{4n+4}{2n+2}\cdot\frac{\pi}{2}. \tag{5}$$

Notice that each of the improper integrals is equal to a positive rational multiple of $\frac{\pi}{2}$, which is the value of $\Pi_{-1} = \int_0^\infty \left(\frac{S_{-1}(x)}{x}\right)^2 dx = \int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx$, where the empty sum that arises in $S_n(x)$ when $n = -1$ is understood to be nil. Two obvious connections between the four improper integrals are $J_n = \Lambda_n$ and $I_n = \Pi_{n-1}$ for all $n \in \mathbb{Z}_{\geqslant 0}$.

The remaining two improper integrals in this family have integrands that are odd functions. Neither Fourier transform methods nor contour integration methods can be used to evaluate these integrals. In the former case, the general forms for the Fourier transforms needed are not known [11]. In the latter case, the integrand being odd precludes contour integration being used. The integrals can instead be found using antidifferentiation and an inductive argument that relies on making use of the cosine integral function. The values are [2]:

$$K_n = \int_0^\infty \frac{C_n^2(x)}{x^{4n+3}}\,dx = \frac{2}{(4n+2)!}\left\{2^{4n}\log 2 - 2^{4n}H_{4n+2} + \sum_{k=n+1}^{2n+1}\binom{4n+2}{2k}H_{2k}\right\}; \tag{6}$$

$$\Psi_n = \int_0^\infty \frac{S_n^2(x)}{x^{4n+5}}\,dx = \frac{2}{(4n+4)!}\left\{2^{4n+2}\log 2 - 2^{4n+2}H_{4n+4} + \sum_{k=n+1}^{2n+1}\binom{4n+4}{2k+1}H_{2k+1}\right\}. \tag{7}$$

Here H_n denotes the nth harmonic number defined by $\sum_{k=1}^{n}\frac{1}{k}$ along with $H_0 \equiv 0$. Notice that each improper integral consists of a sum of a rational part (which may be zero) and a term consisting of a positive rational multiple of $\log 2$.

In moving to the case for improper integrals containing the cube of the tails of the Maclaurin series for the sine and cosine functions, their evaluation becomes more challenging. Fourier transform methods for the case when the integrand is even cannot be used since, unlike the squared case where the presence of the squared term allowed Plancherel's theorem to be used, this is clearly not the case when the integrand is cubed. And while it is possible to use contour integration methods to evaluate those improper integrals where the integrand is even, it again fails for those cases where the integrand is odd. In overcoming these challenges, a recently devised technique known as 'integration by differentiation' can be used. Originally introduced heuristically by Kempf, Jackson, and Morales [6,5] in the context of quantum field theory, it was made rigorous several years later by Jia, Tang, and Kempf [4] and further extended recently by the present authors [3]. Still relatively unknown, using a reformulation of this method we shall see that it is a very powerful method in the evaluation of certain types of definite integrals.

For $a \in \mathbb{R}$, it is easily shown that $\int_0^\infty C_n^3(x)/x^a\,dx$ converges for $6n+1 < a < 6n+7$ while the integral $\int_0^\infty S_n^3(x)/x^a\,dx$ converges for $6n+4 < a < 6n+10$. So for $a \in \mathbb{Z}_{\geqslant 0}$ there are a total of five convergent improper integrals in each of the families

$$\int_0^\infty \frac{C_n^3(x)}{x^a}\,dx,\ a = \begin{cases} 6n+2 \\ 6n+3 \\ 6n+4 \\ 6n+5 \\ 6n+6, \end{cases} \quad \text{and} \quad \int_0^\infty \frac{S_n^3(x)}{x^a}\,dx,\ a = \begin{cases} 6n+5 \\ 6n+6 \\ 6n+7 \\ 6n+8 \\ 6n+9 \end{cases} \tag{8}$$

where $n \in \mathbb{Z}_{\geqslant 0}$. Six of these improper integrals have integrands that are even:

$$C\alpha_n = \int_0^\infty \frac{C_n^3(x)}{x^{6n+2}}\,dx; \tag{9a}$$
$$S\alpha_n = \int_0^\infty \frac{S_n^3(x)}{x^{6n+5}}\,dx; \tag{9b}$$
$$C\gamma_n = \int_0^\infty \frac{C_n^3(x)}{x^{6n+4}}\,dx; \tag{9c}$$
$$S\gamma_n = \int_0^\infty \frac{S_n^3(x)}{x^{6n+7}}\,dx; \tag{9d}$$
$$C\varepsilon_n = \int_0^\infty \frac{C_n^3(x)}{x^{6n+6}}\,dx; \tag{9e}$$
$$S\varepsilon_n = \int_0^\infty \frac{S_n^3(x)}{x^{6n+9}}\,dx. \tag{9f}$$

The remaining four integrals have integrands that are odd. They are:

$$C\mu_n = \int_0^\infty \frac{C_n^3(x)}{x^{6n+3}}\,dx; \tag{10a}$$
$$S\mu_n = \int_0^\infty \frac{S_n^3(x)}{x^{6n+6}}\,dx; \tag{10b}$$
$$C\nu_n = \int_0^\infty \frac{C_n^3(x)}{x^{6n+5}}\,dx; \tag{10c}$$
$$S\nu_n = \int_0^\infty \frac{S_n^3(x)}{x^{6n+8}}\,dx. \tag{10d}$$

For small values of n, exact values for the integrals can be readily found using simple techniques such as integration by parts. For example, when $n=0$, the values of the integrals with even integrands are

$$\int_0^\infty \frac{(\cos x - 1)^3}{x^2}\,dx = -\frac{3}{4}\pi; \qquad \int_0^\infty \frac{(\sin x - x)^3}{x^5}\,dx = -\frac{5}{32}\pi;$$
$$\int_0^\infty \frac{(\cos x - 1)^3}{x^4}\,dx = -\frac{1}{8}\pi; \qquad \int_0^\infty \frac{(\sin x - x)^3}{x^7}\,dx = -\frac{11}{960}\pi;$$
$$\int_0^\infty \frac{(\cos x - 1)^3}{x^6}\,dx = -\frac{11}{160}\pi; \qquad \int_0^\infty \frac{(\sin x - x)^3}{x^9}\,dx = -\frac{181}{53760}\pi;$$

while the values of the integrals with odd integrands are

$$\int_0^\infty \frac{(\cos x-1)^3}{x^3}\,dx = -3\log 2+\frac{9}{8}\log 3;$$
$$\int_0^\infty \frac{(\cos x-1)^3}{x^5}\,dx = \log 2-\frac{27}{32}\log 3;$$
$$\int_0^\infty \frac{(\sin x-x)^3}{x^6}\,dx = \frac{1}{40}-\log 2+\frac{81}{160}\log 3;$$
$$\int_0^\infty \frac{(\sin x-x)^3}{x^8}\,dx = \frac{19}{1680}+\frac{2}{15}\log 2-\frac{243}{2240}\log 3.$$

For larger values of n, the coefficients for the numbers π, $\log 2$, and $\log 3$; which we shall verify are rational; quickly become more involved. For instance, two examples when $n=1$ are

$$\int_0^\infty \frac{(\sin x-x+x^3/6)^3}{x^{15}}\,dx = \frac{73501}{116237721600}\pi;$$
$$\int_0^\infty \frac{(\cos x-1+x^2/2)^3}{x^{11}}\,dx = -\frac{1187}{2419200}-\frac{7}{1350}\log 2+\frac{729}{179200}\log 3.$$

As the value of n grows, eventually all computer algebra systems fail to find exact values for the integrals. The primary goal of this paper is to find the exact values of these integrals for all values of $n\in\mathbb{Z}_{\geqslant 0}$.

In the next section, a reformulation of the method of integration by differentiation that is going to be used in the evaluation of the family of integrals given in (8) is introduced. The contribution of several terms that are going to be repeatedly found in the integrands of those improper integrals we wish to evaluate are explicitly found, thereby easing some of the computational burden associated with the method. Expressions for all ten of the convergent improper integrals (9a) to (9f) and (10a) to (10d) are listed in Section 3 while their evaluation is given in Section 4. In Section 5, a number of inter-relations between the six improper integrals (9a) to (9f) are established and lead to some interesting binomial identities. These are recorded in Section 6.

2. A reformulation of the method of integration by differentiation

When the integration by differentiation method applies, it allows the integral to be found in a systematic fashion without the need of having to resort to special functions, brute force, or some other combination of clever tricks. As a technique, it is naturally well suited to the evaluation of the improper integrals we consider here. It also makes the evaluation of Dirichlet-type integrals very easy. Recall Dirichlet's famous integral is $\int_0^\infty \frac{\sin x}{x}\,dx=\frac{\pi}{2}$. We now present a reformulation of this so-called integration by differentiation method.

