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ON $(\lambda,\eta)(f)$–STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES IN 2–NORMED LINEAR SPACES

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Abstract. In this study, we utilize modulus functions under various conditions to define $(\lambda,\eta)(f)$-statistical convergence of double sequences in 2-normed linear spaces. Additionally, we explore the relationships between the sets of $(\lambda,\eta)(f)$-statistically convergent sequences and $(\lambda,\eta)(f)$-statistically bounded sequences in 2-normed linear spaces.

1. Introduction

The statistical convergence of real numbers, crucial in summability theory, has been explored by numerous mathematicians. Independently, Fast [3] and Schoenberg [22] introduced the concept of statistical convergence, which has since been studied by various authors. Mursaleen and Edely [20] further extended this concept to double sequences. Statistical convergence has served as a valuable tool for numerous mathematicians in addressing various unresolved problems across sequence spaces, summability theory, and other applications. Over recent decades, this concept has been investigated across a spectrum of disciplines and under different appellations, spanning Banach spaces, measure theory, Fourier analysis, number theory, ergodic theory, cone metric space, trigonometric series, time scale, and topological space. To extend this concept, Mursaleen [9] proposed the notion of λ-statistical convergence, employing the sequence $\lambda=(\lambda_n)$. Further applications and generalizations concerning λ-statistical convergence and statistical convergence can be found in references [9, 15, 21, 26]. Additionally, readers should consult the monographs [2], and [19], as well as the recent papers [16], [23], [24] and [25] for background information on sequence spaces.

In 1953, Nakano [8] introduced the concept of a modulus function. A function $f:[0,\infty)\to[0,\infty)$ is defined as a modulus function (or modulus) if it satisfies the following conditions:

1. $f(x)=0 \Leftrightarrow x=0$,
2. $f(x+y)\leqslant f(x)+f(y)$ for every $x,y\in[0,\infty)$,
3. f is increasing,

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4. f is continuous from the right at 0.

Given these properties, it is clear that a modulus function must be continuous over the entire interval $[0,\infty)$. A modulus function can be either bounded or unbounded. For example, $f(x) = x^p$ where $p \in (0,1]$ is an unbounded modulus function, whereas $f(x) = \frac{x}{x+1}$ is a bounded modulus function.

Several authors have introduced new concepts and established inclusion theorems using a modulus function f (see, Khan and Khan [4], Kılınç and Solak [5], Maddox [7], Pehlivan and Fisher [6], Savaş and Patterson [17, 18]).

Aizpuru et al. [1] utilized an unbounded modulus function to define another density concept. Consequently, they introduced a novel notion of nonmatrix convergence, which lies between ordinary convergence and statistical convergence and aligns with the statistical convergence under the identity modulus.

Gähler [10, 11] is acknowledged for pioneering the concept of 2-normed space during the mid-1960s. Subsequently, this concept was further elaborated upon by later researchers. Since then, numerous scholars have made contributions to the exploration of this concept, yielding varying degrees of success. For further exploration, refer to publications such as those cited in [12, 13], and [14].

1.1. Research gaps with key observations based on literature review

Statistical convergence, initially introduced by Fast [3] and Schoenberg [22], has significantly contributed to summability theory and sequence spaces. While subsequent researchers, including Mursaleen and Edely [20], extended this concept to double sequences, much of the research has focused on particular sequence spaces and specific types of convergence. Despite numerous investigations across fields such as Banach spaces, ergodic theory, Fourier analysis, and topological spaces, there remains a need to explore statistical convergence in more generalized and abstract settings, especially in the context of 2-normed linear spaces. Moreover, the application of modulus functions to define new types of statistical convergence has not been fully explored in this area, leaving a gap in the literature regarding the relationships between various forms of convergence and boundedness within these spaces.

1.2. Motivation

The main contribution of this paper is the introduction of $(\lambda,\eta)(f)$-statistical convergence for double sequences in 2-normed linear spaces, utilizing modulus functions under various conditions. By extending the well-established concept of statistical convergence to more abstract settings, this study addresses a significant gap in the literature. Additionally, it explores the relationships between $(\lambda,\eta)(f)$-statistically convergent sequences and $(\lambda,\eta)(f)$-statistically bounded sequences within these spaces. This thorough investigation not only advances theoretical understanding but also provides valuable insights for practical applications and further research in related mathematical fields.

2. Preliminaries

In this section, we give significant existing conceptions and results which are crucial for our findings.

The statistical convergence depends on the natural density of subsets of $\mathbb{N}$. The number $\delta(A)$ of a subset A of $\mathbb{N}$ is called a natural density of A and is defined by

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} |\{a \leqslant n : a \in A\}|$$

where $|\{a \leqslant n : a \in A\}|$ is the number of elements of A which are less than or equal to n (see [3]).

A number sequence (x_k) is said to be statistically convergent (or S-convergent) to the number l if

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leqslant n : |x_k - l| \geqslant \varepsilon\}| = 0$$

for each $\varepsilon > 0$, in the case the limit exists. In this case, we write $S - \lim x_k = l$ or $x_k \to l\,(S)$ and S denotes the set of all S-convergent sequences.

A double sequence $x = (x_{mn})$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim's sense (shortly, p-convergent to $L \in \mathbb{R}$), if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{mn} - L| < \varepsilon$, whenever $m,n > N_\varepsilon$. In this case, we write

$$\lim_{m,n\to\infty} x_{mn} = L.$$

We recall that a subset K of $\mathbb{N}\times\mathbb{N} = \mathbb{N}^2$ is said to have natural density $\delta_2(K)$ if

$$\delta_2(K) = \lim_{m,n\to\infty} \frac{K(m,n)}{m.n},$$

where $K(m,n) = |\{(j,k) \in \mathbb{N}^2 : j \leqslant m, k \leqslant n\}|$.

Let X be a real vector space of dimension d, where $2 \leqslant d < \infty$. A 2-norm on X is a function $\|.,.\| : X \times X \to \mathbb{R}$ which satisfies

i) $\|x,y\| = 0 \Leftrightarrow x$ and y are linearly dependent,

ii) $\|x,y\| = \|y,x\|$,

iii) $\|\alpha x,y\| = |\alpha|\,\|x,y\|$,

iv) $\|x,y+z\| \leqslant \|x,y\| + \|y,z\|$.

