Journal of Classical Analysis
Volume 25, Number 1 (2025), 63–74 doi:10.7153/jca-2025-25-04

REDUCTION AND SUMMATION FORMULAS FOR APPELL FUNCTIONS IN THE SPIRIT OF BURCHNALL–CHAUNDY

THOMAS ERNST* AND PER W. KARLSSON

Abstract. We find some reduction and summation formulas for multiple hypergeometric functions by using Burchnall-Chaundy expansions. The proofs use Legendre duplication, Kummer's second summation formula, Gauss summation formula and Whipple transformation. Many times we simply put a hypergeometric function argument equal to zero in order to cancel this second factor. We conclude by proving a q-analogue of a reduction formula for the third q-Appell function.

1. Introduction

This paper is based on notes of the late Per Karlsson, whose son Henrik kindly provided them to the first author. Karlsson was a leading expert on multiple hypergeometric functions [10], which have many applications as is seen in Exton [7]. After these two books, the paper by Toscano [11], the book by Hari M. Srivastava [9], and the book by Appell et. al. [2], the subject has been dormant for several years. This article is an attempt to revitalize this important subject.

For the convenience of the reader, we state all necessary definitions. The pertinent theorems can be found in [3], [5], [6] and [8].

DEFINITION 1. Throughout, unit arguments in hypergeometric functions are left out [3]. The sign $\equiv$ denotes a definition and $\cong$ denotes a formal equality. The variables

$$a,b,c,a_1,a_2,\ldots,b_1,b_2,\ldots \in \mathbb{C}$$

denote certain parameters in hypergeometric series or q-hypergeometric series. The variables i,j,k,l,m,n,p,r will denote natural numbers except for certain cases where it will be clear from the context that i will denote the imaginary unit.

DEFINITION 2. Let the *Pochhammer symbol* $(a)_n$ be defined by

$$(a)_n = \prod_{m=0}^{n-1}(a+m), \quad (a)_0 = 1. \tag{1}$$

Mathematics subject classification (2020): 33C65, 33C05, 33B15.
Keywords and phrases: Quadratic transformation, reduction and summation formulas, Appell function.

* Corresponding author.

Paper JCA-25-04

Since *products* of Pochhammer symbols occur so often, we shall frequently use the more compact notation

$$(a_1,a_2,\ldots,a_m)_n \equiv \prod_{j=1}^{m}(a_j)_n. \tag{2}$$

THEOREM 1. *A formula for the Pochhammer symbol* [8, p. 22 (2)] *is given by*

$$(a)_{kn} = k^{nk}\prod_{m=0}^{k-1}(\frac{a+m}{k})_n. \tag{3}$$

DEFINITION 3. Let $-b_i \ni \mathbb{N}_0$, $1 \leqslant i \leqslant r$ and $p \leqslant r+1$. Then the *generalized hypergeometric series* ${}_p\mathrm{F}_r$ [6] is defined by

$$\begin{aligned}{}_p\mathrm{F}_r(a_1,\ldots,a_p;b_1,\ldots,b_r;z) &\equiv {}_p\mathrm{F}_r\left[\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_r\end{matrix}\middle| z\right]\\ &\equiv \sum_{n=0}^{\infty}\frac{(a_1,\ldots,a_p)_n}{n!(b_1,\ldots,b_r)_n}z^n.\end{aligned} \tag{4}$$

Several times we shall use that its value for $z=0$ is 1.

The basic formula for ${}_2\mathrm{F}_1(a,b;c;1) \equiv {}_2\mathrm{F}_1(a,b;c)$ is given in the following theorem.

THEOREM 2. *The Gauß summation formula*

$${}_2\mathrm{F}_1(a,b;c) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \Re(c-a-b)>0. \tag{5}$$

We turn to double hypergeometric series. Four double hypergeometric series, known as Appell series [1], [2], are defined by.