We begin by first introducing a function that helps in streamlining the process. Let $z=a+bi$ be a fixed non-zero complex number with $a\leqslant 0$, $b\in\mathbb{R}$, and let n be a non-negative integer. Here i is the imaginary unit. Define a function $E_{(n,z)}(x)$ on $\mathbb{R}$ by $E_{(n,z)}(0) = z^{n+1}/(n+1)!$ and

$$E_{(n,z)}(x) = \frac{1}{x^{n+1}}\left(e^{zx} - \sum_{k=0}^{n} \frac{z^k}{k!}x^k\right)$$

for $x \neq 0$. Note that the function $E_{(n,z)}$ is analytic on $\mathbb{R}$ and bounded on the interval $[0,\infty)$. We now give in the following Lemma an integral representation for the function $E_{(n,z)}$.

LEMMA 1. *The function $E_{(n,z)}$ has the following integral representation*

$$E_{(n,z)}(x) = \frac{z^{n+1}}{n!}\int_0^1 (1-t)^n e^{zxt}\,dt.$$

Proof. Using properties of the beta function, we find that

$$\begin{aligned}
E_{(n,z)}(x) &= \sum_{k=n+1}^{\infty} \frac{z^k}{k!}x^{k-n-1} = \sum_{k=0}^{\infty} \frac{z^{k+n+1}}{(k+n+1)!}x^k \\
&= \frac{z^{n+1}}{n!}\sum_{k=0}^{\infty}\left(\frac{(zx)^k}{k!}\cdot\frac{k!\,n!}{(k+n+1)!}\right) \\
&= \frac{z^{n+1}}{n!}\sum_{k=0}^{\infty}\left(\frac{(zx)^k}{k!}\cdot\int_0^1 (1-t)^n t^k\,dt\right) \\
&= \frac{z^{n+1}}{n!}\int_0^1 \sum_{k=0}^{\infty}\left(\frac{(zxt)^k}{k!}(1-t)^n\,dt\right) \\
&= \frac{z^{n+1}}{n!}\int_0^1 (1-t)^n e^{zxt}\,dt,
\end{aligned}$$

and this completes the proof. Here the interchange made between the integral and sum is valid since the power series for the exponential function converges uniformly on the interval $[0,1]$. $\square$

In what follows, we use the principal branch of the logarithm function, denoted by Log, for complex arguments. Again H_n denotes the nth harmonic number, with $H_0 = 0$. Finally, the Laplace transform of f, denoted by F, is defined by

$$F(s) = \mathscr{L}\{f(t)\}(s) := \int_0^\infty e^{-st} f(t)\,dt,$$

for all values of s for which the improper integral converges. The connection between the Laplace transform and our reformulated method of integration by differentiation will be formalised in Proposition 1. In the next Lemma, we give an important limit for the Laplace transform of the function $E_{(n,z)}$.

LEMMA 2. *The Laplace transform $\Phi_{(n,z)}$ of $E_{(n,z)}$ satisfies the equation*

$$\lim_{s\to 0^+}\left(\Phi_{(n,z)}(s)-\frac{z^n}{n!}\log s\right)=\frac{z^n}{n!}\left\{H_n-\log|z|-i\arctan\left(\frac{b}{a}\right)\right\} \tag{11}$$

for all appropriate n and z values.

Proof. Using Lemma 1, we find

$$\begin{aligned}
\Phi_{(n,z)}(s) &:= \mathscr{L}\{E_{(n,z)}(x)\}(s)\\
&= \int_0^\infty E_{(n,z)}(x)e^{-sx}\,dx\\
&= \int_0^\infty \left(\frac{z^{n+1}}{n!}\int_0^1 (1-t)^n e^{zxt}\,dt\right)e^{-sx}\,dx.
\end{aligned}$$

Fubini's theorem can be applied to change the order of integration since

$$\iint_R \left|(1-t)^n e^{(zt-s)x}\right|\,dA$$

exists, where R is the infinite rectangle $\{(t,x): 0\leqslant t\leqslant 1,\, x\geqslant 0\}$. Doing so, and noting that

$$\lim_{x\to\infty} e^{(zt-s)x}=\lim_{x\to\infty} e^{(at-s)x+bitx}=0$$

for all $t\geqslant 0$ when $s>0$ and $a\leqslant 0$, we find

$$\begin{aligned}
\Phi_{(n,z)}(s) &= \frac{z^{n+1}}{n!}\int_0^1\int_0^\infty (1-t)^n e^{(zt-s)x}\,dx\,dt\\
&= \frac{z^{n+1}}{n!}\int_0^1 (1-t)^n\cdot \left.\frac{e^{(zt-s)x}}{zt-s}\right|_0^\infty dt\\
&= \frac{z^{n+1}}{n!}\int_0^1 \frac{(1-t)^n}{s-zt}\,dt\\
&= \frac{z^{n+1}}{n!}\int_0^1 \frac{1-(1-t)^n}{zt-s}\,dt+\frac{z^{n+1}}{n!}\int_0^1\frac{1}{s-zt}\,dt\\
&= \frac{z^{n+1}}{n!}\int_0^1 \frac{1-(1-t)^n}{zt-s}\,dt-\frac{z^n}{n!}\left.\mathrm{Log}(s-zt)\right|_0^1\\
&= \frac{z^{n+1}}{n!}\int_0^1 \frac{1-(1-t)^n}{zt-s}\,dt-\frac{z^n}{n!}\left(\mathrm{Log}(s-z)-\log s\right);\\
\Phi_{(n,z)}(s)-\frac{z^n}{n!}\log s &= \frac{z^{n+1}}{n!}\int_0^1 \frac{1-(1-t)^n}{zt-s}\,dt-\frac{z^n}{n!}\mathrm{Log}(s-z)
\end{aligned}$$

for all $s>0$. Observing the integrand for the remaining t-integral is continuous for

$0 \leqslant t \leqslant 1$ and $0 \leqslant s \leqslant 1$, on taking the limit as $s \to 0^+$ yields

$$\begin{aligned}
\lim_{s \to 0^+} \left(\Phi_{(n,z)}(s) - \frac{z^n}{n!} \log s \right) &= \frac{z^n}{n!} \left(\int_0^1 \frac{1-(1-t)^n}{t} \, dt - \mathrm{Log}(-z) \right) \\
&= \frac{z^n}{n!} \left(\int_0^1 \frac{1-t^n}{1-t} \, dt - \mathrm{Log}(-a-bi) \right) \\
&= \frac{z^n}{n!} \left\{ H_n - \log(\sqrt{a^2+b^2}) - i \arctan\left(\frac{b}{a}\right) \right\} \\
&= \frac{z^n}{n!} \left\{ H_n - \log|z| - i \arctan\left(\frac{b}{a}\right) \right\}.
\end{aligned}$$

This completes the proof. □

REMARK 1. We note that taking the limit of (11) as $a \to 0^-$ yields

$$\lim_{s \to 0^+} \left(\Phi_{(n,z)}(s) - \frac{(bi)^n}{n!} \log s \right) = \frac{(bi)^n}{n!} \left(H_n - \log|b| + \frac{i\pi}{2} \operatorname{sgn}(b) \right).$$

Here sgn denotes the sign function.

Let S be the collection of all functions that are analytic on $\mathbb{R}$, bounded on $[0,\infty)$, the improper integral $\int_0^\infty f$ exists, and can be expressed as a finite linear combination of terms of the form $t^k e^{\beta t}$, where k is a negative integer and β is a complex number with $\mathrm{Re}\, \beta \leqslant 0$, using complex numbers for the scalars. We now give the main result to be used in evaluating the improper integrals in this paper using. It represents a reformulation of what in the past has been referred to as the method of integration by differentiation.

PROPOSITION 1. *Suppose that $f \in S$ can be expressed as*

$$f(x) = \sum_{k=1}^{n} \frac{c_k e^{z_k x}}{x^{p_k+1}} + \sum_{k=1}^{m} \frac{d_k}{x^{q_k+1}},$$

where $z_k = a_k + b_k i$ with $a_k \leqslant 0$ and $b_k \neq 0$, the values of c_k and d_k are complex numbers, and the values of p_k and q_k are positive integers. Then

$$\int_0^\infty f(x)\, dx = \sum_{k=1}^{n} \frac{c_k z_k^{p_k}}{p_k!} \left\{ H_{p_k} - \log|z_k| - i \arctan\left(\frac{b_k}{a_k}\right) \right\}. \tag{12}$$

Proof. Define a function g on $\mathbb{R}$ by

$$g(x) = \sum_{k=1}^{n} c_k E_{(p_k,z_k)}(x) = \sum_{k=1}^{n} c_k \left(\frac{e^{z_k x}}{x^{p_k+1}} - \sum_{j=0}^{p_k} \frac{z_k^j}{j!} x^{j-p_k-1} \right)$$

and note that

$$f(x) - g(x) = \sum_{k=1}^{m} \left(\frac{d_k}{x^{q_k+1}} + \frac{c_k}{x^{p_k+1}} \sum_{j=0}^{p_k} \frac{z_k^j}{j!} x^j \right).$$