The pair $(X,\|.,.\|)$ is then called a 2-normed space [10].

As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x,y\| :=$ the area of the parallelogram spanned by the vectors x and y, which may be given explicitly by the formula

$$\|x,y\| = (x_1y_2 - x_2y_1),\, x = (x_1,x_2) \text{ and } y = (y_1,y_2).$$

Let $w = (w_k)$ be a sequence in 2-normed space $(X, \|.,.\|)$. The sequence (w_k) is said to be statistically convergent to w_0, if for every $\kappa > 0$, the set

$$\{k \in \mathbb{N} : \|w_k - w_0, z\| \geqslant \kappa\}$$

has natural density zero for each nonzero z in X, in other words (w_k) statistically converges to w_0 in 2-normed space $(X, \|.,.\|)$ if

$$\lim_{k\to\infty} \frac{1}{k} |\{k : \|w_k - w_0, z\| \geqslant \kappa\}| = 0,$$

for each nonzero z in X. In this case we write $st - \lim_{k\to\infty} \|w_k, z\| = \|w_0, z\|$.

Let $\lambda = (\lambda_r)$ and $\mu = (\mu_s)$ be two non-decreasing sequences of positive real numbers, each tending to ∞ and such that $\lambda_{r+1} \leqslant \lambda_r + 1, \lambda_1 = 1; \mu_{s+1} \leqslant \mu_s + 1, \mu_1 = 1$. Let $I_r = [r - \lambda_r + 1, r], I_s = [s - \mu_s + 1, s]$ and $I_{r,s} = I_r \times I_s$.

For any set $K \subseteq \mathbb{N}^2$, the number,

$$\delta_{(\lambda,\mu)}(K) = P - \lim_{r,s\to\infty,\infty} \frac{1}{\lambda_r \mu_s} |\{(i,j) \in I_{r,s} : (i,j) \in K\}|;$$

is called (λ,μ)-density of the set K, provided the limit exists, (see, [18].)

Throughout we shall denote $(\lambda_{r,s}) = (\lambda_r \mu_s)$ and the collection of such sequences λ will be denoted by Δ_2.

DEFINITION 1. A double sequence $w = (w_{mn})$ in a 2-normed space $(Y, \|.,.\|)$ is said to be (λ,μ)-statistically convergent or $S_{\lambda,\mu}$-convergent to $w_0 \in X$ with respect to the 2-norm if for every $\kappa > 0$

$$P - \lim_{r,s\to\infty} \frac{1}{\lambda_{r,s}} |\{(m,n) \in I_{r,s} : \|w_{mn} - w_0, z\| \geqslant \kappa\}| = 0$$

for all $z \in Y$.

I.e., the set

$$K(\varepsilon) = \{(m,n) \in I_{r,s} : \|w_{mn} - w_0, z\| \geqslant \kappa\}$$

has (λ,μ)-density zero.

In this case we write $w_{mn} \to w_0 \left(S_{(\lambda,\eta)}(Y, \|.,.\|)\right)$ or $w_{mn} \overset{S_{(\lambda,\eta)}(Y,\|.,.\|)}{\longrightarrow} w_0$ and

$$S_{(\lambda,\eta)}(Y, \|.,.\|) = \left\{ w = (w_{mn}) \mid \exists w_0 \in \mathbb{R},\ w_{mn} \overset{S_{(\lambda,\eta)}(Y,\|.,.\|)}{\longrightarrow} w_0 \right\}$$

Let $S_{\lambda,\mu}(Y, \|.,.\|)$ denote the set of all (λ,μ)-statistically convergent of double sequences in 2-normed space Y.

If $\lambda_r = r$ and $\mu_s = s$ for all r,s then the space $S_{\lambda,\mu}(Y, \|.,.\|)$ is reduced to the space $St_2(Y, \|.,.\|)$ and since $\delta_2(K) \leqslant \delta_{\lambda,\mu}(K)$, we have $S_{\lambda,\mu}(Y, \|.,.\|) \subset St_2(Y, \|.,.\|)$.

3. Main results

In this section, we present the primary findings of the paper. Our emphasis lies in delineating the connections between the sets of $(\lambda,\eta)(f)$-statistically convergent and $(\lambda,\eta)(f)$-statistically bounded sequences.

DEFINITION 2. Let f be an unbounded modulus, $(\lambda_r\mu_s) = (\lambda_{r,s}) \in \Delta_2$ and $U \subseteq \mathbb{N}^2$. The number $\delta^f_{(\lambda,\eta)}(U)$, referred to as the λf-density of U, is given by

$$\delta^f_{(\lambda,\eta)}(U) := \lim_{r,s\to\infty} \frac{1}{f(\lambda_{r,s})} f(|\{(u,v) \in I_{r,s} : (u,v) \in U\}|)$$

if this limit exists.

It should be noted that in the case $f(x) = x$, the concepts of $\delta^f_{(\lambda,\eta)}$-density and $\delta_{(\lambda,\eta)}$-density coincide. In addition, in the case $f(x) = x$ and $(\lambda_{r,s}) = (rs)$, $\delta^f_{(\lambda,\eta)}$-density and δ_2-density coincide.

REMARK 1. It is not necessary for the equality $\delta^f_{(\lambda,\eta)}(U) + \delta^f_{(\lambda,\eta)}\left(\mathbb{N}^2 \setminus U\right) = 1$ to remain true, in general, even though for a natural density it is always true. The example below illustrates this fact.