DEFINITION 4. The first Appell function is defined by

$$\mathrm{F}_1(a;b,b';c;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty}\frac{(a)_{m_1+m_2}(b)_{m_1}(b')_{m_2}}{m_1!m_2!(c)_{m_1+m_2}}x_1^{m_1}x_2^{m_2},$$
$$\max(|x_1|,|x_2|)<1.$$

The second Appell function is defined by

$$\mathrm{F}_2(a;b,b';c,c';x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty}\frac{(a)_{m_1+m_2}(b)_{m_1}(b')_{m_2}}{m_1!m_2!(c)_{m_1}(c')_{m_2}}x_1^{m_1}x_2^{m_2},$$
$$|x_1|+|x_2|<1.$$

The third Appell function is defined by

$$\mathrm{F}_3(a,a';b,b';c;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty}\frac{(a)_{m_1}(a')_{m_2}(b)_{m_1}(b')_{m_2}}{m_1!m_2!(c)_{m_1+m_2}}x_1^{m_1}x_2^{m_2},$$
$$\max(|x_1|,|x_2|)<1.$$

The fourth Appell function is defined by

$$\mathrm{F}_4(a;b;c,c';x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{(a)_{m_1+m_2}(b)_{m_1+m_2}}{m_1!m_2!(c)_{m_1}(c')_{m_2}} x_1^{m_1} x_2^{m_2},$$
$$|\sqrt{x_1}|+|\sqrt{x_2}|<1.$$

The hypergeometric and Gamma functions are intimately connected and we first state the necessary Gamma function formulas. The Legendre duplication formula [8, p. 24]

$$\Gamma(2x) = \frac{2^{2x-1}\Gamma(x)\Gamma(x+\frac{1}{2})}{\sqrt{\pi}}, \quad x \neq 0, -\frac{1}{2}, -1, -\frac{3}{2} \dots \tag{6}$$

DEFINITION 5. The generalized Γ function is defined as follows:

$$\Gamma\begin{bmatrix} a_1,\dots,a_p \\ b_1,\dots,b_r \end{bmatrix} \equiv \frac{\Gamma(a_1)\dots\Gamma(a_p)}{\Gamma(b_1)\dots\Gamma(b_r)}.$$

THEOREM 3. *The following expansions for Appell functions can be found in the paper by Burchnall-Chaundy* [4, Eqs. (26), (28), (30), (39), (41), (43), (47), (49)]

$$\mathrm{F}_2(a;b,b';c,c';x_1,x_2) = \sum_{r=0}^{\infty} \frac{(a,b,b')_r}{(1,c,c')_r} x_1^r x_2^r$$
$$\times\ {}_2\mathrm{F}_1(a+r,b+r;c+r;x_1)\ {}_2\mathrm{F}_1(a+r,b'+r;c'+r;x_2),$$

$$\mathrm{F}_3(a,a';b,b';c;x_1,x_2) = \sum_{r=0}^{\infty} \frac{(-1)^r(a,a',b,b')_r}{(1,c+r-1)_r(c)_{2r}} x_1^r x_2^r$$
$$\times\ {}_2\mathrm{F}_1(a+r,b+r;c+2r;x_1)\ {}_2\mathrm{F}_1(a'+r,b'+r;c+2r;x_2), \tag{7}$$

$$\mathrm{F}_1(a;b,b';c;x_1,x_2) = \sum_{r=0}^{\infty} \frac{(c-a,a,b,b')_r}{(1,c+r-1)_r(c)_{2r}} x_1^r x_2^r$$
$$\times\ {}_2\mathrm{F}_1(a+r,b+r;c+2r;x_1)\ {}_2\mathrm{F}_1(a+r,b'+r;c+2r;x_2), \tag{8}$$

$$\mathrm{F}_1(a;b,b;c;x_1,x_2) = \sum_{r=0}^{\infty} \frac{(-1)^r(b)_r(a)_{2r}}{(1)_r(c)_{2r}} {x_1}^r {x_2}^r$$
$$\times\ {}_2\mathrm{F}_1(a+2r,b+r;c+2r;x_1+x_2), \tag{9}$$

$$\mathrm{F}_4(a;b;c,c;x_1,x_2) = \sum_{r=0}^{\infty} \frac{(a,b)_{2r}}{(1,c)_r(c)_{2r}} {x_1}^r {x_2}^r$$
$$\times\ {}_2\mathrm{F}_1(a+2r,b+2r;c+2r;x_1+x_2), \tag{10}$$

