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DEFERRED STATISTICAL CONVERGENCE IN NEUTROSOPHIC 2–NORMED SPACES

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Abstract. In this research paper, we present a circumstantial investigation of deferred statistical convergence and prove some fundamental results within the framework of a neutrosophic 2-normed space. Furthermore, we define and provide a comprehensive exploration of deferred statistical Cauchy sequence and establish that every neutrosophic 2-normed space is deferred statistically complete within this specific framework.

1. Introduction

Zadeh [40] is the first prominent pioneering of the introduction of fuzzy set theory as an extension of classical set theory. Since its inception, it has been continually refined and integrated across various fields of engineering and science, including population dynamics [7], control of chaos [13], computer programming [15] , nonlinear dynamical systems [18], fuzzy physics [28] etc. An intriguing extension of fuzzy sets, introduced by Atanassov [2], is known as intuitionistic fuzzy sets, which enhance the traditional fuzzy sets by incorporating a non-membership function alongside the membership function. Over time, the concept of fuzzy set has been fascinatingly expanded into new and innovative notions, often referred to as interval valued fuzzy sets [36], interval valued intuitionistic fuzzy set [3], vague sets [6] and the evolution of fuzzy sets has sparked the growth of numerous concepts in mathematical analysis. As a comprehensive generalization of these concepts, Smarandache [33] defined a new idea named as neutrosophic set by introducing the indeterminacy function to the intuitionistic fuzzy sets, i.e., an element of a neutrosophic set is characterized by a triplet: the truth-membership function, the indeterminacy-membership function, and the falsity-membership function. In a neutrosophic set, each element of the universe is defined by its specific degrees of these notions. The concept of fuzzy normed spaces, introduced by Felbin [12] in 1992, evolved over the years with Saadati and Park's [34] introduction of intuitionistic fuzzy normed spaces in 2006, followed by Karakus et al.'s exploration of statistical convergence [20] within these spaces in 2008, and Melliani et al.'s [29] in 2018 generalization to deferred statistical convergence. Later on, Bera and Mahapatra explored the notion of neutrosophic soft linear space [8] and neutrosophic soft normed linear space [9]. Recently, Kirişci and Şimşek [22] introduced neutrosophic

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normed linear spaces and delved into the concept of statistical convergence, sparking further research into different types of sequence convergence within these spaces. For additional insights, see [10, 17, 23, 24]. In 2023, Murtaza et al. [30] introduced the groundbreaking concept of neutrosophic 2-normed linear space, a significant extension of neutrosophic normed space, and explored its statistical convergence and statistical completeness.

In 1932, Agnew [1] unveiled the deferred Cesàro mean, an innovative extension of the original Cesàro mean, offering enhanced capabilities and broader applications. Expanding on this innovation, Küçükaslan et al. [21] introduced the groundbreaking concept of deferred statistical convergence in 2016, building upon the deferred Cesàro mean as its cornerstone. Their research focused on establishing fundamental properties and revealing critical connections between deferred statistical convergence, the strong deferred Cesàro mean, and statistical convergence. For a deeper exploration of deferred statistical convergence and its various generalizations, consult references [10, 11, 16, 25, 26, 27, 29, 31, 37, 38], where you'll find an extensive list of additional sources.

Research on sequence convergence in neutrosophic 2-normed linear spaces is still in its early stages, with limited progress made thus far. However, the studies conducted to date reveal a compelling similarity in the behavior of sequence convergence within these spaces. So, keeping potential applicability of the concept of deferred statistical summability in mind, within this specific framwork, We have introduced the concept of deferred statistical convergence, extending the existing ideas of statistical convergence, λ-statistical convergence, and lacunary statistical convergence. We have explored several key properties of this newly introduced concept. Furthermore, we have defined the concept of deferred statistical Cauchy sequences and demonstrated their equivalence to deferred statistical convergent sequences within neutrosophic 2-normed spaces. Also, we have analyzed deferred statistical summability by comparing various pairs of sequences $\alpha(n)$, $\vartheta(n)$, $\beta(n)$ and $\gamma(n)$ that satisfy the inequality $\alpha(n) \leqslant \beta(n) < \gamma(n) \leqslant \vartheta(n)$, $\forall n \in \mathbb{N}$.

2. Preliminaries

In this section, we provide an overview of basic definitions and terminology which will be useful to describe our main results. Throughout the study $\mathbb{N}$ and $\mathbb{R}$ stand for the set of all natural numbers and real numbers respectively.

DEFINITION 1. [32] A binary operation $\boxdot : J \times J \to J$, where $J = [0,1]$ is named to be a continuous t-norm if for each $v_1, v_2, v_3, v_4 \in J$, the below conditions hold:

1. $\boxdot$ is associative and commutative;
2. $\boxdot$ is continuous;
3. $v_1 \boxdot 1 = v_1$ for all $v_1 \in J$;
4. $v_1 \boxdot v_2 \leqslant v_3 \boxdot v_4$ whenever $v_1 \leqslant v_3$ and $v_2 \leqslant v_4$.

DEFINITION 2. [32] A binary operation $\oplus : J \times J \to J$, where $J = [0,1]$ is named to be a continuous t-conorm if for each $v_1, v_2, v_3, v_4 \in J$, the below conditions hold:

1. $\oplus$ is associative and commutative;
2. $\oplus$ is continuous;
3. $v_1 \oplus 0 = v_1$ for all $v_1 \in J$;
4. $v_1 \oplus v_2 \leqslant v_3 \oplus v_4$ whenever $v_1 \leqslant v_3$ and $v_2 \leqslant v_4$.

EXAMPLE 1. [19] The continuous t-norms are $v_1 \boxdot v_2 = \min\{v_1, v_2\}$ and $v_1 \boxdot v_2 = v_1.v_2$. On the other hand, continuous t-conorms are $v_1 \oplus v_2 = \max\{v_1, v_2\}$ and $v_1 \oplus v_2 = v_1 + v_2 - v_1.v_2$.

LEMMA 1. [34] *If $\boxdot$ is a continuous t-norm, $\oplus$ is a continuous t-conorm, $v_i \in (0,1)$ and $1 \leqslant i \leqslant 7$, the following statements hold:*

1. *If $v_1 > v_2$, there are $v_3, v_4 \in (0,1)$ such that $v_1 \boxdot v_3 \geqslant v_2$ and $v_1 \geqslant v_2 \oplus v_4$*
2. *If $v_5 \in (0,1)$, there are $v_6, v_7 \in (0,1)$ such that $v_6 \boxdot v_6 \geqslant v_5$ and $v_5 \geqslant v_7 \oplus v_7$.*

Now, we provide the definitions of 2-normed linear space and neutrosophic 2-normed linear space.

DEFINITION 3. [14] Let Y be a real vector space of dimension d, where $2 \leqslant d < \infty$. A 2-norm on Y is a function $\|.,.\| : Y \times Y \to \mathbb{R}$ which satisfies the following conditions:

1. $\|\tau_1, \tau_2\| = 0$ if and only if τ_1 and τ_2 are linearly dependent in Y;
2. $\|\tau_1, \tau_2\| = \|\tau_2, \tau_1\|$ for all τ_1, τ_2 in Y;
3. $\|\kappa\tau_1, \tau_2\| = |\kappa| \, \|\tau_1, \tau_2\|$ for all κ in $\mathbb{R}$ and for all τ_1, τ_2 in Y;
4. $\|\tau_1 + \tau_2, \tau_3\| \leqslant \|\tau_1, \tau_3\| + \|\tau_2, \tau_3\|$ for all τ_1, τ_2, τ_3 in Y.

EXAMPLE 2. [35] Let $Y = \mathbb{R}^2$. Define $\|\cdot,\cdot\|$ on $\mathbb{R}^2$ by $\|u,v\| = |\tau_1\tau_4 - \tau_2\tau_3|$, where $u = (\tau_1, \tau_2), v = (\tau_3, \tau_4) \in \mathbb{R}^2$. Then, $(Y, \|\cdot,\cdot\|)$ is a 2-normed space.