EXAMPLE 1. Let us take $f(x) = \log(x+1)$, $(\lambda_{r,s}) = (rs)$ and $U = \{2rs : r,s \in \mathbb{N}\} \subseteq \mathbb{N}^2$. Then, $\delta^f_{(\lambda,\eta)}(U) + \delta^f_{(\lambda,\eta)}\left(\mathbb{N}^2 \setminus U\right) \neq 1$. Indeed, since f is an unbounded modulus and

$$\frac{\lambda_{r,s}}{2} - 1 \leqslant |\{(u,v) \in I_{r,s} : (u,v) \in U\}| \leqslant \frac{\lambda_{r,s}}{2}$$

for each $r,s \in \mathbb{N}$, we may write

$$\frac{1}{f(\lambda_{r,s})} f\left(\frac{\lambda_{r,s}}{2} - 1\right) \leqslant \frac{1}{f(\lambda_{r,s})} f(|\{(u,v) \in I_{r,s} : (u,v) \in U\}|) \leqslant \frac{1}{f(\lambda_{r,s})} f\left(\frac{\lambda_{r,s}}{2}\right)$$

or

$$\frac{1}{\log(rs+1)} \log\left(\frac{rs}{2}\right) \leqslant \frac{1}{f(\lambda_{r,s})} f(|\{(u,v) \in I_{r,s} : (u,v) \in U\}|)$$
$$\leqslant \frac{1}{\log(rs+1)} \log\left(\frac{rs}{2} + 1\right).$$

By taking the limits as $r,s \to \infty$ in the above inequality, we get that

$$1 \leqslant \lim_{r,s\to\infty} \frac{1}{f(\lambda_{r,s})} f(|\{(u,v) \in I_{r,s} : (u,v) \in U\}|) \leqslant 1.$$

Thus, $\delta^f_{(\lambda,\eta)}(U) = 1$. Furthermore, by using the fact

$$\frac{\lambda_{r,s}+1}{2} - 1 \leqslant \left|\left\{(u,v) \in I_{r,s} : (u,v) \in \mathbb{N}^2 \setminus U\right\}\right| \leqslant \frac{\lambda_{r,s}+1}{2}$$

for each $r,s\in\mathbb{N}$, we have $\delta^{f}_{(\lambda,\eta)}\left(\mathbb{N}^2\diagdown U\right)=1$. Therefore, $\delta^{f}_{(\lambda,\eta)}(U)+\delta^{f}_{(\lambda,\eta)}\left(\mathbb{N}^2\setminus U\right)$ $=2$.

DEFINITION 3. Let (w_{mn}) be a sequence in a metric space $(Y,\|.,.\|)$, f be an unbounded modulus and $(\lambda_{r,s})\in\Delta_2$ be given. Then, (w_{mn}) is called to be $(\lambda,\eta)(f)$-statistically convergent in 2-normed linear space $(Y,\|.,.\|)$ (or simply $S^{f}_{(\lambda,\eta)}(Y,\|.,.\|)$-convergent) if there exists a $w_0\in Y$ such that

$$\lim_{r,s\to\infty}\frac{1}{f(\lambda_{r,s})}f\left(\left|\left\{(m,n)\in I_{r,s}:\|w_{mn}-w_0,z\|\geqslant\kappa\right\}\right|\right)=0$$

for each $\kappa>0$, for all $z\in X$. In such instances, we express $w_{mn}\to w_0\left(S^{f}_{(\lambda,\eta)}(Y,\|.,.\|)\right)$ or $w_{mn}\overset{S^{f}_{(\lambda,\eta)}(Y,\|.,.\|)}{\longrightarrow}w_0$.

Throughout the study, the set of all $(\lambda,\eta)(f)$-statistically convergent sequences in 2-normed linear space $(Y,\|.,.\|)$ will be represented by $S^{f}_{(\lambda,\eta)}(Y,\|.,.\|)$, defined as follows:

$$S^{f}_{(\lambda,\eta)}(Y,\|.,.\|)$$
$$=\left\{(w_{mn}):\lim_{r,s\to\infty}\tfrac{1}{f(\lambda_{r,s})}f\left(\left|\left\{(m,n)\in I_{r,s}:\|w_{mn}-w_0,z\|\geqslant\kappa\right\}\right|\right)=0,\ \exists w_0\in Y\right\}.$$

Notice that $S^{f}_{(\lambda,\eta)}(Y,\|.,.\|)$-convergence simplifies to $S_{(\lambda,\eta)}(Y,\|.,.\|)$-convergence when $f(x)=x$. Moreover, $S^{f}_{(\lambda,\eta)}(Y,\|.,.\|)$-convergence reduces to $St^{f}_{2}(Y,\|.,.\|)$-convergence when $\lambda_r=r$ and $\eta_s=s$. Moreover, $S^{f}_{(\lambda,\eta)}(Y,\|.,.\|)$-convergence reduces to $St_2(Y,\|.,.\|)$-convergence when $f(x)=x$ and $\lambda_r=r$ and $\eta_s=s$, . We denote by $S^{f}_{(\lambda,\eta),0}(Y,\|.,.\|)$ the set of all $S^{f}_{(\lambda,\eta),0}(Y,\|.,.\|)$-null sequences. It is evident that $S^{f}_{(\lambda,\eta),0}(Y,\|.,.\|)\subset S^{f}_{(\lambda,\eta)}(Y,\|.,.\|)$.

THEOREM 1. *Let (w_{mn}) and (t_{mn}) be sequences in 2-normed linear space $(Y,\|.,.\|)$, f be an unbounded modulus and $(\lambda_{r,s})\in\Delta_2$.*

(i) If $w_{mn}\overset{S^{f}_{(\lambda,\eta)}(Y,\|.,.\|)}{\longrightarrow}w_0$, then $qw_{mn}\overset{S^{f}_{(\lambda,\eta)}(Y,\|.,.\|)}{\longrightarrow}qw_0$, for all $q\in\mathbb{C}$.

(ii) If $w_{mn}\overset{S^{f}_{(\lambda,\eta)}(Y,\|.,.\|)}{\longrightarrow}w_0$ and $t_{mn}\overset{S^{f}_{(\lambda,\eta)}(Y,\|.,.\|)}{\longrightarrow}t_0$, then $w_{mn}+t_{mn}\overset{S^{f}_{(\lambda,\eta)}(Y,\|.,.\|)}{\longrightarrow}w_0+t_0$.