$$\mathrm{F}_2(a;b,b;c,c;x_1,x_2) = \sum_{r=0}^{\infty} (-1)^r \frac{(a)_{2r}(b,c-b)_r}{(c)_{2r}(1,c)_r} {x_1}^r {x_2}^r$$
$$\times\ {}_2\mathrm{F}_1(a+2r,b+r;c+2r;x_1+x_2), \tag{11}$$

$$\begin{aligned}\mathrm{F}_1(a;b,b;c;x_1,x_2) = \sum_{r=0}^{\infty} \frac{(a,b,c-a)_r}{r!(c)_{2r}} {x_1}^r {x_2}^r \\ \times \ {}_2\mathrm{F}_1(a+r,b+r;c+2r;x_1+x_2-x_1x_2),\end{aligned} \tag{12}$$

$$\begin{aligned}\mathrm{F}_3(a,a;b,b;c;x_1,x_2) = \sum_{r=0}^{\infty} \frac{(a,b,c-a-b)_r}{r!(c)_{2r}} {x_1}^r {x_2}^r \\ \times \ {}_2\mathrm{F}_1(a+r,b+r;c+2r;x_1+x_2-x_1x_2).\end{aligned} \tag{13}$$

2. Appell function reduction formulas

In this section we will state and prove several Appell function reduction formulas by using the previous Burchnall-Chaundy expansions.

THEOREM 4. *A reduction formula for the third Appell function is given by*

$$\mathrm{F}_3\left(a,a;a-\tfrac{1}{2},a-\tfrac{1}{2};2a;x_1,x_2\right) = \left[\frac{(1+\sqrt{1-x_1})(1+\sqrt{1-x_2})}{4}\right]^{1-2a}$$
$$\times \ {}_2\mathrm{F}_1\left[\begin{matrix} a-\frac{1}{2}, 2a-1 \\ a+\frac{1}{2}\end{matrix} \middle| -\frac{(1-\sqrt{1-x_1})(1-\sqrt{1-x_2})}{(1+\sqrt{1-x_1})(1+\sqrt{1-x_2})}\right].$$

Proof. When

$$a = a', \ \ c = 2a, \ \ b = b', \ \ b = a - \tfrac{1}{2}$$

in formula (7) we obtain for the left hand side

$$\begin{aligned}&\overset{\text{by (7)}}{=} \sum_{r=0}^{\infty} \frac{(-1)^r (a,a,a-\frac{1}{2},a-\frac{1}{2})_r}{(1,2a-1+r)_r (2a)_{2r}} x_1^r x_2^r \\ &\quad \times \ {}_2\mathrm{F}_1(a+r,a-\tfrac{1}{2}+r;2a+2r;x_1) \ {}_2\mathrm{F}_1(a+r,a-\tfrac{1}{2}+r;2a+2r;x_2) \\ &\overset{\text{by [5, §2.8 (6)]}}{=} \sum_{r=0}^{\infty} \frac{(-1)^r (a,a,a-\frac{1}{2},a-\frac{1}{2},2a-1)_r}{r!(2a,2a-1)_{2r}} x_1^r x_2^r \\ &\quad \times \left[\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1-x_1}\right]^{1-2a-2r} \left[\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1-x_2}\right]^{1-2a-2r} \\ &= \left[\frac{(1+\sqrt{1-x_1})(1+\sqrt{1-x_2})}{4}\right]^{1-2a} \\ &\quad \times \sum_{r=0}^{\infty} \frac{(-1)^r (a-\frac{1}{2},2a-1)_r}{r!(a+\frac{1}{2})_r} \left(\frac{x_1}{(1+\sqrt{1-x_1})^2}\right)^r \left(\frac{x_2}{(1+\sqrt{1-x_2})^2}\right)^r,\end{aligned}$$

which is equal to the desired result. □

THEOREM 5. *Assume* $Re(c-a-b) > 0$, $Re(c-a'-b') > 0$. *Then a reduction formula for the third Appell function is given by*

$$\mathrm{F}_3(a,a',b,b';c;1,1) = \Gamma\left[\begin{matrix} c, c-a-b \\ c-a, c-b \end{matrix}\right] \ {}_3\mathrm{F}_2\left[\begin{matrix} a', b', c-a-b \\ c-a, c-b \end{matrix}\right].$$