Since f and g are continuous at zero and the rational function $f-g$ consists of a linear combination of terms of the form $1/x^\ell$, where ℓ is a positive integer, we conclude that $f-g$ must be the zero function. It follows that $f(x)=\sum_{k=1}^{\infty} c_k E_{(p_k,z_k)}(x)$. In addition, since $\int_0^\infty f$ exists, we must have $\sum_{k=1}^{n} c_k z_k^{p_k}/p_k! = 0$ since this is the coefficient of $1/x$ in the expression for f (using the g representation for f). Using this fact together with Lemma 2, we find that

$$\begin{aligned}\int_0^\infty f(t)\,dt &= \lim_{s\to 0^+} \mathscr{L}\{f(t)\}(s)\\ &= \lim_{s\to 0^+} \sum_{k=1}^{n} c_k \Phi_{(p_k,z_k)}(x)\\ &= \lim_{s\to 0^+} \sum_{k=1}^{n} c_k \left(\Phi_{(p_k,z_k)}(x) - \frac{z_k^{p_k}}{p_k!}\log s\right)\\ &= \sum_{k=1}^{n} \frac{c_k z^{p_k}}{p_k!}\left\{H_{p_k} - \log|z_k| - i\arctan\left(\frac{b_k}{a_k}\right)\right\}.\end{aligned}$$

This completes the proof. □

We now give a number of corollaries stemming from Proposition 1. These relate to the contribution particular terms appearing in the integrand of (12) give to the value of the improper integral when evaluating each of the ten improper integrals found in (8). In what follows we use $\bar{z}$ to denote the conjugate of the complex number z.

COROLLARY 1. *Let $z=a+bi$ with $a\leqslant 0$ and $b>0$ and write $z^n=c+di$. If a function f belongs to S, from Proposition 1 a term in f of the form*

$$\frac{e^{ax}\sin(bx)}{x^{n+1}} \quad \textit{contributes} \quad \frac{1}{n!}\left\{dH_n - d\log(\sqrt{a^2+b^2}) - c\arctan\left(\frac{b}{a}\right)\right\} \tag{13}$$

as part of the value of the integral when evaluated, while

$$\frac{e^{ax}\cos(bx)}{x^{n+1}} \quad \textit{contributes} \quad \frac{1}{n!}\left\{cH_n - c\log(\sqrt{a^2+b^2}) + d\arctan\left(\frac{b}{a}\right)\right\} \tag{14}$$

as part of the value of the integral when evaluated.

Proof. From Proposition 1, a term in the integrand of the form

$$\frac{e^{ax}\sin(bx)}{x^{n+1}} = \frac{e^{ax}}{x^{n+1}}\cdot\frac{e^{ibx}-e^{-ibx}}{2i} = \frac{1}{2i}\left(\frac{e^{zx}}{x^{n+1}} - \frac{e^{\bar{z}x}}{x^{n+1}}\right)$$

contributes

$$\begin{aligned}
&\frac{1}{2i}\left[\frac{z^n}{n!}\left\{H_n-\log|z|-i\arctan\left(\frac{b}{a}\right)\right\}-\frac{\bar{z}^n}{n!}\left\{H_n-\log|\bar{z}|-i\arctan\left(\frac{-b}{a}\right)\right\}\right]\\
&\quad=\frac{1}{2in!}\left\{(z^n-\bar{z}^n)H_n-(z^n-\bar{z}^n)\log|z|-i(z^n+\bar{z}^n)\arctan\left(\frac{b}{a}\right)\right\}\\
&\quad=\frac{1}{2in!}\left\{(2diH_n-2di\log|z|-2ci\arctan\left(\frac{b}{a}\right)\right\}\\
&\quad=\frac{1}{n!}\left\{dH_n-d\log(\sqrt{a^2+b^2})-c\arctan\left(\frac{b}{a}\right)\right\}
\end{aligned}$$

as part of the value of the integral when evaluated. Similarly, a term in the integrand of the form

$$\frac{e^{ax}\cos(bx)}{x^{n+1}}=\frac{e^{ax}}{x^{n+1}}\cdot\frac{e^{ibx}+e^{-ibx}}{2}=\frac{1}{2}\left(\frac{e^{zx}}{x^{n+1}}+\frac{e^{\bar{z}x}}{x^{n+1}}\right)$$

contributes

$$\begin{aligned}
&\frac{1}{2}\left[\frac{z^n}{n!}\left\{H_n-\log|z|-i\arctan\left(\frac{b}{a}\right)\right\}+\frac{\bar{z}^n}{n!}\left\{H_n-\log|\bar{z}|-i\arctan\left(\frac{-b}{a}\right)\right\}\right]\\
&\quad=\frac{1}{2n!}\left\{(z^n+\bar{z}^n)H_n-(z^n+\bar{z}^n)\log|z|-i(z^n-\bar{z}^n)\arctan\left(\frac{b}{a}\right)\right\}\\
&\quad=\frac{1}{2n!}\left\{2cH_n-2c\log|z|+2d\arctan\left(\frac{b}{a}\right)\right\}\\
&\quad=\frac{1}{n!}\left\{cH_n-c\log(\sqrt{a^2+b^2})+d\arctan\left(\frac{b}{a}\right)\right\}
\end{aligned}$$

as part of the value of the integral when evaluated. This completes the proof. □

COROLLARY 2. *Taking the limits of (13) and (14) as $a\to 0^-$ in Corollary 1, we find that*

$$\frac{\sin(bx)}{x^{n+1}}\quad\textit{contributes}\quad\frac{1}{n!}\left(dH_n-d\log b+\frac{c\pi}{2}\right)\tag{15}$$

and

$$\frac{\cos(bx)}{x^{n+1}}\quad\textit{contributes}\quad\frac{1}{n!}\left(cH_n-c\log b-\frac{d\pi}{2}\right)\tag{16}$$

as part of their values of the integrals when evaluated.

COROLLARY 3. *Let f belong to S. Then from Proposition 1 a term in f of the form*

$$\frac{e^{zx}}{x^{n+1}}\quad\textit{contributes}\quad\frac{z^n}{n!}\left\{H_n-\log|z|-i\arctan\left(\frac{b}{a}\right)\right\},$$

to the integral $\int_0^\infty f(x)\,dx$, where $z=a+bi$ with $a\leqslant 0$, while a term of the form

$$\frac{1}{x^{n+1}}\quad\textit{contributes}\quad 0,$$

to the same integral. Moreover, the value of $\int_0^\infty f(x)\,dx$ *is equal to the sum of these contributions.*

COROLLARY 4. *If* q *is a positive integer and* b *a real number, then terms in* f *of the form*

$$\begin{array}{lll} \dfrac{\sin(bx)}{x^{2q}} & \textit{contributes} & (-1)^{q+1}\cdot\dfrac{b^{2q-1}}{(2q-1)!}\cdot\big(H_{2q-1}-\log b\big) \\ \dfrac{\cos(bx)}{x^{2q}} & \textit{contributes} & (-1)^{q}\cdot\dfrac{b^{2q-1}}{(2q-1)!}\cdot\dfrac{\pi}{2} \\ \dfrac{\sin(bx)}{x^{2q+1}} & \textit{contributes} & (-1)^{q}\cdot\dfrac{b^{2q}}{(2q)!}\cdot\dfrac{\pi}{2} \\ \dfrac{\cos(bx)}{x^{2q+1}} & \textit{contributes} & (-1)^{q}\cdot\dfrac{b^{2q}}{(2q)!}\cdot\big(H_{2q}-\log b\big) \end{array}$$

to the integral $\int_0^\infty f(x)\,dx$.

Proof. When $n=2q-1$ with q a positive integer, we find that $(ib)^n = -i(-1)^q b^{2q-1}$. Thus $c=0$ and $d=(-1)^{q+1}b^{2q-1}$ and the first and second of the results in Corollary 4 follow from (15) and (16) in Corollary 2.

In a similar manner, when $n=2q$ with q a positive integer, we find that $(ib)^n = (-1)^q b^{2q}$. Thus $c=(-1)^q b^{2q}$ and $d=0$ and the third and fourth results in Corollary 4 follow from (15) and (16) in Corollary 2. This completes the proof. □

As our reformulated integration by differentiation method is not likely to be familiar to many, we illustrate the approach using two simple examples.