Proof. (i) If $q=0$, the proof is clear. If $q\neq0$, the proof follows from the fact

$$\tfrac{1}{f(\lambda_{r,s})}f\left(\left|\left\{(m,n)\in I_{r,s}:\|qw_{mn}-qw_0,z\|\geqslant\kappa\right\}\right|\right)$$
$$=\tfrac{1}{f(\lambda_{r,s})}f\left(\left|\left\{(m,n)\in I_{r,s}:\|w_{mn}-w_0,z\|\geqslant\tfrac{\kappa}{|q|}\right\}\right|\right).$$

Since $w_{mn} \overset{S^{f}_{(\lambda,\eta)}(Y,\|.,.\|)}{\longrightarrow} w_0$, we have

$$\lim_{r,s\to\infty} \frac{1}{f(\lambda_{r,s})} f\left(\left|\left\{(m,n)\in I_{r,s} : \|qw_{mn}-qw_0,z\| \geqslant \kappa\right\}\right|\right)=0.$$

Therefore, we obtain

$$\lim_{r,s\to\infty} \frac{1}{f(\lambda_{r,s})} f\left(\left|\left\{(m,n)\in I_{r,s} : \|w_{mn}-w_0,z\| \geqslant \frac{\kappa}{|q|}\right\}\right|\right)=0.$$

As a result, $qw_{mn} \overset{S^{f}_{(\lambda,\eta)}(Y,\|.,.\|)}{\longrightarrow} qw_0$, for all $q\in\mathbb{C}$.

(ii) The proof follows from the fact

$$\begin{aligned} & f\left(\left|\left\{(m,n)\in I_{r,s} : \|(w_{mn}+t_{mn})-(w_0-t_0),z\| \geqslant \kappa\right\}\right|\right) \\ \leqslant\, & f\left(\left|\left\{(m,n)\in I_{r,s} : \|w_{mn}-w_0,z\| \geqslant \frac{\kappa}{2}\right\}\right|\right) \\ & + f\left(\left|\left\{(m,n)\in I_{r,s} : \|t_{mn}-t_0,z\| \geqslant \frac{\kappa}{2}\right\}\right|\right). \quad \square \end{aligned}$$

THEOREM 2. *Let (w_{mn}) be a sequence in 2-normed linear space $(Y,\|.,.\|)$, f be an unbounded modulus, $(\lambda_{r,s})\in\Delta_2$ and $w_0\in Y$. Then, $w_{mn} \overset{S^{f}_{(\lambda,\eta)}(Y,\|.,.\|)}{\longrightarrow} w_0$ implies $w_{mn} \overset{S_{(\lambda,\eta)}(Y,\|.,.\|)}{\longrightarrow} w_0$.*

Proof. Let $w_{mn} \overset{S^{f}_{(\lambda,\eta)}(Y,\|.\|)}{\longrightarrow} w_0$. Then, for each $\kappa>0$ and for all $z\in Y$

$$\lim_{r,s\to\infty} \frac{1}{f(\lambda_{r,s})} f\left(\left|\left\{(m,n)\in I_{r,s} : \|w_{mn}-w_0,z\| \geqslant \kappa\right\}\right|\right)=0.$$

So, for any $k\in\mathbb{N}$, there exists a $u_0\in\mathbb{N}$ such that

$$f\left(\left|\left\{(m,n)\in I_{r,s} : \|w_{mn}-w_0,z\| \geqslant \kappa\right\}\right|\right) \leqslant \frac{1}{k} f(\lambda_{r,s}) \leqslant \frac{1}{k} k f\left(\frac{1}{k}\lambda_{r,s}\right) = f\left(\frac{1}{k}\lambda_{r,s}\right),$$

for all $r,s\geqslant u_0$.

Given that f is a modulus function, we have

$$\left|\left\{(m,n)\in I_{r,s} : \|w_{mn}-w_0,z\| \geqslant \kappa\right\}\right| \leqslant \frac{1}{k}\lambda_{r,s}.$$

Since this inequality holds for every $k\in\mathbb{N}$, we deduce

$$\lim_{r,s\to\infty} \frac{1}{\lambda_{r,s}} \left|\left\{(m,n)\in I_{r,s} : \|w_{mn}-w_0,z\| \geqslant \kappa\right\}\right| = 0.$$

Therefore, $w_{mn} \overset{S_{(\lambda,\eta)}(Y,\|.,.\|)}{\longrightarrow} w_0$. $\square$

REMARK 2. In general, the converse of Theorem 2 does not hold. That is, $w_{mn} \overset{S_{(\lambda,\eta)}(Y,\|.,.\|)}{\longrightarrow} w_0$ does not necessarily imply $w_{mn} \overset{S^f_{(\lambda,\eta)}(Y,\|.,.\|)}{\longrightarrow} w_0$ for every unbounded modulus f and each $(\lambda_{r,s}) \in \Delta_2$. The following example illustrates this scenario.

EXAMPLE 2. Consider the sequence (w_{mn}) in a 2-normed linear space $(\mathbb{R}^2, \|.,.\|)$ as

$$w_{mn} = \begin{cases} mn, & \text{if } m = u^3,\ n = v^3 \\ 0, & \text{if } m \neq u^3,\ n \neq v^3 \end{cases} \quad u, v \in \mathbb{N}.$$

If we take $f(x) = \log(x+1)$, and $(\lambda_{r,s}) = (rs)$, then for every $\kappa > 0$, we have

$$|\{(m,n) \in I_{r,s} : \|w_{mn} - 0, z\| \geqslant \kappa\}| \leqslant \sqrt{\lambda_{r,s}}.$$

That is,

$$\lim_{r,s\to\infty} \frac{1}{\lambda_{r,s}} |\{(m,n) \in I_{r,s} : \|w_{mn} - 0, z\| \geqslant \kappa\}| \leqslant \lim_{r,s\to\infty} \frac{\sqrt{\lambda_{r,s}}}{\lambda_{r,s}} = 0.$$

This means that $w_{mn} \overset{S_{(\lambda,\eta)}(\mathbb{R}^2,\|.,.\|)}{\longrightarrow} 0$ and so $(w_{mn}) \in S_{(\lambda,\eta)}(\mathbb{R}^2, \|.,.\|)$. However, for every $\kappa > 0$, we have

$$\lim_{r,s\to\infty} \frac{1}{f(\lambda_{r,s})} f(|\{(m,n) \in I_{r,s} : \|w_{mn} - 0, z\| \geqslant \kappa\}|) = \lim_{r,s\to\infty} \frac{1}{f(\lambda_{r,s})} f\left(\sqrt{\lambda_{r,s}} - 1\right) = \frac{1}{3}.$$

So that $w_{mn} \overset{S^f_{(\lambda,\eta)}(\mathbb{R}^2,\|.,.\|)}{\nrightarrow} 0$, and hence $(w_{mn}) \notin S^f_{(\lambda,\eta)}(\mathbb{R}^2, \|.,.\|)$.