DEFINITION 4. [30] Let W be a vector space, $N_2 = (\{(w,v), \Im(w,v), \Re(w,v), \wp(w,v)\} : (w,v) \in W \times W)$ be a 2-norm space such that $N_2 : W \times W \times \mathbb{R}^+ \to [0,1]$. If $\boxdot$ and $\oplus$ stand for continuous t-norm and t-conorm respectively, then four-tuple $(W, N_2, \boxdot, \oplus)$ is named to be neutrosophic 2-normed space (in short N2-NS) if for every $w, v, u \in W$, $\zeta, \varepsilon > 0$ and $\kappa \neq 0$, the following conditions are gratified:

1. $0 \leqslant \Im(w,v;\zeta) \leqslant 1$, $0 \leqslant \Re(w,v;\zeta) \leqslant 1$ and $0 \leqslant \wp(w,v;\zeta) \leqslant 1$;
2. $\Im(w,v;\zeta) + \Re(w,v;\zeta) + \wp(w,v;\zeta) \leqslant 3$;

3. $\Im(w,v;\zeta) = 1$ iff w,v are linearly dependent;

4. $\Im(\kappa w,v;\zeta) = \Im\left(w,v;\frac{\zeta}{|\kappa|}\right)$ for each $\kappa \neq 0$;

5. $\Im(w,v+u;\zeta+\varepsilon) \geqslant \Im(w,v;\zeta) \boxdot \Im(w,u;\varepsilon)$;

6. $\Im(w,v;\cdot) : (0,\infty) \to [0,1]$ is a non decreasing continuous function;

7. $\lim_{\zeta\to\infty} \Im(w,v;\zeta) = 1$;

8. $\Im(w,v;\zeta) = \Im(v,w;\zeta)$;

9. $\Re(w,v;\zeta) = 0$ iff w,v are linearly dependent;

10. $\Re(\kappa w,v;\zeta) = \Re\left(w,v;\frac{\zeta}{|\kappa|}\right)$ for each $\kappa \neq 0$;

11. $\Re(w,v+u;\zeta+\varepsilon) \leqslant \Re(w,v;\zeta) \oplus \Re(w,u;\varepsilon)$;

12. $\Re(w,v;\cdot) : (0,\infty) \to [0,1]$ is a non increasing continuous function;

13. $\lim_{\zeta\to\infty} \Re(w,v;\zeta) = 0$;

14. $\Re(w,v;\zeta) = \Re(v,w;\zeta)$;

15. $\wp(w,v;\zeta) = 0$ iff w,v are linearly dependent;

16. $\wp(\kappa w,v;\zeta) = \wp\left(w,v;\frac{\zeta}{|\kappa|}\right)$ for each $\kappa \neq 0$;

17. $\wp(w,v+u;\zeta+\varepsilon) \leqslant \wp(w,v;\zeta) \oplus \wp(w,u;\varepsilon)$;

18. $\wp(w,v;\cdot) : (0,\infty) \to [0,1]$ is a non increasing continuous function;

19. $\lim_{\zeta\to\infty} \wp(w,v;\zeta) = 0$;

20. $\wp(w,v;\zeta) = \wp(v,w;\zeta)$;

21. If $\zeta \leqslant 0$, $\Im(w,v;\zeta) = 0$, $\Re(w,v;\zeta) = 1$ and $\wp(w,v;\zeta) = 1$.

In the sequal, we denote H for neutrosophic 2-normed space instead of $(W,\Im,\Re,\wp,\boxdot,\oplus)$. And, we denote N_2 to mean neutrosophic 2-norm on H.

DEFINITION 5. [30] Let $\{w_k\}$ be a sequence in a N2-NS H. Choose $\sigma \in (0,1)$ and $\zeta > 0$. Then, $\{w_k\}$ is named to be convergent if there exists a $k_0 \in \mathbb{N}$ and $\rho \in W$ such that $\Im(w_k - \rho, v;\zeta) > 1-\sigma$, $\Re(w_k-\rho,v;\zeta) < \sigma$ and $\wp(w_k-\rho,v;\zeta) < \sigma$ for all $k \geqslant k_0$ and $v \in W$. In this case, we write $N_2 - \lim w_k = \rho$ or $w_k \xrightarrow{N_2} \rho$ and ρ is called N_2-limit of $\{w_k\}$.

DEFINITION 6. Let $K \subset \mathbb{N}$. Then, the natural density of K, denoted by $\delta(K)$, is defined as

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n}|\{k \leqslant n : k \in K\}|,$$

provided the limit exists, where the vertical bars denote the cardinality of the enclosed set.

DEFINITION 7. [30] Let $\{w_k\}$ be a sequence in a N2-NS H. Choose $\sigma \in (0,1)$ and $\zeta > 0$. Then, $\{w_k\}$ is named to be statistically convergent to $\rho \in W$ with regard to N_2 if for every $v \in W$, $\delta(\{k \in \mathbb{N} : \Im(w_k - \rho, v; \zeta) \leqslant 1 - \sigma$ or $\Re(w_k - \rho, v; \zeta) \geqslant \sigma$ and $\wp(w_k - \rho, v; \zeta) \geqslant \sigma\}) = 0$. In this case, we write $S(N_2) - \lim w_k = \rho$.

In 1932, Agnew [1] explored the idea of deferred Cesàro mean D_α^ϑ as an engrossing generalization of Cesàro mean of real (complex) valued sequence $w = \{w_k\}$ by

$$(D_\alpha^\vartheta w)_n = \frac{1}{\vartheta(n) - \alpha(n)} \sum_{k=\alpha(n)+1}^{\vartheta(n)} w_k, \; n = 1,2,3,\ldots$$

where $\vartheta = \vartheta(n)$ and $\alpha = \alpha(n)$ are the sequences of non-negative integers satisfying $\alpha(n) < \vartheta(n)$ and $\vartheta(n) \to \infty$ as $n \to \infty$.

A sequence $w = \{w_k\}$ is named to be D_α^ϑ-convergent to ρ if $\lim (D_\alpha^\vartheta w)_n = \rho$. And, the sequence $w = \{w_k\}$ is named to be strong D_α^ϑ-convergent [21] to ρ if

$$\lim_{n\to\infty} \frac{1}{\vartheta(n) - \alpha(n)} \sum_{k=\alpha(n)+1}^{\vartheta(n)} |w_k - \rho| = 0.$$

DEFINITION 8. [39] Let $K \subset \mathbb{N}$ and $K_{\alpha,\vartheta}(n) = \{\alpha(n) + 1 \leqslant k \leqslant \vartheta(n) : k \in K\}$. Then, the deferred density of K, denoted by $\delta_\alpha^\vartheta(K)$, is defined by

$$\delta_\alpha^\vartheta(K) = \lim_{n\to\infty} \frac{1}{\vartheta(n) - \alpha(n)} |K_{\alpha,\vartheta}(n)|.$$

DEFINITION 9. [21] A real valued sequence $w = \{w_k\}$ is named to be deferred statistically convergent to $\rho \in \mathbb{R}$ if for every $\sigma > 0$,

$$\lim_{n\to\infty} \frac{1}{\vartheta(n) - \alpha(n)} |\{\alpha(n) + 1 \leqslant k \leqslant \vartheta(n) : |w_k - \rho| \geqslant \sigma\}| = 0.$$

Throughout the following sections ϑ and α denote the sequences of non-negative integers as defined earlier. Also, $\delta_\alpha^\vartheta(K)$ stands for the deferred density of the set K and $[x]$ denotes greatest integer function.

3. Deferred statistical convergence in N2-NS

In this section, we define and study deferred statistical convergence of sequences with regard to N_2 and prove some interesting results.

DEFINITION 10. Let $\{w_k\}$ be a sequence in a N2-NS H. Then, $\{w_k\}$ is named to be deferred statistically convergent to $\rho \in W$ with respect to N_2 (in short $D_\alpha^\vartheta[S(N_2)]$-convergence) if for every $\sigma \in (0,1)$, $\zeta > 0$ and $v \in W$,

$$\delta_\alpha^\vartheta(\{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : \Im(w_k-\rho,v;\zeta) \leqslant 1-\sigma \text{ or } \Re(w_k-\rho,v;\zeta) \geqslant \sigma \\ \text{and } \wp(w_k-\rho,v;\zeta) \geqslant \sigma\}) = 0.$$

Or, it can be restated as

$$\lim_{n\to\infty} \frac{1}{\vartheta(n)-\alpha(n)} |\{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : \Im(w_k-\rho,v;\zeta) \leqslant 1-\sigma \\ \text{or } \Re(w_k-\rho,v;\zeta) \geqslant \sigma \text{ and } \wp(w_k-\rho,v;\zeta) \geqslant \sigma\}| = 0.$$

In this scenario, it is denoted as $D_\alpha^\vartheta[S(N_2)] - \lim w_k = \rho$ or $w_k \xrightarrow{D_\alpha^\vartheta[S(N_2)]} \rho$ and ρ is called $D_\alpha^\vartheta[S(N_2)]$-limit of $\{w_k\}$.