EXAMPLE 1. We shall verify that (Eq. (9b) when $n=0$)

$$\int_0^\infty \frac{(\sin x - x)^3}{x^5}\,dx = -\frac{5}{32}\pi.$$

From elementary trigonometric identities, we see that the integrand can be expressed as

$$\frac{(\sin x - x)^3}{x^5} = \frac{\frac{3}{4}\sin(x)}{x^5} - \frac{\frac{1}{4}\sin(3x)}{x^5} - \frac{3}{2x^4} + \frac{\frac{3}{2}\cos(2x)}{x^4} + \frac{3\sin(x)}{x^3} - \frac{1}{x^2}.$$

Using Proposition 1, by Corollary 3, two of the corresponding terms will be zero. These are the third and sixth terms. We use Corollary 4 to evaluate the four remaining terms. We have:

$$\begin{array}{lll} \dfrac{\frac{3}{4}\sin(x)}{x^5} & \text{contributes} & \dfrac{3}{4}(-1)^2\cdot\dfrac{1^4}{4!}\cdot\dfrac{\pi}{2} = \dfrac{\pi}{64}; \\ \dfrac{\frac{1}{4}\sin(3x)}{x^5} & \text{contributes} & \dfrac{1}{4}(-1)^2\cdot\dfrac{3^4}{4!}\cdot\dfrac{\pi}{2} = \dfrac{27\pi}{64}; \end{array}$$

$$\frac{\frac{3}{2}\cos(2x)}{x^4} \quad \text{contributes} \quad \frac{3}{2}(-1)^2\cdot\frac{2^3}{3!}\cdot\frac{\pi}{2}=\pi;$$
$$\frac{3\sin(x)}{x^3} \quad \text{contributes} \quad 3(-1)^1\cdot\frac{1^2}{2!}\cdot\frac{\pi}{2}=-\frac{3\pi}{4}.$$

So the value for the integral is

$$\int_0^\infty \frac{(\sin x - x)^3}{x^5}\,dx = \frac{\pi}{64}-\frac{27\pi}{64}+\pi-\frac{3\pi}{4}=-\frac{5\pi}{32},$$

as expected. Note that a direct approach using elementary methods to evaluate this integral is possible but involves considerably more work.

EXAMPLE 2. We shall verify that (Eq. (10a) when $n=0$)

$$\int_0^\infty \frac{(\cos x-1)^3}{x^3}\,dx = -3\log 2+\frac{9}{8}\log 3.$$

From elementary trigonometric identities, we see that the integrand can be expressed as

$$\frac{(\cos x-1)^3}{x^3}=\frac{\frac{1}{4}\cos(3x)}{x^3}+\frac{\frac{15}{4}\cos(x)}{x^3}-\frac{\frac{3}{2}\cos(2x)}{x^3}-\frac{5}{2x^3}.$$

Using Proposition 1, by Corollary 3, one of the corresponding terms evaluates to zero. This is the fourth term. We then use Corollary 4 to evaluate the three remaining terms. We have:

$$\frac{\frac{1}{4}\cos(3x)}{x^3} \quad \text{contributes} \quad \frac{1}{4}(-1)^1\cdot\frac{3^2}{2!}\cdot(H_2-\log 3)=\frac{9}{8}(\log 3-H_2);$$
$$\frac{\frac{15}{4}\cos(x)}{x^3} \quad \text{contributes} \quad \frac{15}{4}(-1)^1\cdot\frac{1^2}{2!}\cdot(H_2-\log 1)=-\frac{15}{8}H_2$$
$$\frac{\frac{3}{2}\cos(2x)}{x^3} \quad \text{contributes} \quad \frac{3}{2}(-1)^1\cdot\frac{2^2}{2!}\cdot(H_2-\log 2)=3(\log 2-H_2).$$

So the value for the integral is

$$\begin{aligned}\int_0^\infty \frac{(\cos x-1)^3}{x^3}\,dx &= \frac{9}{8}(\log 3-H_2)-\frac{15}{8}H_2-3(\log 2-H_2)\\ &= -3\log 2+\frac{9}{8}\log 3,\end{aligned}$$

as expected.

3. Main results

In this section, we list results for all ten convergent improper integrals in the class given by (8). Six of these results involve even integrands. For those containing $S_n(x)$ in the integrand, if r has the value 2, 3, or 4, then:

$$\int_0^\infty \frac{S_n^3(x)}{x^{6n+2r+1}}\,dx = \frac{(-1)^{3n+r}}{(6n+2r)!}\left\{1-3^{6n+2r-1}+\sum_{k=0}^{n}\binom{6n+2r}{2k+1}2^{6n-2k+2r}\right.$$
$$\left.-4\sum_{k=1}^{2n+1}\binom{6n+2r}{2k}\sum_{j=k-n-1}^{n}\binom{2k}{2j+1}\right\}\cdot\frac{3\pi}{8}. \tag{17}$$

For those containing $C_n(x)$ in the integrand, if r has the value 1, 2, or 3, then:

$$\int_0^\infty \frac{C_n^3(x)}{x^{6n+2r}}\,dx = \frac{(-1)^{3n+r}}{(6n+2r-1)!}\left\{1+3^{6n+2r-2}-\sum_{k=0}^{n}\binom{6n+2r-1}{2k}2^{6n-2k+2r}\right.$$
$$\left.+4\sum_{k=0}^{2n}\binom{6n+2r-1}{2k}\sum_{j=k-n}^{n}\binom{2k}{2j}\right\}\cdot\frac{3\pi}{8}. \tag{18}$$

REMARK 2. As was the case for the square of the tails when the integrand was even, notice the value for each of the improper integrals is equal to a rational multiple of $\frac{3\pi}{8}$; the value of $S\varepsilon_{-1}=\int_0^\infty\left(\frac{\sin x}{x}\right)^3 dx$.

The remaining four results involve odd integrands. For those containing $S_n(x)$ in the integrand, if r has the value 3 or 4, then:

$$\int_0^\infty \frac{S_n^3(x)}{x^{6n+2r}}\,dx = \frac{3(-1)^{3n+r-1}}{4(6n+2r-1)!}\left\{H_{6n+2r-1}+3^{6n+2r-2}\left(\log 3-H_{6n+2r-1}\right)\right.$$
$$-\sum_{k=0}^{n}\binom{6n+2r-1}{2k+1}2^{6n-2k+2r-1}\left(\log 2-H_{6n-2k+2r-2}\right)$$
$$\left.-4\sum_{k=1}^{2n+1}\binom{6n+2r-1}{2k}\sum_{j=k-n-1}^{n}\binom{2k}{2j+1}H_{6n-2k+2r-1}\right\}. \tag{19}$$

For those containing $C_n(x)$ in the integrand, if r has the value 1 or 2, then:

$$\int_0^\infty \frac{C_n^3(x)}{x^{6n+2r+1}}\,dx = \frac{3(-1)^{3n+r}}{4(6n+2r)!}\left\{H_{6n+2r}-3^{6n+2r-1}\left(\log 3-H_{6n+2r}\right)\right.$$
$$+\sum_{k=0}^{n}\binom{6n+2r}{2k}2^{6n-2k+2r+1}\left(\log 2-H_{6n-2k+2r}\right)$$
$$\left.+4\sum_{k=0}^{2n}\binom{6n+2r}{2k}\sum_{j=k-n}^{n}\binom{2k}{2j}H_{6n-2k+2r}\right\}. \tag{20}$$

REMARK 3. Notice in these four cases, the value for the improper integral consists of a sum of three terms: a rational part (which may be zero) and two terms consisting of a rational multiple of $\log 2$ and $\log 3$.

4. Evaluations using Proposition 1

In this section, we use the reformulated method of integration by differentiation as established in Proposition 1 to prove the integral equalities listed in the previous section. The results follow in a relatively straight forward manner on application of Corollaries 3 and 4. As the evaluations depend on a term containing the square of a finite sum, we consider this first.

4.1. Square of a finite sum

For $n \in \mathbb{Z}_{\geqslant 0}$, if we take the Cauchy product of the series $\sum_{k=0}^{\infty} \alpha_k x^k$ with itself, assuming that $\alpha_k = 0$ for all $k > n$, we find

$$\left(\sum_{k=0}^{n} \alpha_k x^k\right)^2 = \sum_{k=0}^{n}\sum_{j=0}^{k} \alpha_j \alpha_{k-j} x^k + \sum_{k=n+1}^{2n}\sum_{j=k-n}^{n} \alpha_j \alpha_{k-j} x^k.$$

We shall write this as

$$\left(\sum_{k=0}^{n} \alpha_k x^k\right)^2 = \sum_{k=0}^{2n} \gamma_k x^k, \quad \text{where } \gamma_k = \begin{cases} \sum_{j=0}^{k} \alpha_j \alpha_{k-j}, & 0 \leqslant k \leqslant n; \\ \sum_{j=k-n}^{n} \alpha_j \alpha_{k-j}, & n < k \leqslant 2n. \end{cases}$$

Now define the terms $b_c(n,k)$ for $n \geqslant 0$ and $0 \leqslant k \leqslant 2n$ so that the equation

$$c_n^2(x) = \left(\sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k}\right)^2 = \sum_{k=0}^{2n} b_c(n,k) x^{2k}$$

is satisfied. Noting that $b_c(n,k)$ is the coefficient of x^k in the expression $c_n^2(\sqrt{x})$ and that

$$\frac{(-1)^j}{(2j)!} \cdot \frac{(-1)^{k-j}}{(2k-2j)!} = \frac{(-1)^k}{(2k)!}\binom{2k}{2j},$$

it follows that

$$b_c(n,k) = \begin{cases} \dfrac{(-1)^k}{(2k)!}\displaystyle\sum_{j=0}^{k}\binom{2k}{2j}, & 0 \leqslant k \leqslant n; \\[2ex] \dfrac{(-1)^k}{(2k)!}\displaystyle\sum_{j=k-n}^{n}\binom{2k}{2j}, & n < k \leqslant 2n. \end{cases}$$