Proof. Let's choose $x_1 = x_2 = 1$ in formula (7) to obtain for the left hand side

$$\overset{\text{by (3),(7)}}{=} \sum_{r=0}^{\infty} \frac{(-1)^r (a,a',b,b',c-1)_r}{r!(c,c-1)_{2r}} \times \Gamma\left[\begin{matrix} c+2r, c+2r, c-a-b, c-a'-b' \\ c-a+r, c-b+r, c-a'+r, c-b'+r \end{matrix}\right]$$

$$= \Gamma\left[\begin{matrix} c,c,c-a-b,c-a'-b' \\ c-a,c-b,c-a',c-b' \end{matrix}\right] \sum_{r=0}^{\infty} \frac{(-1)^r (a,a',b,b',c-1,\frac{c}{2},\frac{c+1}{2})_r}{r!(c-a,c-b,c-a',c-b',\frac{c-1}{2},\frac{c}{2})_r}$$

$$= \Gamma\left[\begin{matrix} c,c,c-a-b,c-a'-b' \\ c-a,c-b,c-a',c-b' \end{matrix}\right] {}_6\mathrm{F}_5\left[\begin{matrix} a,a',b,b',c-1,\frac{c+1}{2} \\ c-a,c-b,c-a',c-b',\frac{c-1}{2} \end{matrix}\middle| -1\right].$$

This is equal to the desired result by [3, p. 28 (2)]. □

THEOREM 6. *Assume* $Re\,(c-a-b) > 0$. *Then a summation formula for the third Appell function is given by*

$$\mathrm{F}_3(a,1-a,b,1-b;c;1,1) = \frac{1}{2}(c-1)\Gamma\left[\begin{matrix} \frac{c+a+b-2}{2}, \frac{c-a-b}{2} \\ \frac{c+a-b}{2}, \frac{c-a+b}{2} \end{matrix}\right].$$

Proof. Let's choose $x = a - \frac{1}{2}$, $y = b - \frac{1}{2}$ in [3, p. 97, Ex. 3(III)]. Then choose $x_1 = x_2 = 1$ in formula (7) to obtain for the left hand side

$$\overset{\text{by (5), (7)}}{=} \sum_{r=0}^{\infty} \frac{(-1)^r (a,1-a,b,1-b)_r}{(1,c+r-1)_r (c)_{2r}} \times \Gamma\left[\begin{matrix} c+2r, c+2r, c-a-b, c+a+b-2 \\ c-a+r, c-b+r, c+a-1+r, c+b-1+r \end{matrix}\right]$$

$$\overset{\text{by (3)}}{=} \Gamma\left[\begin{matrix} c,c,c-a-b,c+a+b-2 \\ c-a,c-b,c+a-1,c+b-1 \end{matrix}\right] \times {}_6\mathrm{F}_5\left[\begin{matrix} \frac{c+1}{2}, c-1, a, 1-a, b, 1-b \\ \frac{c-1}{2}, c-a, c-b, c+a-1, c+b-1 \end{matrix}\middle| -1\right]$$

$$\overset{\text{by [3, p. 97, Ex. 3(III)]}}{=} \Gamma\left[\begin{matrix} c, c-a-b, c+a+b+2, \frac{1}{2}, \frac{1}{2}, c+a+1, c-b, c+b-1 \\ c-b, c+a-1, c+b-1, c-1, \frac{c+a+b-1}{2}, \frac{c+a-b}{2}, \frac{c-a+b}{2}, \frac{c+1-a-b}{2} \end{matrix}\right]$$

$$\overset{\text{by (6)}}{=} \frac{1}{2}\Gamma\left[\begin{matrix} c, \frac{c+a+b-2}{2}, \frac{c-a-b}{2} \\ c-1, \frac{c+a-b}{2}, \frac{c-a+b}{2} \end{matrix}\right],$$