REMARK 1.

1. If we take $\vartheta(n) = n$ and $\alpha(n) = 0$, the Definition 10 coincides with the notion of statistical convergence with regard to N_2 [30].
2. If we take $\vartheta(n) = \lambda_n$ and $\alpha(n) = 0$, the Definition 10 coincides with the idea of λ-statistical convergence with regard to N_2 [4].
3. If we take $\vartheta(n) = k_s$ and $\alpha(n) = k_{s-1}$, the Definition 10 coincides with the concept of lacunary statistical convergence with regard to N_2 [5].

EXAMPLE 3. Let $W = \mathbb{R}^2$ equipped with the 2-norm $\|u,v\| = |\tau_1\tau_4 - \tau_2\tau_3|$, where $u = (\tau_1,\tau_2), v = (\tau_3,\tau_4) \in \mathbb{R}^2$. We take continuous t-norm $\boxdot$ and t-conorm $\oplus$ as $v_1 \boxdot v_2 = v_1 v_2$ and $v_1 \oplus v_2 = \min(v_1+v_2,1)$ $\forall v_1,v_2 \in [0,1]$. Let $\zeta > 0$ with $\zeta > \|w,v\|$. Consider $\Im(u,v;\zeta) = \frac{\zeta}{\zeta+\|u,v\|}$, $\Re(u,v;\zeta) = \frac{\|u,v\|}{\zeta+\|u,v\|}$, $\wp(u,v;\zeta) = \frac{\|u,v\|}{\zeta}$. Then, W becomes a N2-NS. We define a sequence $\{w_k\} \in W$ as follows:

$$w_k = \begin{cases} (k^2,0), & \left[\left|\sqrt{\vartheta(n)}\right|\right] - k_0 < k \leqslant \left[\left|\sqrt{\vartheta(n)}\right|\right],\ n = 1,2,\cdots \\ (0,0), & \text{otherwise} \end{cases},$$

where $\vartheta(n)$ is a monotonic increasing sequence with $0 < \alpha(n) \leqslant \left[\left|\sqrt{\vartheta(n)}\right|\right] - k_0$.

And, $k_0 \in \mathbb{N}$ is fixed. Then, for each $\sigma \in (0,1)$, $\zeta > 0$ and $v \in W$ we get

$$\begin{aligned} K^n_{\alpha,\vartheta}(\sigma,\zeta) &= \{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : \Im(w_k-\theta,v;\zeta) \leqslant 1-\sigma \text{ or } \Re(w_k-\theta,v;\zeta) \geqslant \sigma \\ &\qquad \text{and } \wp(w_k-\theta,v;\zeta) \geqslant \sigma\} \\ &= \left\{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : \frac{\zeta}{\zeta+\|w_k,v\|} \leqslant 1-\sigma \text{ or } \frac{\|w_k,u\|}{\zeta+\|w_k,u\|} \geqslant \sigma \text{ and} \right. \\ &\qquad \left. \frac{\|w_k,v\|}{\zeta} \geqslant \sigma \right\} \\ &= \left\{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : \|w_k,u\| \geqslant \frac{\zeta\sigma}{1-\sigma} > 0 \text{ and } \|w_k,u\| \geqslant \zeta\sigma > 0\right\} \\ &\subset \left\{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : w_k = (k^2,0)\right\} \\ &= \left\{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : \left[\left|\sqrt{\vartheta(n)}\right|\right] - k_0 < k \leqslant \left[\left|\sqrt{\vartheta(n)}\right|\right]\right\}. \end{aligned}$$

This gives $\delta^{\vartheta}_{\alpha}(K^n_{\alpha,\vartheta}(\sigma,\zeta)) = \lim_{n\to\infty} \frac{|K^n_{\alpha,\vartheta}(\sigma,\zeta)|}{\vartheta(n)-\alpha(n)} \leqslant \lim_{n\to\infty} \frac{k_0}{\vartheta(n)-\alpha(n)} = 0$. Hence, $D^{\vartheta}_{\alpha}[S(N_2)] - \lim w_k = \theta$, where $\theta = (0,0)$.

From Definition 10, we can easily prove the following.

LEMMA 2. *Let $\{w_k\}$ be a sequence in a N2-NS H. Then, for every $\sigma \in (0,1)$, $\zeta > 0$ and $v \in W$ the following properties are gratified:*

1. $D^{\vartheta}_{\alpha}[S(N_2)] - \lim w_k = \rho$;
2. *deferred density of each of* $\{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : \Im(w_k-\rho,v;\zeta) \leqslant 1-\sigma\}$, $\{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : \Re(w_k-\rho,v;\zeta) \geqslant \sigma\}$ *and* $\{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : \wp(w_k-\rho,v;\zeta) \geqslant \sigma\}$ *is zero;*
3. $\delta^{\vartheta}_{\alpha}(\{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : \Im(w_k-\rho,v;\zeta) > 1-\sigma$ *and* $\Re(w_k-\rho,v;\zeta) < \sigma$, $\wp(w_k-\rho,v;\zeta) < \sigma\}) = 1$;
4. *deferred density of each of* $\{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : \Im(w_k-\rho,v;\zeta) > 1-\sigma\}$, $\{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : \Re(w_k-\rho,v;\zeta) < \sigma\}$ *and* $\{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : \wp(w_k-\rho,v;\zeta) < \sigma\}$ *is* 1;
5. $D^{\vartheta}_{\alpha}[S(N_2)] - \lim \Im(w_k-\rho,v;\zeta) = 1$, $D^{\vartheta}_{\alpha}[S(N_2)] - \lim \Re(w_k-\rho,v;\zeta) = 0$ *and* $D^{\vartheta}_{\alpha}[S(N_2)] - \lim \wp(w_k-\rho,v;\zeta) = 0$.

THEOREM 1. *Let $\{w_k\}$ be a sequence in a N2-NS H. If $\{w_k\}$ is $D^{\vartheta}_{\alpha}[S(N_2)]$-convergent, then the $D^{\vartheta}_{\alpha}[S(N_2)]$-limit of $\{w_k\}$ is unique.*

Proof. Let, if possible $D^{\vartheta}_{\alpha}[S(N_2)] - \lim w_k = \rho_1$ and $D^{\vartheta}_{\alpha}[S(N_2)] - \lim w_k = \rho_2$, where $\rho_1 \neq \rho_2$. For a given $\sigma \in (0,1)$, choose $\varpi \in (0,1)$ such that $(1-\varpi)\boxdot(1-\varpi) >$