Observing the convention that $\binom{n}{k} = 0$ whenever $k > n$ or $k < 0$, we may write

$$(-1)^k (2k)! b_c(n,k) = \sum_{j=k-n}^{n} \binom{2k}{2j} \tag{21}$$

for $0 \leqslant k \leqslant 2n$. We also note that

$$(-1)^k(2k)!b_c(n,k) = \begin{cases} 1, & k=0, \\ 2^{2k-1}, & 1 \leqslant k \leqslant n. \end{cases}$$

Next, define the terms $b_s(n,k)$ for $n \geqslant 0$ and $1 \leqslant k \leqslant 2n+1$ so that the equation

$$s_n^2(x) = \left(\sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \right)^2 = \sum_{k=1}^{2n+1} b_s(n,k) x^{2k},$$

is satisfied. We begin by noting that

$$\frac{s_n(\sqrt{x})}{\sqrt{x}} = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^k,$$

and

$$\frac{(-1)^j}{(2j+1)!} \cdot \frac{(-1)^{k-j}}{(2k-2j+1)!} = \frac{(-1)^k}{(2k+2)!} \binom{2k+2}{2j+1}.$$

Since $b_s(n,k)$ is the coefficient of x^{k-1} in the expression $s_n^2(\sqrt{x})/x$, we find that

$$b_s(n,k) = \begin{cases} \dfrac{(-1)^{k+1}}{(2k)!} \displaystyle\sum_{j=0}^{k-1} \binom{2k}{2j+1}, & 1 \leqslant k \leqslant n+1; \\ \dfrac{(-1)^{k+1}}{(2k)!} \displaystyle\sum_{j=k-n-1}^{n} \binom{2k}{2j+1}, & n+1 < k \leqslant 2n+1. \end{cases}$$

Again observing the convention that $\binom{n}{k} = 0$ whenever $k > n$ or $k < 0$, we may write

$$(-1)^{k+1}(2k)!b_s(n,k) = \sum_{j=k-n-1}^{n} \binom{2k}{2j+1} \tag{22}$$

for $1 \leqslant k \leqslant 2n+1$. We also note that

$$(-1)^{k+1}(2k)!b_s(n,k) = 2^{2k-1}$$

for $1 \leqslant k \leqslant n+1$.

4.2. Proofs for the family of ten improper integrals

We now prove our main results given in Section 3. To verify the values for the family of ten improper integrals containing the cube of the tails of the Maclaurin series for the sine and cosine functions, we require four separate proofs.

We commence by giving proofs for the three improper integrals $C\alpha_n$, $C\gamma_n$, and $C\varepsilon_n$ that have an even integrand and contain a cosine term.

Proof. Consider the integrals

$$C\chi_{r,n} = \int_0^\infty \frac{(\cos x - c_n(x))^3}{x^{6n+2r}}\,dx,$$

where n is a fixed non-negative integer and r has the value 1, 2, or 3. It then follows that $C\chi_{1,n} = C\alpha_n$, $C\chi_{2,n} = C\gamma_n$, and $C\chi_{3,n} = C\varepsilon_n$ and these correspond to the integrals (9a), (9c), and (9e) respectively. Expanding the cube term using elementary trigonometric identities, we have

$$\begin{aligned}
(\cos x - c_n(x))^3 &= \cos^3 x - 3c_n(x)\cos^2 x + 3c_n^2(x)\cos x - c_n^3(x)\\
&= \cos^3 x - 3\cos^2 x\sum_{k=0}^{n} a_c(n,k)x^{2k} + 3\cos x\sum_{k=0}^{2n} b_c(n,k)x^{2k} - \sum_{k=0}^{3n} c_c(n,k)x^{2k}\\
&= \frac{3}{4}\cos x + \frac{1}{4}\cos(3x) - \frac{3}{2}\sum_{k=0}^{n} a_c(n,k)x^{2k} - \frac{3}{2}\cos(2x)\sum_{k=0}^{n} a_c(n,k)x^{2k}\\
&\quad + 3\cos x\sum_{k=0}^{2n} b_c(n,k)x^{2k} - \sum_{k=0}^{3n} c_c(n,k)x^{2k},
\end{aligned} \tag{23}$$

where the values of $a_c(n,k)$ are $(-1)^k/(2k)!$, the values of $b_c(n,k)$ are given by (21), and (as we shall see) the values for $c_c(n,k)$ are not needed. Dividing throughout by x^{6n+2r} yields

$$\begin{aligned}
\frac{(\cos x - c_n(x))^3}{x^{6n+2r}} &= \frac{\frac{3}{4}\cos x}{x^{6n+2r}} + \frac{\frac{1}{4}\cos(3x)}{x^{6n+2r}} - \sum_{k=0}^{n}\frac{\frac{3}{2}a_c(n,k)}{x^{6n-2k+2r}} - \sum_{k=0}^{n}\frac{\frac{3}{2}a_c(n,k)\cos(2x)}{x^{6n-2k+2r}}\\
&\quad + \sum_{k=0}^{2n}\frac{3b_c(n,k)\cos x}{x^{6n-2k+2r}} - \sum_{k=0}^{3n}\frac{c_c(n,k)}{x^{6n-2k+2r}}.
\end{aligned} \tag{24}$$

In evaluating the improper integral, on applying Proposition 1, due to Corollary 3 we see the contribution to the improper integral from two of the corresponding terms will be zero. These are the third and sixth terms of (24). The contribution to the improper integral from the remaining four terms follow from the second of the results given in Corollary 4. We have:

$$\begin{aligned}
C\chi_{r,n} &= \frac{\pi}{2}\left\{\frac{\frac{3}{4}(-1)^{3n+r}}{(6n+2r-1)!} + \frac{\frac{1}{4}(-1)^{3n+r}3^{6n+2r-1}}{(6n+2r-1)!}\right.\\
&\quad \left. - \sum_{k=0}^{n}\frac{\frac{3}{2}a_c(n,k)(-1)^{3n-k+r}2^{6n-2k+2r-1}}{(6n-2k+2r-1)!} + \sum_{k=0}^{2n}\frac{3b_c(n,k)(-1)^{3n-k+r}}{(6n-2k+2r-1)!}\right\}\\
&= (-1)^{3n+r}\left\{\frac{1}{(6n+2r-1)!} + \frac{3^{6n+2r-2}}{(6n+2r-1)!}\right.\\
&\quad \left. - \sum_{k=0}^{n}\frac{(-1)^k a_c(n,k)2^{6n-2k+2r}}{(6n-2k+2r-1)!} + 4\sum_{k=0}^{2n}\frac{b_c(n,k)(-1)^k}{(6n-2k+2r-1)!}\right\}\cdot\frac{3\pi}{8}.
\end{aligned}$$

Using the values for $a_c(n,k)$ and $b_c(n,k)$, we obtain

$$C\chi_{r,n} = \frac{(-1)^{3n+r}}{(6n+2r-1)!}\left\{1+3^{6n+2r-2}-\sum_{k=0}^{n}\binom{6n+2r-1}{2k}2^{6n-2k+2r}\right.$$
$$\left.+4\sum_{k=0}^{2n}\binom{6n+2r-1}{2k}\sum_{j=k-n}^{n}\binom{2k}{2j}\right\}\cdot\frac{3\pi}{8},$$

as the value for the integral, with the values for $C\alpha_n$, $C\gamma_n$, and $C\varepsilon_n$ following on setting $r=1,2$, and 3, respectively. This completes the proof. $\square$

Next we give proofs for the three improper integrals $S\alpha_n$, $S\gamma_n$, and $S\varepsilon_n$ that have an even integrand and contain a sine term.