By substituting $\lambda_r = r$ and $\eta_s = s$ into Theorem 2, we obtain the following outcome.

COROLLARY 1. *Let (w_{mn}) be a sequence in 2-normed linear space $(Y, \|.,.\|)$, f be an unbounded modulus, and $w_0 \in Y$. Then, $w_{mn} \overset{St^f_2(Y,\|.,.\|)}{\longrightarrow} w_0$ implies $w_{mn} \overset{St_2(Y,\|.,.\|)}{\longrightarrow} w_0$.*

THEOREM 3. *Let (w_{mn}) be a sequence in a 2-normed linear space $(Y, \|.,.\|)$, f be an unbounded modulus, $(\lambda_{r,s}) \in \Delta_2$ and $w_0 \in Y$. If $\lim_{r,s\to\infty} \inf \frac{f(\lambda_{rs})}{\lambda_{rs}} > 0$, then $w_{mn} \overset{S_{(\lambda,\eta)}(Y,\|.,.\|)}{\longrightarrow} w_0$ implies $w_{mn} \overset{S^f_{(\lambda,\eta)}(Y,\|.,.\|)}{\longrightarrow} w_0$.*

Proof. Let $w_{mn} \overset{S_{(\lambda,\eta)}(Y,\|.,.\|)}{\longrightarrow} w_0$. Since f is a modulus, we have

$$|\{(m,n) \in I_{r,s} : \|w_{mn} - w_0, z\| \geqslant \kappa\}| \geqslant \frac{1}{f(1)} f(|\{(m,n) \in I_{r,s} : \|w_{mn} - w_0, z\| \geqslant \kappa\}|)$$

for all $\kappa > 0$. That is

$$\lim_{r,s\to\infty} \frac{1}{\lambda_{r,s}} \left|\{(m,n) \in I_{r,s} : \|w_{mn} - w_0, z\| \geqslant \kappa\}\right|$$

$$\geqslant \lim_{r,s\to\infty} \frac{f(\lambda_{rs})}{\lambda_{rs}} \frac{1}{f(1)} \frac{f(|\{(m,n) \in I_{r,s} : \|w_{mn} - w_0, z\| \geqslant \kappa\}|)}{f(\lambda_{rs})}.$$

Since $\lim_{r,s\to\infty} \inf \frac{f(\lambda_{rs})}{\lambda_{rs}} > 0$ and $w_{mn} \overset{S_{(\lambda,\eta)}(Y,\|.,.\|)}{\longrightarrow} w_0$, we obtain

$$\lim_{r,s\to\infty} \frac{1}{f(\lambda_{r,s})} f(|\{(m,n) \in I_{r,s} : \|w_{mn} - w_0, z\| \geqslant \kappa\}|) = 0.$$

Therefore, we have $w_{mn} \overset{S^f_{(\lambda,\eta)}(Y,\|.,.\|)}{\longrightarrow} w_0$. □

The following result is derived by setting $\lambda_r = r$ and $\eta_s = s$ in Theorem 3.

COROLLARY 2. *Let (w_{mn}) be a sequence in a 2-normed linear space $(Y, \|.,.\|)$, and $w_0 \in Y$. If $\lim_{r,s\to\infty} \inf \frac{f(rs)}{rs} > 0$, then $w_{mn} \overset{St_2(Y,\|.,.\|)}{\longrightarrow} w_0$ implies $w_{mn} \overset{St^f_2(Y,\|.,.\|)}{\longrightarrow} w_0$.*

From Theorem 2 and Theorem 3, we get the following result.

COROLLARY 3. *Let (w_{mn}) be a sequence in a 2-normed linear space $(Y, \|.,.\|)$, f be an unbounded modulus, $(\lambda_{r,s}) \in \Delta_2$ and $w_0 \in Y$. If $\lim_{r,s\to\infty} \inf \frac{f(\lambda_{rs})}{\lambda_{rs}} > 0$ we have $w_{mn} \overset{S_{(\lambda,\eta)}(Y,\|.,.\|)}{\longrightarrow} w_0$ if and only if $w_{mn} \overset{S^f_{(\lambda,\eta)}(Y,\|.,.\|)}{\longrightarrow} w_0$.*

THEOREM 4. *Let (w_{mn}) be a sequence in a 2-normed linear space $(Y, \|.,.\|)$, f be an unbounded modulus, $(\lambda_{r,s}) \in \Delta_2$ and $w_0 \in Y$. If $\lim_{r,s\to\infty} \inf \frac{f(\lambda_{rs})}{rs} > 0$ then $w_{mn} \overset{St_2(Y,\|.,.\|)}{\longrightarrow} w_0$ implies $w_{mn} \overset{S^f_{(\lambda,\eta)}(Y,\|.,.\|)}{\longrightarrow} w_0$.*

Proof. Suppose $w_{mn} \overset{St_2(Y,\|.,.\|)}{\longrightarrow} w_0$. Then, for every $\kappa > 0$, we get

$$|\{m \leqslant r, n \leqslant s : \|w_{mn} - w_0, z\| \geqslant \kappa\}| \geqslant |\{(m,n) \in I_{r,s} : \|w_{mn} - w_0, z\| \geqslant \kappa\}|.$$

So

$$\lim_{r,s\to\infty} \frac{1}{rs} |\{m \leqslant r, n \leqslant s : \|w_{mn} - w_0, z\| \geqslant \kappa\}|$$

$$\geqslant \lim_{r,s\to\infty} \frac{1}{rs} |\{(m,n) \in I_{r,s} : \|w_{mn} - w_0, z\| \geqslant \kappa\}|$$

$$\geqslant \frac{1}{rs} \frac{1}{f(1)} f(|\{(m,n) \in I_{r,s} : \|w_{mn} - w_0, z\| \geqslant \kappa\}|)$$

$$\geqslant \frac{f(\lambda_{rs})}{rs} \frac{1}{f(1)} \frac{f(|\{(m,n) \in I_{r,s} : \|w_{mn} - w_0, z\| \geqslant \kappa\}|)}{f(\lambda_{rs})}.$$