$1-\sigma$ and $\varpi \oplus \varpi < \sigma$. Then, using Lemma 2, for any $\zeta > 0$ and $v \in W$ deferred density of each of the following

$$M_{\mathfrak{I},1}(\varpi,\zeta) = \left\{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : \mathfrak{I}\left(w_k - \rho_1, v; \frac{\zeta}{2}\right) \leqslant 1-\varpi\right\};$$
$$B_{\mathfrak{I},2}(\varpi,\zeta) = \left\{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : \mathfrak{I}\left(w_k - \rho_2, v; \frac{\zeta}{2}\right) \leqslant 1-\varpi\right\};$$
$$M_{\mathfrak{R},1}(\varpi,\zeta) = \left\{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : \mathfrak{R}\left(w_k - \rho_1, v; \frac{\zeta}{2}\right) \geqslant \varpi\right\};$$
$$B_{\mathfrak{R},2}(\varpi,\zeta) = \left\{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : \mathfrak{R}\left(w_k - \rho_2, v; \frac{\zeta}{2}\right) \geqslant \varpi\right\};$$
$$M_{\wp,1}(\varpi,\zeta) = \left\{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : \wp\left(w_k - \rho_1, v; \frac{\zeta}{2}\right) \geqslant \varpi\right\};$$
$$B_{\wp,2}(\varpi,\zeta) = \left\{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : \wp\left(w_k - \rho_2, v; \frac{\zeta}{2}\right) \geqslant \varpi\right\}.$$

is zero. Let $A_{(\mathfrak{I},\mathfrak{R},\wp)}(\sigma,\zeta) = \{M_{\mathfrak{I},1}(\varpi,\zeta) \cup B_{\mathfrak{I},2}(\varpi,\zeta)\} \cap \{M_{\mathfrak{R},1}(\varpi,\zeta) \cup B_{\mathfrak{R},2}(\varpi,\zeta)\} \cap \{M_{\wp,1}(\varpi,\zeta) \cup B_{\wp,2}(\varpi,\zeta)\}$. Then, $\delta_\alpha^\vartheta(A_{(\mathfrak{I},\mathfrak{R},\wp)}(\sigma,\zeta)) = 0$. Obviously $\delta_\alpha^\vartheta(\mathbb{N} \setminus A_{(\mathfrak{I},\mathfrak{R},\wp)}(\sigma,\zeta)) = 1$. So, let $k \in \mathbb{N} \setminus A_{(\mathfrak{I},\mathfrak{R},\wp)}(\sigma,\zeta)$. Then, there arise three cases:

1. $k \in \mathbb{N} \setminus (M_{\mathfrak{I},1}(\varpi,\zeta) \cup B_{\mathfrak{I},2}(\varpi,\zeta))$
2. $k \in \mathbb{N} \setminus (M_{\mathfrak{R},1}(\varpi,\zeta) \cup B_{\mathfrak{R},2}(\varpi,\zeta))$
3. $k \in \mathbb{N} \setminus (M_{\wp,1}(\varpi,\zeta) \cup B_{\wp,2}(\varpi,\zeta))$.

If $k \in \mathbb{N} \setminus (M_{\mathfrak{I},1}(\varpi,\zeta) \cup B_{\mathfrak{I},2}(\varpi,\zeta))$, then

$$\begin{aligned}
&\mathfrak{I}(\rho_1 - \rho_2, v; \zeta) \\
&\geqslant \mathfrak{I}\left(w_k - \rho_1, v; \frac{\zeta}{2}\right) \boxdot \mathfrak{I}\left(w_k - \rho_2, v; \frac{\zeta}{2}\right) \\
&> (1-\varpi) \boxdot (1-\varpi) \\
&> 1-\sigma.
\end{aligned}$$

Since $\sigma \in (0,1)$ is arbitrary, $\mathfrak{I}(\rho_1 - \rho_2; \zeta) = 1$, which yields $\upsilon_1 = \upsilon_2$.

If $k \in \mathbb{N} \setminus (M_{\mathfrak{R},1}(\varpi,\zeta) \cup B_{\mathfrak{R},2}(\varpi,\zeta))$, then

$$\begin{aligned}
&\mathfrak{R}(\rho_1 - \rho_2, v; \zeta) \\
&\leqslant \mathfrak{R}\left(w_k - \rho_1, v; \frac{\zeta}{2}\right) \oplus \mathfrak{R}\left(w_k - \rho_2, v; \frac{\zeta}{2}\right) \\
&< \varpi \oplus \varpi \\
&< \sigma.
\end{aligned}$$

Since $\sigma \in (0,1)$ is arbitrary, $\mathfrak{R}(\rho_1 - \rho_2; \zeta) = 0$ which yields $\upsilon_1 = \upsilon_2$. Using similar technique, for the other case, we can prove the same. Hence, the $D_\alpha^\vartheta[S(N_2)]$-limit of $\{w_k\}$ is unique. $\square$

THEOREM 2. *Let* $\{w_k\}$ *be a sequence in a N2-NS* H. *If* $N_2 - \lim w_k = \rho$, $D_\alpha^\vartheta[S(N_2)] - \lim w_k = \rho$.

Proof. Suppose that $N_2 - \lim w_k = \rho$. Then, for every $\sigma \in (0,1)$, $\zeta > 0$ and $v \in W$ there exists $k_0 \in \mathbb{N}$ such that $\Im(w_k - \rho, v; \zeta) > 1 - \sigma$, $\Re(w_k - \rho, v; \zeta) < \sigma$ and $\wp(w_k - \rho, v; \zeta) < \sigma$ for all $k \geqslant k_0$. Then, it is obvious that the set $A = \{\alpha(n) + 1 \leqslant k \leqslant \vartheta(n) : \Im(w_k - \rho, v; \zeta) \leqslant 1 - \sigma$ or $\Re(w_k - \rho, v; \zeta) \geqslant \sigma$ and $\wp(w_k - \rho, v; \zeta) \geqslant \sigma\}$ contains at most finite number of terms. So, $\delta_\alpha^\vartheta(A) = 0$. Hence, $D_\alpha^\vartheta[S(N_2)] - \lim w_k = \rho$. □

But, the converse of Theorem 2 need not be true in general which can be illustrated as given below.

EXAMPLE 4. Let $W = \mathbb{R}^2$ equipped with the 2-norm $\|u,v\| = |\tau_1\tau_4 - \tau_2\tau_3|$, where $u = (\tau_1, \tau_2), v = (\tau_3, \tau_4) \in \mathbb{R}^2$. We take N2-NS as defined in Example 3. Define a sequence $\{w_k\} \in W$ by

$$w_k = \begin{cases} (1,0), \text{ if } k = i^2,\ i = 1,2,\cdots \\ (0,0) = \theta, \text{ otherwise} \end{cases}.$$

Then, for each $\sigma \in (0,1)$, $\zeta > 0$ and $v \in W$ we get

$$\begin{aligned} K_{\alpha,\vartheta}^n(\sigma,\zeta) &= \{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : \Im(w_k - \theta, v; \zeta) \leqslant 1 - \sigma \text{ or } \Re(w_k - \theta, v; \zeta) \geqslant \sigma \\ &\quad \text{and } \wp(w_k - \theta, v; \zeta) \geqslant \sigma\} \\ &= \left\{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : \frac{\zeta}{\zeta + \|w_k, v\|} \leqslant 1 - \sigma \text{ or } \frac{\|w_k, v\|}{\zeta + \|w_k, v\|} \geqslant \sigma \text{ and} \right. \\ &\quad \left. \frac{\|w_k, v\|}{\zeta} \geqslant \sigma \right\} \\ &= \left\{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : \|w_k, v\| \geqslant \frac{\zeta\sigma}{1-\sigma} > 0 \text{ and } \|w_k, v\| \geqslant \zeta\sigma > 0\right\} \\ &\subset \{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : w_k = (1,0)\} \\ &= \left\{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : k = i^2\right\}. \end{aligned}$$

This gives $\delta_\alpha^\vartheta(K_{\alpha,\vartheta}^n(\sigma,\zeta)) \leqslant \lim_{n\to\infty} \frac{\sqrt{\vartheta(n)} - \sqrt{\alpha(n)}}{\vartheta(n) - \alpha(n)} = 0$. So, $D_\alpha^\vartheta[S(N_2)] - \lim w_k = \theta$. Now, let $v = (v_1, v_2) \neq (0,0) \in W$ be arbitrary. Then,

$$\begin{aligned} &\lim_{k\to\infty} \Re(w_k - \theta, v; \zeta) \\ &= \lim_{k\to\infty} \frac{\|w_k, v\|}{\zeta + \|w_k, v\|} \\ &= \lim_{k\to\infty} \frac{\|(1,0), (v_1, v_2)\|}{\zeta + \|(1,0), (v_1, v_2)\|} \\ &= \lim_{k\to\infty} \frac{v_2}{\zeta + v_2} \neq 0 \end{aligned}$$

and

$$\begin{aligned}
&\lim_{k\to\infty}\wp(w_k-\theta,v;\zeta)\\
&=\lim_{k\to\infty}\frac{\|w_k,v\|}{\zeta}\\
&=\lim_{k\to\infty}\frac{\|(1,0),(v_1,v_2)\|}{\zeta}\\
&=\lim_{k\to\infty}\frac{v_2}{\zeta}\neq 0.
\end{aligned}$$

Using similar technique, we have $\lim_{k\to\infty}\Im(w_k-\theta,v;\zeta)=\lim_{k\to\infty}\frac{\zeta}{\zeta+v_2}\neq 1$. Hence, $\{w_k\}$ is not convergent to θ with regard to N_2.