Proof. Consider the integrals

$$S\chi_{r-1,n} = \int_0^\infty \frac{(\sin x - s_n(x))^3}{x^{6n+2r+1}}\,dx,$$

where n is a fixed non-negative integer and r has the value 2, 3, or 4. It follows that $S\chi_{1,n} = S\alpha_n$, $S\chi_{2,n} = S\gamma_n$, and $S\chi_{3,n} = S\varepsilon_n$ and these correspond to the integrals (9b), (9d), and (9f) respectively. Expanding the cube term using elementary trigonometric identities, we have

$$\begin{aligned}
(\sin x - s_n(x))^3 &= \sin^3 x - 3s_n(x)\sin^2 x + 3s_n^2(x)\sin x - s_n^3(x)\\
&= \sin^3 x - 3\sin^2 x\sum_{k=0}^{n} a_s(n,k)x^{2k+1} + 3\sin x\sum_{k=1}^{2n+1} b_s(n,k)x^{2k}\\
&\quad - \sum_{k=1}^{3n+1} c_s(n,k)x^{2k+1}\\
&= \frac{3}{4}\sin x - \frac{1}{4}\sin(3x) - \frac{3}{2}\sum_{k=0}^{n} a_s(n,k)x^{2k+1} + \frac{3}{2}\cos 2x\sum_{k=0}^{n} a_s(n,k)x^{2k+1}\\
&\quad + 3\sin x\sum_{k=1}^{2n+1} b_s(n,k)x^{2k} - \sum_{k=1}^{3n+1} c_s(n,k)x^{2k+1},
\end{aligned} \tag{25}$$

where the values of $a_s(n,k)$ are $(-1)^k/(2k+1)!$, the values of $b_s(n,k)$ are given by (22), and (as before) the values of $c_s(n,k)$ will not be needed. Dividing throughout by $x^{6n+2r+1}$ yields

$$\begin{aligned}
\frac{(\sin x - s_n(x))^3}{x^{6n+2r+1}} &= \frac{\frac{3}{4}\sin x}{x^{6n+2r+1}} - \frac{\frac{1}{4}\sin(3x)}{x^{6n+2r+1}} - \sum_{k=0}^{n}\frac{\frac{3}{2}a_s(n,k)}{x^{6n-2k+2r}} + \sum_{k=0}^{n}\frac{\frac{3}{2}a_s(n,k)\cos(2x)}{x^{6n-2k+2r}}\\
&\quad + \sum_{k=1}^{2n+1}\frac{3b_s(n,k)\sin x}{x^{6n-2k+2r+1}} - \sum_{k=1}^{3n+1}\frac{c_s(n,k)}{x^{6n-2k+2r}}.
\end{aligned} \tag{26}$$

In evaluating the improper integral, on applying Proposition 1, due to Corollary 3 we see the contribution from two of the corresponding terms to the improper integral will be zero. These are the third and sixth terms of (26). The contribution to the improper integral from the remaining four terms follow from the second and third results given in Corollary 4. We have:

$$
\begin{aligned}
S\chi_{r-1,n} &= \frac{\pi}{2}\left\{\frac{\frac{3}{4}(-1)^{3n+r}}{(6n+2r)!} - \frac{\frac{1}{4}(-1)^{3n+r}3^{6n+2r}}{(6n+2r)!}\right.\\
&\qquad \left. + \sum_{k=0}^{n}\frac{\frac{3}{2}a_s(n,k)(-1)^{3n-k+r}2^{6n-2k+2r-1}}{(6n-2k+2r-1)!} + \sum_{k=1}^{2n+1}\frac{3b_s(n,k)(-1)^{3n-k+r}}{(6n-2k+2r)!}\right\}\\
&= (-1)^{3n+r}\left\{\frac{1}{(6n+2r)!} - \frac{3^{6n+2r-1}}{(6n+2r)!} + \sum_{k=0}^{n}\frac{(-1)^k a_s(n,k)2^{6n-2k+2r}}{(6n-2k+2r-1)!}\right.\\
&\qquad \left. + 4\sum_{k=1}^{2n+1}\frac{b_s(n,k)(-1)^k}{(6n-2k+2r)!}\right\}\cdot\frac{3\pi}{8}.
\end{aligned}
$$

Using the values for $a_s(n,k)$ and $b_s(n,k)$, we find that

$$
\begin{aligned}
S\chi_{r-1,n} = \frac{(-1)^{3n+r}}{(6n+2r)!}&\left\{1 - 3^{6n+2r-1} + \sum_{k=0}^{n}\binom{6n+2r}{2k+1}2^{6n-2k+2r}\right.\\
&\left. - 4\sum_{k=1}^{2n+1}\binom{6n+2r}{2k}\sum_{j=k-n-1}^{n}\binom{2k}{2j+1}\right\}\cdot\frac{3\pi}{8},
\end{aligned}
$$

as the value for the integral, with the values for $S\alpha_n$, $S\gamma_n$, and $S\varepsilon_n$ following on setting $r = 2, 3$, and 4, respectively. This completes the proof. □

We now give proofs for the two improper integrals $C\mu_n$ and $C\nu_n$ that have an odd integrand and contain a cosine term.

Proof. Consider the integrals

$$
C\psi_{r,n} = \int_0^\infty \frac{(\cos x - c_n(x))^3}{x^{6n+2r+1}}\,dx,
$$

where n is a fixed non-negative integer and r has the value 1 or 2. It follows that $C\psi_{1,n} = C\mu_n$ and $C\psi_{2,n} = C\nu_n$ and these correspond to the integrals (10a) and (10c), respectively. If the expansion for $(\cos x - c_n(x))^3$ given in (23) is divided by $x^{6n+2r+1}$, then

$$
\begin{aligned}
\frac{(\cos x - c_n(x))^3}{x^{6n+2r+1}} &= \frac{\frac{3}{4}\cos x}{x^{6n+2r+1}} + \frac{\frac{1}{4}\cos(3x)}{x^{6n+2r+1}} - \sum_{k=0}^{n}\frac{\frac{3}{2}a_c(n,k)}{x^{6n-2k+2r+1}} - \sum_{k=0}^{n}\frac{\frac{3}{2}a_c(n,k)\cos(2x)}{x^{6n-2k+2r+1}}\\
&\quad + \sum_{k=0}^{2n}\frac{3b_c(n,k)\cos x}{x^{6n-2k+2r+1}} - \sum_{k=0}^{3n}\frac{c_c(n,k)}{x^{6n-2k+2r+1}}. \qquad (27)
\end{aligned}
$$

Using Corollary 4 to evaluate the four non-zero contributions to the improper integral, we find:

$$
\begin{aligned}
C\psi_{r,n} &= -\frac{\frac{3}{4}(-1)^{3n+r}}{(6n+2r)!}(\log 1 - H_{6n+2r}) - \frac{\frac{1}{4}(-1)^{3n+r}3^{6n+2r}}{(6n+2r)!}(\log 3 - H_{6n+2r}) \\
&\quad + \sum_{k=0}^{n} \frac{\frac{3}{2}a_c(n,k)(-1)^{3n-k+r}2^{6n-2k+2r}}{(6n-2k+2r)!}(\log 2 - H_{6n-2k+2r}) \\
&\quad - \sum_{k=0}^{2n} \frac{3b_c(n,k)(-1)^{3n-k+r}}{(6n-2k+2r)!}(\log 1 - H_{6n-2k+2r}) \\
&= \frac{3(-1)^{3n+r}}{4(6n+2r)!}\Bigg\{H_{6n+2r} - 3^{6n+2r-1}(\log 3 - H_{6n+2r}) \\
&\quad + \sum_{k=0}^{n} \frac{(-1)^k a_c(n,k)(6n+2r)!}{(6n-2k+2r)!}2^{6n-2k+2r+1}(\log 2 - H_{6n-2k+2r}) \\
&\quad + 4\sum_{k=0}^{2n} \frac{(-1)^k b_c(n,k)(6n+2r)!}{(6n-2k+2r)!}H_{6n-2k+2r}\Bigg\}.
\end{aligned}
$$

Substituting the values for $a_c(n,k)$ and $b_c(n,k)$ gives

$$
\begin{aligned}
C\psi_{r,n} = \frac{3(-1)^{3n+r}}{4(6n+2r)!}\Bigg\{ & H_{6n+2r} - 3^{6n+2r-1}(\log 3 - H_{6n+2r}) \\
& + \sum_{k=0}^{n} \binom{6n+2r}{2k} 2^{6n-2k+2r+1}(\log 2 - H_{6n-2k+2r}) \\
& +4\sum_{k=0}^{2n} \binom{6n+2r}{2k} \sum_{j=k-n}^{n} \binom{2k}{2j} H_{6n-2k+2r}\Bigg\},
\end{aligned}
$$

as the value for the integral, with the values for $C\mu_n$ and $C\nu_n$ following on setting $r=1$ and 2, respectively. This completes the proof. □

Lastly, we give proofs for the two improper integrals $S\mu_n$ and $S\nu_n$ that have an odd integrand and contain a sine term.