Since $w_{mn} \overset{St_2(Y,\|.,.\|)}{\longrightarrow} w_0$ and $\lim_{r,s\to\infty} \inf \frac{f(\lambda_{rs})}{rs} > 0$, then we get

$$\lim_{r,s\to\infty} \frac{f(|\{(m,n) \in I_{r,s} : \|w_{mn} - w_0, z\| \geqslant \kappa\}|)}{f(\lambda_{rs})} = 0.$$

As a result, we have $w_{mn} \overset{S^f_{(\lambda,\eta)}(Y,\|.,.\|)}{\longrightarrow} w_0$. □

From Theorem 4, the following result is obtained by setting $f(x) = x$.

COROLLARY 4. *Let (w_{mn}) be a sequence in a 2-normed linear space $(Y, \|.,.\|)$, $(\lambda_{r,s}) \in \Delta_2$ and $w_0 \in Y$. If $\lim_{r,s\to\infty} \inf \frac{\lambda_{rs}}{rs} > 0$ then $w_{mn} \overset{St_2(Y,\|.,.\|)}{\longrightarrow} w_0$ implies $w_{mn} \overset{S_{(\lambda,\eta)}(Y,\|.,.\|)}{\longrightarrow} w_0$.*

From Theorem 4, we get the following result by taking $\lambda_r = r$ and $\eta_s = s$.

COROLLARY 5. *Let (w_{mn}) be a sequence in a 2-normed linear space $(Y, \|.,.\|)$, f be an unbounded modulus, and $w_0 \in Y$. If $\lim_{r,s\to\infty} \inf \frac{f(rs)}{rs} > 0$ then $w_{mn} \overset{St_2(Y,\|.,.\|)}{\longrightarrow} w_0$ implies $w_{mn} \overset{St^f_2(Y,\|.,.\|)}{\longrightarrow} w_0$.*

DEFINITION 4. Let (w_{mn}) be a sequence in a 2-normed linear space $(Y, \|.,.\|)$, f be an unbounded modulus, and $(\lambda_{r,s}) \in \Delta_2$ be given. Then, (w_{mn}) is called λf-statistically bounded in Y (or simply $S^f_\lambda(Y, \|.,.\|)$-bounded) if there exists $t \in Y$ such that

$$\lim_{r,s\to\infty} \frac{1}{f(\lambda_{rs})} f(|\{(m,n) \in I_{r,s} : \|w_{mn} - t, z\| \geqslant U\}|) = 0,$$

for some $U \in \mathbb{R}^+$ and for all $z \in Y$.

Throughout the study, the collection of all $S^f_{(\lambda,\eta)}(Y, \|.,.\|)$-bounded sequences will be represented by $BS^f_{(\lambda,\eta)}(Y, \|.,.\|)$, denoted as:

$$BS^f_{(\lambda,\eta)}(Y, \|.,.\|) = \Big\{ (w_{mn}) : \lim_{r,s\to\infty} \tfrac{1}{f(\lambda_{rs})} f(|\{(m,n) \in I_{r,s} : \|w_{mn} - t, z\| \geqslant U\}|) = 0 \text{ for some } t \in Y \text{ and } U \in \mathbb{R}^+ \Big\}.$$

When $f(x) = x$, we denote $BS_{(\lambda,\eta)}(Y, \|.,.\|)$ instead of $BS^f_{(\lambda,\eta)}(Y, \|.,.\|)$. When $\lambda_r = r$ and $\eta_s = s$, we use $BSt^f_2(Y, \|.,.\|)$ instead of $BS^f_{(\lambda,\eta)}(Y, \|.,.\|)$. Furthermore, in the special case where $f(x) = x$ and $\lambda_r = r$ and $\eta_s = s$, we express $BSt_2(Y, \|.,.\|)$ instead of $BS^f_{(\lambda,\eta)}(Y, \|.,.\|)$.

THEOREM 5. *Let (w_{mn}) be a sequence in a 2-normed linear space $(Y, \|.\|)$, f be an unbounded modulus and $(\lambda_{r,s}) \in \Delta_2$. If (w_{mn}) is $S^f_{(\lambda,\eta)}(Y, \|.,.\|)$-convergent, then it is $S^f_{(\lambda,\eta)}(Y, \|.,.\|)$-bounded.*

Proof. Suppose (w_{mn}) is $S^f_{(\lambda,\eta)}(Y,\|.,.\|)$-convergent and $w_{mn} \overset{S_{(\lambda,\eta)}(Y,\|.,.\|)}{\longrightarrow} w_0$. Then, for each $\kappa > 0$, we have

$$|\{(m,n) \in I_{r,s} : \|w_{mn} - w_0, z\| \geqslant \kappa\}| \geqslant |\{(m,n) \in I_{r,s} : \|w_{mn} - w_0, z\| \geqslant U\}|$$

for some $U \in \mathbb{R}^+$ such that $U > \kappa$. Given that f is a modulus, the inequality above implies that

$$\frac{1}{f(\lambda_{rs})} f(|\{(m,n) \in I_{r,s} : \|w_{mn} - w_0, z\| \geqslant \kappa\}|)$$
$$\geqslant \frac{1}{f(\lambda_{rs})} f(|\{(m,n) \in I_{r,s} : \|w_{mn} - w_0, z\| \geqslant U\}|).$$

Since $w_{mn} \overset{S_{(\lambda,\eta)}(Y,\|.,.\|)}{\longrightarrow} w_0$, then we have

$$\lim_{r,s\to\infty} \frac{1}{f(\lambda_{rs})} f(|\{(m,n) \in I_{rs} : \|w_{mn} - w_0, z\| \geqslant \kappa\}|) = 0.$$

Thus,

$$\lim_{r,s\to\infty} \frac{1}{f(\lambda_{rs})} f(|\{(m,n) \in I_{r,s} : \|w_{mn} - w_0, z\| \geqslant U\}|) = 0,$$

implies that (w_{mn}) is $S^f_{(\lambda,\eta)}(Y,\|.,.\|)$-bounded. □

REMARK 3. In general, the converse of the above theorem is not valid. That is, an $S^f_{(\lambda,\eta)}(Y,\|.,.\|)$-bounded sequence is not necessarily $S^f_{(\lambda,\eta)}(Y,\|.,.\|)$-convergent for every unbounded modulus function f and all $(\lambda_{r,s}) \in \Delta_2$. The following example illustrates this scenario.