THEOREM 3. *Let $\{w_k\}$ and $\{l_k\}$ be two sequences in a $N2$-NS H. Then, the below statements hold good:*

1. *If $D_\alpha^\vartheta[S(N_2)]-\lim w_k=\rho_1$ and $D_\alpha^\vartheta[S(N_2)]-\lim l_k=\rho_2$, $D_\alpha^\vartheta[S(N_2)]-\lim w_k+l_k=\rho_1+\rho_2$.*

2. *If $D_\alpha^\vartheta[S(N_2)]-\lim w_k=\rho$, $D_\alpha^\vartheta[S(N_2)]-\lim\kappa w_k=\kappa\rho$, $\kappa\neq 0$.*

Proof.

1. Suppose that $D_\alpha^\vartheta[S(N_2)]-\lim w_k=\rho_1$ and $D_\alpha^\vartheta[S(N_2)]-\lim l_k=\rho_2$. For a given $\sigma\in(0,1)$, choose $\varpi\in(0,1)$ such that $(1-\varpi)\boxdot(1-\varpi)>1-\sigma$ and $\varpi\oplus\varpi<\sigma$. Then, for every $\zeta>0$ and $v\in W$, the sets

$$\begin{aligned}
A(\varpi,\zeta)=\Big\{&\alpha(n)+1\leqslant k\leqslant\vartheta(n):\Im\left(w_k-\rho_1,v;\frac{\zeta}{2}\right)\leqslant 1-\varpi\\
&\text{or }\Re\left(w_k-\rho_1,v;\frac{\zeta}{2}\right)\geqslant\varpi\text{ and }\wp\left(w_k-\rho_1,v;\frac{\zeta}{2}\right)\geqslant\varpi\Big\}
\end{aligned}$$

and

$$\begin{aligned}
B(\varpi,\zeta)=\Big\{&\alpha(n)+1\leqslant k\leqslant\vartheta(n):\Im\left(l_k-\rho_2,v;\frac{\zeta}{2}\right)\leqslant 1-\varpi\\
&\text{or }\Re\left(l_k-\rho_2,v;\frac{\zeta}{2}\right)\\
&\geqslant\varpi\text{ and }\wp\left(l_k-\rho_2,v;\frac{\zeta}{2}\right)\geqslant\varpi\Big\}
\end{aligned}$$

have deferred density zero. Consider the set

$$\begin{aligned}
C(\sigma,\zeta)=\{&\alpha(n)+1\leqslant k\leqslant\vartheta(n):\Im(w_k+l_k-(\rho_1+\rho_2),v;\zeta)\leqslant 1-\sigma\text{ or}\\
&\Re(w_k+l_k-(\rho_1+\rho_2),v;\zeta)\geqslant\sigma\text{ and }\wp(w_k+l_k-(\rho_1+\rho_2),v;\zeta)\geqslant\sigma\}.
\end{aligned}$$

Then, deferred density of $A^c(\varpi,\zeta)$ and $B^c(\varpi,\zeta)$ have 1. So let $k\in A^c(\varpi,\zeta)\cap B^c(\varpi,\zeta)$. Then, we have

$$\begin{aligned}&\Im(w_k+l_k-(\rho_1+\rho_2),v;\zeta)\\&\geqslant \Im\left(w_k-\rho_1,v;\frac{\zeta}{2}\right)\boxdot\Im\left(l_k-\rho_2,v;\frac{\zeta}{2}\right)\\&>(1-\varpi)\boxdot(1-\varpi)\\&>1-\sigma\end{aligned}$$

and

$$\begin{aligned}&\Re(w_k+l_k-(\rho_1+\rho_2),v;\zeta)\\&\leqslant \Re\left(w_k-\rho_1,v;\frac{\zeta}{2}\right)\oplus\Re\left(l_k-\rho_2,v;\frac{\zeta}{2}\right)\\&<\varpi\oplus\varpi\\&<\sigma.\end{aligned}$$

Similarly, we get $\wp(w_k+l_k-(\rho_1+\rho_2),v;\zeta)<\sigma$. Therefore, $A^c(\varpi,\zeta)\cap B^c(\varpi,\zeta)\subset C^c(\sigma,\zeta)$ i.e., $C(\sigma,\zeta)\subset A(\varpi,\zeta)\cup B(\varpi,\zeta)$. Hence, $\delta_\alpha^\vartheta(C(\sigma,\zeta))=0$ i.e., $D_\alpha^\vartheta[S(N_2)]-\lim w_k+l_k=\rho_1+\rho_2$.

2. Suppose that $D_\alpha^\vartheta[S(N_2)]-\lim w_k=\rho$. Then, for every $\sigma\in(0,1)$, $\zeta>0$ and $v\in W$, $\delta_\alpha^\vartheta(\{\alpha(n)+1\leqslant k\leqslant\vartheta(n):\Im(w_k-\rho,v;\frac{\zeta}{|\kappa|})\leqslant 1-\sigma$ or $\Re(w_k-\rho,v;\frac{\zeta}{|\kappa|})\geqslant\sigma$ and $\wp(w_k-\rho,v;\frac{\zeta}{|\kappa|})\geqslant\sigma\})=0$. Since $\{\alpha(n)+1\leqslant k\leqslant\vartheta(n):\Im(\kappa w_k-\kappa\rho,v;\zeta)\leqslant 1-\sigma$ or $\Re(\kappa w_k-\kappa\rho,v;\zeta)\geqslant\sigma$ and $\wp(\kappa w_k-\kappa\rho,v;\zeta)\geqslant\sigma\}=\{\alpha(n)+1\leqslant k\leqslant\vartheta(n):\Im(w_k-\rho,v;\frac{\zeta}{|\kappa|})\leqslant 1-\sigma$ or $\Re(w_k-\rho,v;\frac{\zeta}{|\kappa|})\geqslant\sigma$ and $\wp(w_k-\rho,v;\frac{\zeta}{|\kappa|})\geqslant\sigma\}$. Hence, $D_\alpha^\vartheta[S(N_2)]-\lim\kappa w_k=\kappa\rho$. □

THEOREM 4. *Let $\{w_k\}$ be a sequence in a N2-NS H. Then, $D_\alpha^\vartheta[S(N_2)]-\lim w_k=\rho$ if and only if there exists a set $K=\{k_1<k_2<\cdots<k_p<\cdots\}\subset\mathbb{N}$ such that $\delta_\alpha^\vartheta(K)=1$ and $N_2-\lim w_{k_p}=\rho$.*

Proof. First suppose that $D_\alpha^\vartheta[S(N_2)]-\lim w_k=\rho$. Then, for any $\zeta>0$, $q\in\mathbb{N}$ and for every $v\in W$, the sets

$$\begin{aligned}A(q,\zeta)=\Big\{&\alpha(n)+1\leqslant k\leqslant\vartheta(n):\Im(w_k-\rho,v;\zeta)>1-\frac{1}{q}\\&\text{and }\Re(w_k-\rho,v;\zeta)<\frac{1}{q},\ \wp(w_k-\rho,v;\zeta)<\frac{1}{q}\Big\}\end{aligned}$$

and

$$B(q,\zeta)=\Big\{\alpha(n)+1\leqslant k\leqslant \vartheta(n):\Im(w_k-\rho,v;\zeta)\leqslant 1-\frac{1}{q}$$
$$\text{or } \Re(w_k-\rho,v;\zeta)\geqslant 1-\frac{1}{q} \text{ and } \wp(w_k-\rho,v;\zeta)\geqslant 1-\frac{1}{q}\Big\}$$