Proof. Consider the integrals

$$
S\psi_{r-2,n} = \int_0^\infty \frac{(\sin x - s_n(x))^3}{x^{6n+2r}}\,dx,
$$

where n is a fixed non-negative integer and r has the value 3 or 4. It follows that $S\psi_{1,n} = S\mu_n$ and $S\psi_{2,n} = S\nu_n$ and these correspond to the integrals (10b) and (10d), respectively. If the expansion for $(\sin x - s_n(x))^3$ given in (25) is divided by x^{6n+2r}, then

$$\frac{(\sin x - s_n(x))^3}{x^{6n+2r}} = \frac{\frac{3}{4}\sin x}{x^{6n+2r}} - \frac{\frac{1}{4}\sin(3x)}{x^{6n+2r}} - \sum_{k=0}^{n} \frac{\frac{3}{2}a_s(n,k)}{x^{6n-2k+2r-1}} + \sum_{k=0}^{n} \frac{\frac{3}{2}a_s(n,k)\cos(2x)}{x^{6n-2k+2r-1}} + \sum_{k=1}^{2n+1} \frac{3b_s(n,k)\sin x}{x^{6n-2k+2r}} - \sum_{k=1}^{3n+1} \frac{c_s(n,k)}{x^{6n-2k+2r-1}}. \tag{28}$$

Using again Corollary 4 to evaluate the four non-zero contributions to the improper integral, we find:

$$\begin{aligned} S\psi_{r-2,n} &= \frac{-\frac{3}{4}(-1)^{3n+r+1}}{(6n+2r-1)!}(\log 1 - H_{6n+2r-1}) \\ &\quad + \frac{\frac{1}{4}(-1)^{3n+r+1}3^{6n+2r-1}}{(6n+2r-1)!}(\log 3 - H_{6n+2r-1}) \\ &\quad - \sum_{k=0}^{n} \frac{\frac{3}{2}a_s(n,k)(-1)^{3n-k+r+1}2^{6n-2k+2r-2}}{(6n-2k+2r-2)!}(\log 2 - H_{6n-2k+2r-2}) \\ &\quad + \sum_{k=1}^{2n+1} \frac{3b_s(n,k)(-1)^{3n-k+r}}{(6n-2k+2r-1)!}(\log 1 - H_{6n-2k+2r-1}) \\ &= \frac{3(-1)^{3n+r-1}}{4(6n+2r-1)!}\Bigg\{ H_{6n+2r-1} + 3^{6n+2r-2}(\log 3 - H_{6n+2r-1}) \\ &\quad - \sum_{k=0}^{n} \frac{(-1)^k a_s(n,k)(6n+2r-1)!}{(6n-2k+2r-2)!} 2^{6n-2k+2r-1}(\log 2 - H_{6n-2k+2r-2}) \\ &\quad + 4\sum_{k=1}^{2n+1} \frac{(-1)^k b_s(n,k)(6n+2r-1)!}{(6n-2k+2r-1)!} H_{6n-2k+2r-1} \Bigg\}. \end{aligned}$$

Using the values for $a_s(n,k)$ and $b_s(n,k)$, we find that

$$\begin{aligned} S\psi_{r-2,n} = \frac{3(-1)^{3n+r-1}}{4(6n+2r-1)!}\Bigg\{ & H_{6n+2r-1} + 3^{6n+2r-2}(\log 3 - H_{6n+2r-1}) \\ & - \sum_{k=0}^{n} \binom{6n+2r-1}{2k+1} 2^{6n-2k+2r-1}(\log 2 - H_{6n-2k+2r-2}) \\ & - 4\sum_{k=1}^{2n+1} \binom{6n+2r-1}{2k} \sum_{j=k-n-1}^{n} \binom{2k}{2j+1} H_{6n-2k+2r-1} \Bigg\}, \end{aligned}$$

as the value for the integral with the values for $S\mu_n$ and $S\nu_n$ following on setting $r = 3$ and 4, respectively. This completes the proof. $\square$

5. Interconnections between some of the improper integrals

As mentioned earlier, for the family of improper integrals containing the square of the tails of the Maclaurin series for the sine and cosine functions listed in (1), two simple connections between the integrals with even integrands exist, namely $J_n = \Lambda_n$ and $I_n = \Pi_{n-1}$, respectively [11]. For the family of improper integrals containing the cube of the tails of the Maclaurin series for the sine and cosine functions, a far richer structure in terms of interconnections between the improper integrals (9a) to (9f) containing even integrands exists. We list these and show how each can be established using methods of contour integration. Some intriguing binomial identities that are consequences of these results are presented in the section that follows.

5.1. Results

Six interconnections between the improper integrals (9a) to (9f) containing even integrands exist. For each $n \in \mathbb{Z}_{\geqslant 0}$, they are

$$C\varepsilon_n = (6n+6)S\gamma_n; \tag{29}$$

$$S\alpha_n + (6n+5)C\varepsilon_n = \frac{3(-1)^{n+1}}{(2n+1)!}J_n; \tag{30}$$

$$S\alpha_n + (6n+5)(6n+6)S\gamma_n = \frac{3(-1)^{n+1}}{(2n+1)!}J_n; \tag{31}$$

$$S\varepsilon_{n-1} = -(6n+3)C\gamma_n; \tag{32}$$

$$C\alpha_n - (6n+2)S\varepsilon_{n-1} = \frac{3(-1)^{n+1}}{(2n)!}I_n; \tag{33}$$

$$C\alpha_n + (6n+2)(6n+3)C\gamma_n = \frac{3(-1)^{n+1}}{(2n)!}I_n. \tag{34}$$

Here I_n and J_n are given by (2) and (3), respectively. We also note that (31) and (34) are simple consequences of the previous two connections appearing above them.

5.2. Proofs

For each non-negative integer n, let

$$e_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}, \qquad c_n(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}, \quad s_n(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!},$$
$$E_n(x) = e^x - e_n(x), \quad C_n(x) = \cos x - c_n(x), \quad S_n(x) = \sin x - s_n(x).$$

Recalling $S_{-1}(x) = \sin x$, note that the equations $s_n'(x) = c_n(x)$ and $S_n'(x) = C_n(x)$, and $c_n'(x) = -s_{n-1}(x)$ (with $s_{-1}(x) = 0$) and $C_n'(x) = -S_{n-1}(x)$ are valid for all $n \in \mathbb{Z}_{\geqslant 0}$. It is also easy to verify that

$$e_{2n+1}(ix) = c_n(x) + i s_n(x) \quad \text{and} \quad E_{2n+1}(ix) = C_n(x) + i S_n(x).$$

Here $i=\sqrt{-1}$ is the imaginary unit. Also, we see that the four functions

$$\frac{(E_{2n+1}(iz))^3}{z^{6n+5}}, \quad \frac{(E_{2n+1}(iz))^3}{z^{6n+6}}, \quad \frac{(E_{2n}(iz))^3}{z^{6n+2}}, \quad \text{and} \quad \frac{(E_{2n}(iz))^3}{z^{6n+3}},$$

with $z\in\mathbb{C}$ are analytic in $\mathbb{C}$. By considering the powers on the polynomial terms, one can see that the integrals for all four of these functions over the semi-circular contour C_ρ in the upper half of the complex plane with end-points on the x-axis at $x=\pm\rho$ go to zero as $\rho\to\infty$. We are now in a position to prove each of the six results given in section 5.1. We give proofs for (29) and (30); proofs for (32) and (33) are similar.

Proof. Since

$$\int_{-\infty}^{\infty}\frac{(E_{2n+1}(ix))^3}{x^{6n+6}}\,dx=0=\int_{-\infty}^{\infty}\frac{(E_{2n+1}(ix))^3}{x^{6n+5}}\,dx,$$

as

$$(E_{2n+1}(ix))^3=C_n^3(x)-3C_n(x)S_n^2(x)+i\left(3C_n^2(x)S_n(x)-S_n^3(x)\right),$$

equating real and imaginary parts in the respective integrals yields

$$\int_{-\infty}^{\infty}\frac{C_n^3(x)}{x^{6n+6}}\,dx=\int_{-\infty}^{\infty}\frac{3S_n^2(x)C_n(x)}{x^{6n+6}}\,dx \tag{35}$$

and

$$\int_{-\infty}^{\infty}\frac{S_n^3(x)}{x^{6n+5}}\,dx=\int_{-\infty}^{\infty}\frac{3C_n^2(x)S_n(x)}{x^{6n+5}}\,dx. \tag{36}$$

As the two integrands in (35) are even, it follows that

$$C\varepsilon_n=\int_0^{\infty}\frac{C_n^3(x)}{x^{6n+6}}\,dx=\int_0^{\infty}\frac{3S_n^2(x)C_n(x)}{x^{6n+6}}\,dx=\int_0^{\infty}\frac{(S_n^3(x))'}{x^{6n+6}}\,dx,$$

since $S_n'(x)=C_n(x)$. Using integration by parts on the right-most integral produces

$$C\varepsilon_n=\left[\frac{S_n^3(x)}{x^{6n+6}}\right]_0^{\infty}+(6n+6)\int_0^{\infty}\frac{S_n^3(x)}{x^{6n+7}}\,dx.$$

As $S_n^3(x)/x^{6n+6}$ behaves as $\mathscr{O}(x^3)$ as $x\to0^+$ and as $\mathscr{O}(x^{-3})$ as $x\to\infty$, we have

$$C\varepsilon_n=(6n+6)\int_0^{\infty}\frac{S_n^3(x)}{x^{6n+7}}\,dx=(6n+6)S\gamma_n,$$

thereby proving (29). Similarly, since the two integrands in (36) are even

$$\begin{aligned}
S\alpha_n&=\int_0^{\infty}\frac{S_n^3(x)}{x^{6n+5}}\,dx=\int_0^{\infty}\frac{3C_n^2(x)S_n(x)}{x^{6n+5}}\,dx\\
&=\int_0^{\infty}\frac{3C_n^2(x)\left(S_{n-1}-\frac{(-1)^n}{(2n+1)!}x^{2n+1}\right)}{x^{6n+5}}\,dx\\
&=\int_0^{\infty}\frac{3C_n^2(x)S_{n-1}(x)}{x^{6n+5}}\,dx+\frac{3(-1)^{n+1}}{(2n+1)!}\int_0^{\infty}\frac{C_n^2(x)}{x^{4n+4}}\,dx\\
&=\int_0^{\infty}\frac{-(C_n^3(x))'}{x^{6n+5}}\,dx+\frac{3(-1)^{n+1}}{(2n+1)!}J_n.
\end{aligned}$$

Integrating by parts, omitting the limit step, produces

$$S\alpha_n = -(6n+5)\int_0^\infty \frac{C_n^3(x)}{x^{6n+6}}\,dx + \frac{3(-1)^{n+1}}{(2n+1)!}J_n = -(6n+5)C\varepsilon_n + \frac{3(-1)^{n+1}}{(2n+1)!}J_n.$$

Rearranging terms completes the proof for (30). $\square$

REMARK 4. We are not aware of a proof of these six integral relationships without the use of contour integration and complex functions as we have given here; such a proof would be interesting.