which is equal to the desired result. □

THEOREM 7. *A reduction formula for the first Appell function is given by*

$$\mathrm{F}_1(a;a-\tfrac{1}{2},a-\tfrac{1}{2};2a;x_1,x_2)=\left[\frac{1}{4}(1+\sqrt{1-x_1})(1+\sqrt{1-x_2})\right]^{1-2a}$$
$$\times\ {}_2\mathrm{F}_1\left[\begin{matrix}a-\frac{1}{2},2a-1\\ a+\frac{1}{2}\end{matrix}\middle|\frac{(1-\sqrt{1-x_1})(1-\sqrt{1-x_2})}{(1+\sqrt{1-x_1})(1+\sqrt{1-x_2})}\right].$$

Proof. Put

$$c=2a,\quad b=b',\quad b=a-\tfrac{1}{2}$$

in formula (8) to obtain for the left hand side:

$$\overset{\text{by (8)}}{=}\sum_{r=0}^{\infty}\frac{(a,a,a-\frac{1}{2},a-\frac{1}{2},2a-1)_r}{r!(2a,2a-1)_{2r}}x_1^r x_2^r$$
$$\times\ {}_2\mathrm{F}_1(a+r,a+r-\tfrac{1}{2};2(a+r);x_1)\ {}_2\mathrm{F}_1(a+r,a+r-\tfrac{1}{2};2(a+r);x_2)$$
$$\overset{\text{by [5, §2.8 (6)]}}{=}\sum_{r=0}^{\infty}\frac{(a-\frac{1}{2},2a-1)_r}{r!2^{4r}(a+\frac{1}{2})_r}\left(\frac{1+\sqrt{1-x_1}}{2}\right)^{1-2a-2r}$$
$$\times\left(\frac{1+\sqrt{1-x_2}}{2}\right)^{1-2a-2r}x_1^r x_2^r$$
$$=\left[\frac{1}{4}(1+\sqrt{1-x_1})(1+\sqrt{1-x_2})\right]^{1-2a}$$
$$\times\ {}_2\mathrm{F}_1\left[\begin{matrix}a-\frac{1}{2},2a-1\\ a+\frac{1}{2}\end{matrix}\middle|\frac{x_1x_2}{(1+\sqrt{1-x_1})^2(1+\sqrt{1-x_2})^2}\right].$$

This is equal to the desired result. □

THEOREM 8. *For* $x<\frac{1}{2}-\frac{1}{\sqrt{2}}$ *we use analytic continuation of* ${}_3\mathrm{F}_2$.
For terminating series, that is

$$b=-n,\ \ c\neq -n,\ \ (c-b-a)\ni\mathbb{Z},$$

a reduction formula for a special first Appell function is given by

$$\mathrm{F}_1(a;b,b;c;x,1-x)$$
$$=\Gamma\left[\begin{matrix}c,c-b-a\\ c-a,c-b\end{matrix}\right]\ {}_3\mathrm{F}_2\left[\begin{matrix}\frac{1}{2}a,\frac{a+1}{2},b\\ c-b,1+a+b-c\end{matrix}\middle|4x(1-x)\right].$$

Proof. Let us choose $x_1+x_2=1$ in formula (9), then use Gauß's summation formula to obtain for the left hand side

$$\overset{\text{by (9), (5)}}{=}\sum_{r=0}^{\infty}\frac{(-1)^r(b)_r(a)_{2r}}{r!(c)_{2r}}[x(1-x)]^r\,\Gamma\left[\begin{matrix}c+2r,c-a-b-r\\ c-a,c-b+r\end{matrix}\right]$$
$$=\Gamma\left[\begin{matrix}c,c-b-a\\ c-a,c-b\end{matrix}\right]\sum_{r=0}^{\infty}\frac{(-1)^r(b)_r(c-b-a)_{-r}(a)_{2r}}{r!(c-b)_r}[x(1-x)]^r,$$

which is equal to the desired result by (3). □

THEOREM 9. *Assume* $|\sqrt{x}|+\left|\sqrt{\frac{1}{2}-x}\right|<1$. *Then a reduction formula for a special* F_4 *is given by*

$$\mathrm{F}_4\left(a;b;\tfrac{a+b+1}{2},\tfrac{a+b+1}{2};x,\tfrac{1}{2}-x\right)=\Gamma\left[\begin{matrix}\frac{a+b+1}{2},\frac{1}{2}\\ \frac{a+1}{2},\frac{b+1}{2}\end{matrix}\right]\,{}_2\mathrm{F}_1\left[\left.\begin{matrix}a,b\\ \frac{a+b+1}{2}\end{matrix}\right|2x\right].$$

This is a particular case of Watson's F_4 reduction formula.