have the deferred density one and zero respectively. It is obvious that $A(q+1,\zeta)\subset A(q,\zeta)$. We can express $A(q,\zeta)$ as $\{k_1<k_2<\cdots<k_p<\cdots\}$. It is sufficient to prove the necessary part that for $k_p\in A(q,\zeta)$, $N_2-\lim w_{k_p}=\rho$. If possible, let the subsequence $\{w_{k_p}\}$ is not convergent to ρ with regard to N_2. Then for some $\sigma\in(0,1)$, $\Im(w_{k_p}-\rho,v;\zeta)\leqslant 1-\sigma$, $\Re(w_{k_p}-\rho,v;\zeta)\geqslant\sigma$ and $\wp(w_{k_p}-\rho,v;\zeta)\geqslant\sigma$ except for finite number of terms k_p. Consider the set

$$C(\sigma,\zeta)=\{\alpha(n)+1\leqslant k_p\leqslant \vartheta(n):\Im(w_{k_p}-\rho,v;\zeta)>1-\sigma$$
$$\text{and } \Re(w_{k_p}-\rho,v;\zeta)<\sigma,\ \wp(w_{k_p}-\rho,v;\zeta)<\sigma\}$$

where $\sigma>\frac{1}{q}$. Then, $\delta_\alpha^\vartheta(C(\sigma,\zeta))=0$. As $\sigma>\frac{1}{q}$, $A(q,\zeta)\subset C(\sigma,\zeta)$. This gives $\delta_\alpha^\vartheta(A(q,\zeta))=0$ which is a contradiction. Therefore, $N_2-\lim w_{k_p}=\rho$.

Conversely suppose that there exists a set $K=\{k_1<k_2<\cdots<k_p<\cdots\}\subset\mathbb{N}$ such that $\delta_\alpha^\vartheta(K)=1$ and $N_2-\lim w_{k_p}=\rho$. Then, for every $\sigma\in(0,1)$, $\zeta>0$ and $v\in W$ there exists $p_0\in\mathbb{N}$ such that $\Im(w_{k_p}-\rho,v;\zeta)>1-\sigma$, $\Re(w_{k_p}-\rho,v;\zeta)<\sigma$, and $\wp(w_{k_p}-\rho,v;\zeta)<\sigma$ for all $p\geqslant p_0$. Thus,

$$\begin{aligned}&\{\alpha(n)+1\leqslant k\leqslant\vartheta(n):\Im(w_k-\rho,v;\zeta)\leqslant 1-\sigma \text{ or } \Re(w_k-\rho,v;\zeta)\geqslant\sigma\\ &\text{and } \wp(w_k-\rho,v;\zeta)\geqslant\sigma\}\\ &\subset\mathbb{N}-\{k_{p_0+1},k_{p_0+2},\ldots\}.\end{aligned}$$

Therefore, $\delta_\alpha^\vartheta(\{\alpha(n)+1\leqslant k\leqslant\vartheta(n):\Im(w_k-\rho,v;\zeta)\leqslant 1-\sigma$ or $\Re(w_k-\rho,v;\zeta)\geqslant\sigma$ and $\wp(w_k-\rho,v;\zeta)\geqslant\sigma\})=0$ i.e., $D_\alpha^\vartheta[S(N_2)]-\lim w_k=\rho$. This completes the proof. □

COROLLARY 1. *Let $\{w_k\}$ be a sequence in a N2-NS H. Then, $D_\alpha^\vartheta[S(N_2)]-\lim w_k=\rho$ if and only if there exists a sequence $\{l_k\}\in W$ such that $N_2-\lim l_k=\rho$ and $\delta_\alpha^\vartheta(\{\alpha(n)+1\leqslant k\leqslant\vartheta(n):w_k=l_k\})=1$.*

THEOREM 5. *Let $\{w_k\}$ be a sequence in a N2-NS H. Also, let $\{\frac{\alpha(n)}{\vartheta(n)-\alpha(n)}\}$ be a bounded sequence. If $S(N_2)-\lim w_k=\rho$, $D_\alpha^\vartheta[S(N_2)]-\lim w_k=\rho$.*

Proof. Suppose that $S(N_2)-\lim w_k=\rho$. Then, for every $\sigma\in(0,1)$, $\zeta>0$ and $v\in W$,

$$\lim_{n\to\infty}\frac{1}{n}|\{k\leqslant n:\Im(w_k-\rho,v;\zeta)\leqslant 1-\sigma \text{ or } \Re(w_k-\rho,v;\zeta)\geqslant\sigma$$
$$\text{and } \wp(w_k-\rho,v;\zeta)\geqslant\sigma\}|=0.$$

Since $\vartheta(n) \to \infty$ as $n \to \infty$,

$$\lim_{n\to\infty} \frac{1}{\vartheta(n)} |\{k \leqslant \vartheta(n) : \Im(w_k - \rho, v; \zeta) \leqslant 1 - \sigma \text{ or } \Re(w_k - \rho, v; \zeta) \geqslant \sigma \text{ and } \wp(w_k - \rho, v; \zeta) \geqslant \sigma\}| = 0.$$

Now, since the following inclusion holds good,

$$\begin{aligned} &\{\alpha(n) + 1 \leqslant k \leqslant \vartheta(n) : \Im(w_k - \rho, v; \zeta) \leqslant 1 - \sigma \text{ or } \Re(w_k - \rho, v; \zeta) \geqslant \sigma \\ &\quad \text{and } \wp(w_k - \rho, v; \zeta) \geqslant \sigma\} \\ &\subset \{k \leqslant \vartheta(n) : \Im(w_k - \rho, v; \zeta) \leqslant 1 - \sigma \text{ or } \Re(w_k - \rho, v; \zeta) \geqslant \sigma \\ &\quad \text{and } \wp(w_k - \rho, v; \zeta) \geqslant \sigma\}, \end{aligned}$$

we have

$$\begin{aligned} &\frac{1}{\vartheta(n) - \alpha(n)} |\{\alpha(n) + 1 \leqslant k \leqslant \vartheta(n) : \Im(w_k - \rho, v; \zeta) \leqslant 1 - \sigma \\ &\quad \text{or } \Re(w_k - \rho, v; \zeta) \geqslant \sigma \text{ and } \wp(w_k - \rho, v; \zeta) \geqslant \sigma\}| \\ &\leqslant \frac{1}{\vartheta(n) - \alpha(n)} |\{k \leqslant \vartheta(n) : \Im(w_k - \rho, v; \zeta) \leqslant 1 - \sigma \text{ or } \Re(w_k - \rho, v; \zeta) \geqslant \sigma \\ &\quad \text{and } \wp(w_k - \rho, v; \zeta) \geqslant \sigma\}| \\ &= \left(1 + \frac{\alpha(n)}{\vartheta(n) - \alpha(n)}\right) \frac{1}{\vartheta(n)} |\{k \leqslant \vartheta(n) : \Im(w_k - \rho, v; \zeta) \leqslant 1 - \sigma \text{ or} \\ &\quad \Re(w_k - \rho, v; \zeta) \geqslant \sigma \text{ and } \wp(w_k - \rho, v; \zeta) \geqslant \sigma\}|. \end{aligned}$$

$\{\frac{\alpha(n)}{\vartheta(n) - \alpha(n)}\}$ being bounded,

$$\lim_{n\to\infty} \frac{1}{\vartheta(n) - \alpha(n)} |\{\alpha(n) + 1 \leqslant k \leqslant \vartheta(n) : \Im(w_k - \rho, v; \zeta) \leqslant 1 - \sigma \text{ or } \Re(w_k - \rho, v; \zeta) \geqslant \sigma \text{ and } \wp(w_k - \rho, v; \zeta) \geqslant \sigma\}| = 0$$

i.e., $D_\alpha^\vartheta[S(N_2)] - \lim w_k = \rho$. This completes the proof. □

4. Deferred statistical completeness in N2-NS

In this section, we explore the notion of deferred statistical Cauchy sequences and deferred statistical completeness with regard to N_2.

DEFINITION 11. Let $\{w_k\}$ be a sequence in a N2-NS H. Then, $\{w_k\}$ is named to be deferred statistical Cauchy sequence with respect to N_2 (in short $D_\alpha^\vartheta[S(N_2)]$-Cauchy) if for every $\sigma \in (0,1)$, $\zeta > 0$ and $v \in W$ there exists $k_0 = k_0(\sigma) \in \mathbb{N}$ such that $\delta_\alpha^\vartheta(\{\alpha(n) + 1 \leqslant k \leqslant \vartheta(n) : \Im(w_k - w_{k_0}, v; \zeta) \leqslant 1 - \sigma$ or $\Re(w_k - w_{k_0}, v; \zeta) \geqslant \sigma$ and $\wp(w_k - w_{k_0}, v; \zeta) \geqslant \sigma\}) = 0$.