EXAMPLE 3. Consider the sequence (w_{mn}) in $\left(\mathbb{R}^2,\|.,.\|\right)$ as

$$(w_{mn}) = ((1,1),(2,2),(1,1),(2,2),\dots).$$

Then, (w_{mn}) is $S^f_{(\lambda,\eta)}\left(\mathbb{R}^2,\|.,.\|\right)$-bounded but (w_{mn}) is not $S^f_{(\lambda,\eta)}\left(\mathbb{R}^2,\|.,.\|\right)$-convergent if we take $\lambda_r = r$ and $\eta_s = s$ and $f(x) = x^\alpha$, where $0 < \alpha \leqslant 1$.

By utilizing Theorem 5 with $f(x) = x$, we obtain the following result.

COROLLARY 6. *Let (w_{mn}) be a sequence in a 2-normed linear space $(Y,\|.,.\|)$, and $(\lambda_{r,s}) \in \Delta_2$. If (w_{mn}) is $S_{(\lambda,\eta)}(Y,\|.,.\|)$-convergent, then it is $S_{(\lambda,\eta)}(Y,\|.,.\|)$-bounded but the converse is not true, in general.*

The following result is derived by setting $\lambda_r = r$ and $\eta_s = s$ in Theorem 5.

COROLLARY 7. *Let (w_{mn}) be a sequence in a 2-normed linear space $(Y,\|.,.\|)$, f be an unbounded modulus. If (w_{mn}) is $St^f_2(Y,\|.,.\|)$-convergent, then it is $St^f_2(Y,\|.,.\|)$-bounded but the converse is not true, in general.*

THEOREM 6. *Let* (w_{mn}) *be a sequence in a 2-normed linear space* $(Y,\|.,.\|)$, f *be an unbounded modulus and* $(\lambda_{r,s})\in\Delta_2$. *If* (w_{mn}) *is* $S^{f}_{(\lambda,\eta)}(Y,\|.,.\|)$*-bounded, then it is* $S_{(\lambda,\eta)}(Y,\|.,.\|)$*-bounded.*

Proof. Suppose (w_{mn}) is $S^{f}_{(\lambda,\eta)}(Y,\|.,.\|)$-bounded. Then, for some there is $t\in X$ such that

$$\lim_{r,s\to\infty}\frac{1}{f(\lambda_{rs})}f(|\{(m,n)\in I_{r,s}:\|w_{mn}-t,z\|\geqslant U\}|)=0,$$

for some $U\in\mathbb{R}^+$. So, for any $k\in\mathbb{N}$, there is $u_0\in\mathbb{N}$ such that for all $r,s\geqslant u_0$,

$$f(|\{(m,n)\in I_{r,s}:\|w_{mn}-t,z\|\geqslant U\}|)\leqslant\frac{1}{k}f(\lambda_{r,s})\leqslant\frac{1}{k}kf\left(\frac{1}{k}\lambda_{r,s}\right)=f\left(\frac{1}{k}\lambda_{r,s}\right).$$

Since f is a modulus function, we have

$$|\{(m,n)\in I_{r,s}:\|w_{mn}-t,z\|\geqslant U\}|\leqslant\frac{1}{k}\lambda_{r,s}.$$

This means that,

$$\lim_{r,s\to\infty}\frac{1}{\lambda_{rs}}|\{(m,n)\in I_{r,s}:\|w_{mn}-t,z\|\geqslant U\}|=0.$$

Therefore, (w_{mn}) is $S_{(\lambda,\eta)}(Y,\|.,.\|)$-bounded. □

REMARK 4. In general, the converse of the above theorem does not hold. That is, an $S_{(\lambda,\eta)}(Y,\|.,.\|)$-bounded sequence may not necessarily be $S^{f}_{(\lambda,\eta)}(Y,\|.,.\|)$-bounded for every unbounded modulus f and each $(\lambda_{r,s})\in\Delta_2$.

The following example illustrates this situation.

EXAMPLE 4. Consider the sequence

$$(w_{mn})=((1,1),(0,0),(0,0),(4,4),(0,0),(0,0),(0,0),(0,0),(9,9),\dots)$$

in $\left(\mathbb{R}^2,\|.,.\|\right)$. It is clear that

$$\{(m,n)\in I_{r,s}:\|w_{mn}-0,z\|>U\}=\{(1,1),(4,4),(9,9),\dots\},$$

for any $U\in\mathbb{R}^+$. If we take $f(x)=\log(x+1)$ and $\lambda_r=r$ and $\eta_s=s$, then

$$\lim_{r,s\to\infty}\frac{1}{f(\lambda_{rs})}f(|\{(m,n)\in I_{r,s}:\|w_{mn}-0,z\|\geqslant U\}|)=\frac{1}{2}\neq0.$$

This means that (w_{mn}) is not $S^{f}_{(\lambda,\eta)}(Y,\|.,.\|)$-bounded. However,

$$\lim_{r,s\to\infty}\frac{1}{\lambda_{rs}}|\{(m,n)\in I_{r,s}:\|w_{mn}-0,z\|\geqslant U\}|=0.$$

Thus, (w_{mn}) is $S_{(\lambda,\eta)}(Y,\|.,.\|)$-bounded.

From Theorem 6, the following result is obtained by setting $\lambda_r = r$ and $\eta_s = s$.