6. Some intriguing binomial identities

We now discuss some interesting binomial identities that arise from the integral connections discovered in the previous section. Suppose n is a fixed positive integer. Let

$$A_k = \sum_{j=k-n-1}^{n} \binom{2k}{2j+1} \text{ for } 1 \leqslant k \leqslant 2n+1, \quad B_k = \sum_{j=k-n}^{n} \binom{2k}{2j} \text{ for } 0 \leqslant k \leqslant 2n,$$

these being related to $b_c(n,k)$ and $b_s(n,k)$ appearing in (21) and (22) respectively. Note that $A_k = 2^{2k-1}$ when $1 \leqslant k \leqslant n+1$ and $B_k = 2^{2k-1}$ when $1 \leqslant k \leqslant n$. With this notation, along with the general equations (17) and (18), we find that

$$S\alpha_n = \frac{(-1)^{3n+2}}{(6n+4)!}\left\{1 - 3^{6n+3} + \sum_{k=0}^{n}\binom{6n+4}{2k+1}2^{6n-2k+4} - 4\sum_{k=1}^{2n+1}\binom{6n+4}{2k}A_k\right\}\frac{3\pi}{8};$$

$$S\gamma_n = \frac{(-1)^{3n+3}}{(6n+6)!}\left\{1 - 3^{6n+5} + \sum_{k=0}^{n}\binom{6n+6}{2k+1}2^{6n-2k+6} - 4\sum_{k=1}^{2n+1}\binom{6n+6}{2k}A_k\right\}\frac{3\pi}{8};$$

$$C\varepsilon_n = \frac{(-1)^{3n+3}}{(6n+5)!}\left\{1 + 3^{6n+4} - \sum_{k=0}^{n}\binom{6n+5}{2k}2^{6n-2k+6} + 4\sum_{k=0}^{2n}\binom{6n+5}{2k}B_k\right\}\frac{3\pi}{8}.$$

Using the fact that $C\varepsilon_n = (6n+6)S\gamma_n$, we find that

$$1 + 3^{6n+4} - \sum_{k=0}^{n}\binom{6n+5}{2k}2^{6n-2k+6} + 4\sum_{k=0}^{2n}\binom{6n+5}{2k}B_k$$

must be equal to

$$1 - 3^{6n+5} + \sum_{k=0}^{n}\binom{6n+6}{2k+1}2^{6n-2k+6} - 4\sum_{k=1}^{2n+1}\binom{6n+6}{2k}A_k.$$

In full form, after a little simplification, this identity states that

$$\sum_{k=0}^{n}\left(\binom{6n+5}{2k} + \binom{6n+6}{2k+1}\right)2^{6n-2k+4}$$

is equal to

$$3^{6n+4}+\sum_{k=0}^{2n}\binom{6n+5}{2k}\sum_{j=k-n}^{n}\binom{2k}{2j}+\sum_{k=1}^{2n+1}\binom{6n+6}{2k}\sum_{j=k-n-1}^{n}\binom{2k}{2j+1}.$$

For example, for the $n=3$ case, we see that

$$\begin{aligned}\sum_{k=0}^{3}\left(\binom{23}{2k}+\binom{24}{2k+1}\right)2^{22-2k}=3^{22}&+\sum_{k=0}^{6}\binom{23}{2k}\sum_{j=k-3}^{3}\binom{2k}{2j}\\&+\sum_{k=1}^{7}\binom{24}{2k}\sum_{j=k-4}^{3}\binom{2k}{2j+1}.\end{aligned}$$

Since we know the sum $S\alpha_n+(6n+5)C\varepsilon_n$, it follows that the sum of

$$\frac{(-1)^{3n+2}}{(6n+4)!}\left(1-3^{6n+3}+\sum_{k=0}^{n}2^{6n-2k+4}\binom{6n+4}{2k+1}-4\sum_{k=1}^{2n+1}A_k\binom{6n+4}{2k}\right)$$

and

$$\frac{(-1)^{3n+3}}{(6n+4)!}\left(1+3^{6n+4}-\sum_{k=0}^{n}2^{6n-2k+6}\binom{6n+5}{2k}+4\sum_{k=0}^{2n}B_k\binom{6n+5}{2k}\right)$$

should equal

$$\frac{4}{4n+3}\cdot\frac{(-1)^{n+1}}{((2n+1)!)^3}$$

This equality can be expressed as the sum of

$$-1+3^{6n+3}-\sum_{k=0}^{n}2^{6n-2k+4}\binom{6n+4}{2k+1}+4\sum_{k=1}^{2n+1}A_k\binom{6n+4}{2k}$$

and

$$1+3^{6n+4}-\sum_{k=0}^{n}2^{6n-2k+6}\binom{6n+5}{2k}+4\sum_{k=0}^{2n}B_k\binom{6n+5}{2k}$$

is equal to

$$\begin{aligned}4\cdot\frac{(6n+4)!}{4n+3}\cdot\frac{1}{((2n+1)!)^3}&=4\cdot\frac{(6n+4)!}{(4n+3)!(2n+1)!}\cdot\frac{(4n+2)!}{((2n+1)!)^2}\\&=4\cdot\binom{6n+4}{2n+1}\cdot\binom{4n+2}{2n+1}.\end{aligned}$$

It then follows that

$$\begin{aligned}3^{6n+3}-\sum_{k=0}^{n}2^{6n-2k+2}\binom{6n+4}{2k+1}-\sum_{k=0}^{n}2^{6n-2k+4}\binom{6n+5}{2k}+\sum_{k=1}^{2n+1}A_k\binom{6n+4}{2k}&\\+\sum_{k=0}^{2n}B_k\binom{6n+5}{2k}=\binom{6n+4}{2n+1}\binom{4n+2}{2n+1}&\end{aligned}$$

a rather involved but intriguing identity.

Since we know the sum $S\alpha_n + (6n+6)(6n+5)S\gamma_n$, it follows that the sum of

$$-1+3^{6n+3}-\sum_{k=0}^{n}2^{6n-2k+4}\binom{6n+4}{2k+1}+4\sum_{k=1}^{2n+1}A_k\binom{6n+4}{2k}$$

and

$$1-3^{6n+5}+\sum_{k=0}^{n}2^{6n-2k+6}\binom{6n+6}{2k+1}-4\sum_{k=1}^{2n+1}A_k\binom{6n+6}{2k}$$

should equal (as shown with the last identity)

$$4\cdot\binom{6n+4}{2n+1}\cdot\binom{4n+2}{2n+1}.$$

It follows that

$$\begin{aligned}-2\cdot 3^{6n+3}-\sum_{k=0}^{n}2^{6n-2k+2}\binom{6n+4}{2k+1}+\sum_{k=1}^{2n+1}A_k\left(\binom{6n+4}{2k}-\binom{6n+6}{2k}\right)&\\ +\sum_{k=0}^{n}2^{6n-2k+4}\binom{6n+6}{2k+1}=\binom{6n+4}{2n+1}\cdot\binom{4n+2}{2n+1}&.\end{aligned}$$

This equation has the advantage of only requiring A_k values rather than both A_k and B_k values.

We have thus shown that the connections between some of our cubed integrals as listed in equations (29), (30), and (31) lead to some unexpected identities. It might be interesting to try to verify these identities without using integrals. Similarly, the equations (32), (33), and (34) lead to three more binomial identities. We will not take the time or space to write them out here.

7. Conclusion

Using a reformulation of a recently devised method referred to as integration by differentiation, we have determined the values for a family of improper integrals that would otherwise be difficult to achieve using more standard approaches. In an effort to alleviate much of the computational burden that arises in applying the method, the contribution to the value of the improper integral due to several terms that repeatedly appear in the integrands of those improper integrals we wish to evaluate have been explicitly found. As demonstrated here, we contend that when the method applies, it is a powerful general approach that can be used in the evaluation of a wide class of improper integrals.

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