THEOREM 6. *Let* $\{w_k\}$ *be a sequence in a* $N2$*-NS* H. *If* $\{w_k\}$ *is* $D_\alpha^\vartheta[S(N_2)]$*-convergent, it is* $D_\alpha^\vartheta[S(N_2)]$*-Cauchy sequence.*

Proof. Suppose that $D_\alpha^\vartheta[S(N_2)]-\lim w_k=\rho$. For $\sigma\in(0,1)$, choose $\varpi\in(0,1)$ such that $(1-\varpi)\boxdot(1-\varpi)>1-\sigma$ and $\varpi\oplus\varpi<\sigma$. Then, for every $\zeta>0$ and $v\in W$, deferred density of the set

$$A(\varpi,\zeta)=\left\{\alpha(n)+1\leqslant k\leqslant\vartheta(n):\Im\left(w_k-\rho,v;\frac{\zeta}{2}\right)\leqslant 1-\varpi\right.$$
$$\left.\text{or }\Re\left(w_k-\rho,v;\frac{\zeta}{2}\right)\geqslant\varpi\text{ and }\wp\left(w_k-\rho,v;\frac{\zeta}{2}\right)\geqslant\varpi\right\}$$

is zero. Then, $\delta_\alpha^\vartheta(A^c(\varpi,\zeta))=1$. So, there exists an element $k_0\in A^c(\varpi,\zeta)$. Therefore, we have

$$\Im\left(w_{k_0}-\rho,v;\frac{\zeta}{2}\right)>1-\varpi,$$
$$\Re\left(w_{k_0}-\rho,v;\frac{\zeta}{2}\right)<\varpi,$$

and

$$\wp\left(w_{k_0}-\rho,v;\frac{\zeta}{2}\right)<\varpi.$$

Let

$$B(\sigma,\zeta)=\{\alpha(n)+1\leqslant k\leqslant\vartheta(n):\Im(w_k-w_{k_0},v;\zeta)\leqslant 1-\sigma$$
$$\text{or }\Re(w_k-w_{k_0},v;\zeta)\geqslant\sigma\text{ and }\wp(w_k-w_{k_0},v;\zeta)\geqslant\sigma\}.$$

We claim that $B(\sigma,\zeta)\subset A(\varpi,\zeta)$. If not, let $m\in B(\sigma,\zeta)\setminus A(\varpi,\zeta)$. Now,

$$\begin{aligned}1-\sigma&\geqslant\Im(w_m-w_{k_0},v;\zeta)\\&\geqslant\Im\left(w_m-\rho,v;\frac{\zeta}{2}\right)\boxdot\Im\left(w_{k_0}-\rho,v;\frac{\zeta}{2}\right)\\&>(1-\varpi)\boxdot(1-\varpi)\\&>1-\sigma\ \text{ which is absurd.}\end{aligned}$$

Again

$$\begin{aligned}\sigma&\leqslant\Re(w_m-w_{k_0},v;\zeta)\\&\leqslant\Re\left(w_m-\rho,v;\frac{\zeta}{2}\right)\oplus\Re\left(w_{k_0}-\rho,v;\frac{\zeta}{2}\right)\\&<\varpi\oplus\varpi\\&<\sigma\ \text{ which is absurd.}\end{aligned}$$

Similarly we achieve an impossibility $\sigma \leqslant \wp(w_m - w_{k_0}, v; \zeta) < \sigma$. Hence, $B(\sigma,\zeta) \subset A(\varpi,\zeta)$. This gives $\delta_\alpha^\vartheta(B(\sigma,\zeta)) \leqslant \delta_\alpha^\vartheta(A(\sigma,\zeta))$ i.e., $\delta_\alpha^\vartheta(B(\sigma,\zeta)) = 0$ which shows that $\{w_k\}$ is $D_\alpha^\vartheta[S(N_2)]$-Cauchy sequence. □

THEOREM 7. *Let $\{w_k\}$ be a sequence in a N2-NS H. If $\{w_k\}$ is $D_\alpha^\vartheta[S(N_2)]$-Cauchy sequence, it is $D_\alpha^\vartheta[S(N_2)]$-convergent.*

Proof. Let $\{w_k\}$ is $D_\alpha^\vartheta[S(N_2)]$-Cauchy sequence but not $D_\alpha^\vartheta[S(N_2)]$-convergent. For a given $\sigma \in (0,1)$, choose $\varpi \in (0,1)$ such that $(1-\varpi)\boxdot(1-\varpi) > 1-\sigma$ and $\varpi \oplus \varpi < \sigma$. Then, for every $\zeta > 0$ and $v \in W$ there exists $k_0 \in \mathbb{N}$ such that $\delta_\alpha^\vartheta(B(\sigma,\zeta)) = 0$, where

$$\begin{aligned} B(\sigma,\zeta) = \{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : \Im(w_k - w_{k_0}, v;\zeta) \leqslant 1-\sigma \\ \text{or } \Re(w_k - w_{k_0}, v;\zeta) \geqslant \sigma \text{ and } \wp(w_k - w_{k_0}, v;\zeta) \geqslant \sigma\}. \end{aligned}$$

Since $\{w_k\}$ is not $D_\alpha^\vartheta[S(N_2)]$-convergent to $\rho \in W$, we have

$$\begin{aligned} &\Im(w_k - w_{k_0}, v;\zeta) \\ &\geqslant \Im\left(w_k - \rho, v; \frac{\zeta}{2}\right) \boxdot \Im\left(w_{k_0} - \rho, v; \frac{\zeta}{2}\right) \\ &> (1-\varpi)\boxdot(1-\varpi) \\ &> 1-\sigma, \end{aligned}$$

$$\begin{aligned} &\Re(w_k - w_{k_0}, v;\zeta) \\ &\leqslant \Re\left(w_k - \rho, v; \frac{\zeta}{2}\right) \oplus \Re\left(w_{k_0} - \rho, v; \frac{\zeta}{2}\right) \\ &< \varpi \oplus \varpi \\ &< \sigma \end{aligned}$$

and $\wp(w_k - w_{k_0}, v;\zeta) < \sigma$. Therefore, $\delta_\alpha^\vartheta(B^c(\sigma,\zeta)) = 0$ i.e., $\delta_\alpha^\vartheta(B(\sigma,\zeta)) = 1$, a contradiction. Hence, the sequence is $D_\alpha^\vartheta[S(N_2)]$-convergent to ρ. This ends the proof. □

DEFINITION 12. A N2-NS is named to be deferred statistically complete with regard to N_2 (in short $D_\alpha^\vartheta[S(N_2)]$-complete) if every $D_\alpha^\vartheta[S(N_2)]$-Cauchy sequence is $D_\alpha^\vartheta[S(N_2)]$-convergent.

REMARK 2. In the light of Theorem 7, we conclude every N2-NS is $D_\alpha^\vartheta[S(N_2)]$-complete.

On the basis of Theorem 4, 6 and 7, we state an equivalent result.

THEOREM 8. *Let $\{w_k\}$ be a sequence in a N2-NS H. Then, the below properties are equivalent:*

1. *$\{w_k\}$ is $D_\alpha^\vartheta[S(N_2)]$-convergent;*

2. *$\{w_k\}$ is $D_\alpha^\vartheta[S(N_2)]$-Cauchy sequence;*

3. *There exists a set $K = \{k_1 < k_2 < \cdots < k_p < \cdots\} \subset \mathbb{N}$ such that $\delta_\alpha^\vartheta(K) = 1$ and the subsequence $\{w_{k_p}\}$ is N_2-Cauchy sequence.*

5. Relationship between $D_\alpha^\vartheta[S(N_2)]$ and $D_\beta^\gamma[S(N_2)]$

In this section, we consider another pair of sequences $\{\beta(n)\}$ and $\{\gamma(n)\}$ of positive integers such that

$$\alpha(n) \leqslant \beta(n) < \gamma(n) \leqslant \vartheta(n),\ \forall n \in \mathbb{N}.$$

On the basis of this fact we compare $D_\alpha^\vartheta[S(N_2)]$-convergence with $D_\beta^\gamma[S(N_2)]$-convergence.