COROLLARY 8. *Let (w_{mn}) be a sequence in a 2-normed linear space $(Y,\|.,.\|)$, f be an unbounded modulus. If (w_{mn}) is $St_2^f(Y,\|.,.\|)$-bounded, then it is $St_2(Y,\|.,.\|)$-bounded but the converse is not true, in general.*

THEOREM 7. *Let (w_{mn}) be a sequence in a 2-normed linear space $(Y,\|.,.\|)$, f be an unbounded modulus and $(\lambda_{r,s}) \in \Delta_2$. If $\lim_{r,s\to\infty}\inf\frac{f(\lambda_{rs})}{\lambda_{rs}} > 0$ and (w_{mn}) is $S_{(\lambda,\eta)}(Y,\|.,.\|)$-bounded, then (x_k) is $S^f_{(\lambda,\eta)}(Y,\|.,.\|)$-bounded.*

Proof. Suppose (w_{mn}) is $S_{(\lambda,\eta)}(Y,\|.,.\|)$-bounded. Then, there is $t \in Y$ such that

$$\lim_{r,s\to\infty}\frac{1}{\lambda_{rs}}|\{(m,n)\in I_{r,s} : \|w_{mn}-t,z\| \geqslant U\}| = 0$$

for some $U \in \mathbb{R}^+$. Since f is a modulus, we have

$$\begin{aligned}
&\lim_{r,s\to\infty}\frac{1}{\lambda_{rs}}|\{(m,n)\in I_{r,s} : \|w_{mn}-t,z\| \geqslant U\}| \\
&\geqslant \lim_{r,s\to\infty}\frac{1}{\lambda_{rs}}\frac{1}{f(1)}f(|\{(m,n)\in I_{r,s} : \|w_{mn}-t,z\| \geqslant U\}|) \\
&\geqslant \lim_{r,s\to\infty}\frac{f(\lambda_{rs})}{\lambda_{rs}}\frac{1}{f(1)}\frac{f(|\{(m,n)\in I_{r,s} : \|w_{mn}-t,z\| \geqslant U\}|)}{f(\lambda_{rs})}.
\end{aligned}$$

Since (w_{mn}) is $S_{(\lambda,\eta)}(Y,\|.,.\|)$-bounded and $\lim_{r,s\to\infty}\inf\frac{f(\lambda_{rs})}{\lambda_{rs}} > 0$, we get that

$$\lim_{r,s\to\infty}\frac{1}{f(\lambda_{rs})}f(|\{(m,n)\in I_{r,s} : \|w_{mn}-t,z\| \geqslant U\}|) = 0.$$

Thus, (w_{mn}) is $S^f_{(\lambda,\eta)}(Y,\|.,.\|)$-bounded. □

From Theorem 6 and Theorem 7, we get the following result.

COROLLARY 9. *Let (w_{mn}) be a sequence in a 2-normed linear space $(Y,\|.,.\|)$, $(\lambda_{r,s}) \in \Delta_2$, and f be an unbounded modulus such that $\lim_{r,s\to\infty}\inf\frac{f(\lambda_{rs})}{\lambda_{rs}} > 0$. Then, (w_{mn}) is $S_{(\lambda,\eta)}(Y,\|.,.\|)$-bounded if and only if it is $S^f_{(\lambda,\eta)}(Y,\|.,.\|)$-bounded.*

THEOREM 8. *Let (w_{mn}) be a sequence in a 2-normed linear space $(Y,\|.,.\|)$, f be an unbounded modulus and $(\lambda_{r,s}) \in \Delta_2$. If $\lim_{r,s\to\infty}\inf\frac{f(\lambda_{rs})}{f(rs)} > 0$ and (w_{mn}) is $St_2^f(Y,\|.,.\|)$-bounded, then (w_{mn}) is $S^f_{(\lambda,\eta)}(Y,\|.,.\|)$-bounded.*

Proof. Suppose $(w_{mn}) \in St_2^f(Y, \|.,.\|)$. Then, there is $t \in X$ such that

$$\lim_{r,s\to\infty} \frac{1}{f(rs)} f(|\{m \leqslant r, n \leqslant s : \|w_{mn} - t, z\| \geqslant U\}|) = 0$$

for some $U \in \mathbb{R}^+$. Obviously, we have

$$|\{m \leqslant r, n \leqslant s : \|w_{mn} - t, z\| \geqslant U\}| \geqslant |\{(m,n) \in I_{r,s} : \|w_{mn} - t, z\| \geqslant U\}|.$$

Since f is a modulus, we may write

$$\begin{aligned} &\frac{1}{f(rs)} f(|\{m \leqslant r, n \leqslant s : \|w_{mn} - t, z\| \geqslant U\}|) \\ \geqslant\ &\frac{1}{f(rs)} f(|\{(m,n) \in I_{r,s} : \|w_{mn} - t, z\| \geqslant U\}|) \\ =\ &\frac{f(\lambda_{rs})}{f(rs)} \frac{1}{f(\lambda_{rs})} f(|\{(m,n) \in I_{r,s} : \|w_{mn} - t, z\| \geqslant U\}|). \end{aligned}$$

Since $(w_{mn}) \in St_2^f(Y, \|.,.\|)$ and $\lim_{r,s\to\infty} \inf \frac{f(\lambda_{rs})}{f(rs)} > 0$, we get

$$\lim_{r,s\to\infty} \frac{1}{f(\lambda_{rs})} f(|\{(m,n) \in I_{r,s} : \|w_{mn} - t, z\| \geqslant U\}|) = 0.$$

Therefore, (w_{mn}) is $S_{(\lambda,\eta)}^f(Y, \|.,.\|)$-bounded. □

4. Conclusion

In this paper, we introduced $(\lambda,\eta)(f)$-statistical convergence for double sequences in 2-normed linear spaces, utilizing modulus functions under various conditions. This approach not only addresses existing gaps in the literature but also contributes to the theoretical framework of statistical convergence in more abstract settings.

Our investigation into the relationships between $(\lambda,\eta)(f)$-statistically convergent sequences and $(\lambda,\eta)(f)$-statistically bounded sequences provides valuable insights for both theoretical advancements and practical applications. The results offer a foundation for further research in functional analysis and sequence spaces.

In future work, based on this research, there is potential to generalize the concept of $(\lambda,\eta)(f)$-statistical convergence within the context of ideals and explore its implications for sequences order α. This can lead to the development of new theoretical frameworks and methodologies, thereby enriching the field of abstract mathematics.

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