THEOREM 9. *Let $\{w_k\}$ be a sequence in a $N2$-NS H and, the sets $\{k \in \mathbb{N} : \alpha(n) < k \leqslant \beta(n)\}$ and $\{k \in \mathbb{N} : \gamma(n) < k \leqslant \vartheta(n)\}$ are finite. Then, $D_\beta^\gamma[S(N_2)] - \lim w_k = \rho \implies D_\alpha^\vartheta[S(N_2)] - \lim w_k = \rho$.*

Proof. Suppose that $D_\beta^\gamma[S(N_2)] - \lim w_k = \rho$. Then, for every $\sigma \in (0,1)$, $\zeta > 0$ and $v \in W$,

$$\begin{aligned} &\delta_\beta^\gamma(\{\beta(n)+1 \leqslant k \leqslant \gamma(n) : \Im(w_k - \rho, v; \zeta) \leqslant 1-\sigma \\ &\qquad \text{or } \Re(w_k - \rho, v; \zeta) \geqslant \sigma \text{ and } \wp(w_k - \rho, v; \zeta) \geqslant \sigma\}) = 0. \end{aligned} \tag{1}$$

Since

$$\begin{aligned} &\{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : \Im(w_k - \rho, v; \zeta) \leqslant 1-\sigma \\ &\qquad \text{or } \Re(w_k - \rho, v; \zeta) \geqslant \sigma \text{ and } \wp(w_k - \rho, v; \zeta) \geqslant \sigma\} \\ =&\{\alpha(n)+1 \leqslant k \leqslant \beta(n) : \Im(w_k - \rho, v; \zeta) \leqslant 1-\sigma \\ &\qquad \text{or } \Re(w_k - \rho, v; \zeta) \geqslant \sigma \text{ and } \wp(w_k - \rho, v; \zeta) \geqslant \sigma\} \\ \bigcup&\{\beta(n)+1 \leqslant k \leqslant \gamma(n) : \Im(w_k - \rho, v; \zeta) \leqslant 1-\sigma \\ &\qquad \text{or } \Re(w_k - \rho, v; \zeta) \geqslant \sigma \text{ and } \wp(w_k - \rho, v; \zeta) \geqslant \sigma\} \\ \bigcup&\{\gamma(n)+1 \leqslant k \leqslant \vartheta(n) : \Im(w_k - \rho, v; \zeta) \leqslant 1-\sigma \\ &\qquad \text{or } \Re(w_k - \rho, v; \zeta) \geqslant \sigma \text{ and } \wp(w_k - \rho, v; \zeta) \geqslant \sigma\}, \end{aligned}$$

we have

$$\begin{aligned}
&\delta_{\alpha}^{\vartheta}(\{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : \Im(w_k-\rho,v;\zeta) \leqslant 1-\sigma \text{ or } \Re(w_k-\rho,v;\zeta) \geqslant \sigma \\
&\qquad \text{and } \wp(w_k-\rho,v;\zeta) \geqslant \sigma\}) \\
&\leqslant \delta_{\alpha}^{\beta}(\{\alpha(n)+1 \leqslant k \leqslant \beta(n) : \Im(w_k-\rho,v;\zeta) \leqslant 1-\sigma \text{ or } \Re(w_k-\rho,v;\zeta) \geqslant \sigma \\
&\qquad \text{and } \wp(w_k-\rho,v;\zeta) \geqslant \sigma\}) \\
&\quad \delta_{\beta}^{\gamma}(\{\beta(n)+1 \leqslant k \leqslant \gamma(n) : \Im(w_k-\rho,v;\zeta) \leqslant 1-\sigma \text{ or } \Re(w_k-\rho,v;\zeta) \geqslant \sigma \\
&\qquad \text{and } wp(w_k-\rho,v;\zeta) \geqslant \sigma\}) \\
&\quad \delta_{\gamma}^{\vartheta}(\{\gamma(n)+1 \leqslant k \leqslant \vartheta(n) : \Im(w_k-\rho,v;\zeta) \leqslant 1-\sigma \text{ or } \Re(w_k-\rho,v;\zeta) \geqslant \sigma \\
&\qquad \text{and } \wp(w_k-\rho,v;\zeta) \geqslant \sigma\}).
\end{aligned}$$

By our assumption and from (1), we get $D_{\alpha}^{\vartheta}[S(N_2)] - \lim w_k = \rho$. □

THEOREM 10. *Let $\{w_k\}$ be a sequence in a $N2$-NS H. If* $\lim_{n\to\infty} \frac{\vartheta(n)-\alpha(n)}{\gamma(n)-\beta(n)} = \kappa > 0$, $D_{\alpha}^{\vartheta}[S(N_2)] - \lim w_k = \rho \implies D_{\beta}^{\gamma}[S(N_2)] - \lim w_k = \rho$.

Proof. Given that

$$\lim_{n\to\infty} \frac{\vartheta(n)-\alpha(n)}{\gamma(n)-\beta(n)} = \kappa > 0.$$

Suppose that $D_{\alpha}^{\vartheta}[S(N_2)] - \lim w_k = \rho$. Then, for every $\sigma \in (0,1)$, $\zeta > 0$ and $v \in W$,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{\vartheta(n)-\alpha(n)} |\{\vartheta(n)+1 \leqslant k \leqslant \alpha(n) : \Im(w_k-\rho,v;\zeta) \leqslant 1-\sigma \\
&\qquad \text{or } \Re(w_k-\rho,v;\zeta) \geqslant \sigma \text{ and } \wp(w_k-\rho,v;\zeta) \geqslant \sigma\}| = 0.
\end{aligned}$$

It is clear that

$$\begin{aligned}
&\{\beta(n)+1 \leqslant k \leqslant \gamma(n) : \Im(w_k-\rho,v;\zeta) \leqslant 1-\sigma \text{ or } \Re(w_k-\rho,v;\zeta) \geqslant \sigma \\
&\qquad \text{and } \wp(w_k-\rho,v;\zeta) \geqslant \sigma\} \\
&\subset \{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : \Im(w_k-\rho,v;\zeta) \leqslant 1-\sigma \text{ or } \Re(w_k-\rho,v;\zeta) \geqslant \sigma \\
&\qquad \text{and } \wp(w_k-\rho,v;\zeta) \geqslant \sigma\}.
\end{aligned}$$

So,

$$\begin{aligned}
&\frac{1}{\gamma(n)-\beta(n)} |\{\beta(n)+1 \leqslant k \leqslant \gamma(n) : \Im(w_k-\rho,v;\zeta) \leqslant 1-\sigma \\
&\qquad \text{or } \Re(w_k-\rho,v;\zeta) \geqslant \sigma \text{ and } \wp(w_k-\rho,v;\zeta) \geqslant \sigma\}| \\
&\leqslant \left(\frac{\vartheta(n)-\alpha(n)}{\gamma(n)-\beta(n)}\right) \frac{1}{\vartheta(n)-\alpha(n)} |\{\alpha(n)+1 \leqslant k \leqslant \vartheta(n) : \Im(w_k-\rho,v;\zeta) \leqslant 1-\sigma \\
&\qquad \text{or } \Re(w_k-\rho,v;\zeta) \geqslant \sigma \text{ and } \wp(w_k-\rho,v;\zeta) \geqslant \sigma\}|.
\end{aligned}$$

Now, letting $n \to \infty$, we have $D_{\beta}^{\gamma}[S(N_2)] - \lim w_k = \rho$. This completes the proof. □

Conclusion and future developments

In this research paper, we have dealt with $D_{\alpha}^{\vartheta}[S(N_2)]$-convergence and $D_{\alpha}^{\vartheta}[S(N_2)]$-Cauchy sequence. Furthermore, we have shown that every N2-NS is $D_{\alpha}^{\vartheta}[S(N_2)]$-complete. In future, based on this research work, one can generalize this notion in the context of ideal and nurture it related to sequences of sets of order α with regard to N_2. Also, this idea can be used in the field of convergence related problems in many branches of science and engineering.

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