Proof. Put $x_1+x_2=\frac{1}{2}$, $c=\frac{a+b+1}{2}$ in formula (10) to obtain

$$\begin{aligned}
&\mathrm{F}_4\left(a;b;\tfrac{a+b+1}{2},\tfrac{a+b+1}{2};x,\tfrac{1}{2}-x\right)\\
&\overset{\text{by }(10)}{=}\sum_{r=0}^{\infty}\frac{(a,b)_{2r}}{(1,\frac{a+b+1}{2})_r(\frac{a+b+1}{2})_{2r}}\left(x(\tfrac{1}{2}-x)\right)^r\,{}_2\mathrm{F}_1\left[\left.\begin{matrix}a+2r,b+2r\\ \frac{a+b+1}{2}+2r\end{matrix}\right|\frac{1}{2}\right]\\
&=\sum_{r=0}^{\infty}\frac{(a,b)_{2r}}{(1,\frac{a+b+1}{2})_r(\frac{a+b+1}{2})_{2r}}\left(x(\tfrac{1}{2}-x)\right)^r\Gamma\left[\begin{matrix}\frac{a+b+1}{2}+2r,\frac{1}{2}\\ \frac{a+1}{2}+r,\frac{b+1}{2}+r\end{matrix}\right]\\
&=\Gamma\left[\begin{matrix}\frac{a+b+1}{2},\frac{1}{2}\\ \frac{a+1}{2},\frac{b+1}{2}\end{matrix}\right]\sum_{r=0}^{\infty}\frac{(\frac{a}{2},\frac{a+1}{2},\frac{b}{2},\frac{b+1}{2})_r4^{2r}}{(1,\frac{a+b+1}{2},\frac{a+1}{2},\frac{b+1}{2})_r}\left(x(\tfrac{1}{2}-x)\right)^r\\
&=\Gamma\left[\begin{matrix}\frac{a+b+1}{2},\frac{1}{2}\\ \frac{a+1}{2},\frac{b+1}{2}\end{matrix}\right]\,{}_2\mathrm{F}_1\left[\left.\begin{matrix}\frac{a}{2},\frac{b}{2}\\ \frac{a+b+1}{2}\end{matrix}\right|8x(1-2x)\right]\overset{\text{by [5, §2.11 (2)]}}{=},
\end{aligned}$$

which proves the desired result. □

THEOREM 10. *A reduction formula for a special second Appell function is given by*

$$\begin{aligned}
&\mathrm{F}_2(a;b,b;c,c;x,1-x)\\
&=\Gamma\left[\begin{matrix}c,c-a-b\\ c-a,c-b\end{matrix}\right]\,{}_3\mathrm{F}_2\left[\left.\begin{matrix}\frac{a}{2},\frac{a+1}{2},b\\ c,1+a+b-c\end{matrix}\right|4x(1-x)\right],\\
&b=-n\vee Re(c-a)>0.
\end{aligned}$$

Proof. Let's put $x_1+x_2=1$ in formula (11). Then use Gauss' summation formula to obtain for the left hand side

$$\overset{\text{by }(11)}{=}\sum_{r=0}^{\infty}(-1)^r\frac{(a)_{2r}(b,c-b)_r}{(c)_{2r}(1,c)_r}\Gamma\left[\begin{matrix}c+2r,c-b-a-r\\ c-a,c-b+r\end{matrix}\right][x(1-x)]^r.$$

Again, this is equal to the desired result. □

THEOREM 11. *Assume* $Re(c-a-b)>0$. *Then*

$$F_1(a;b,b;c;1,y)=\Gamma\begin{bmatrix}c,c-a-b\\c-a,c-b\end{bmatrix}\ {}_2F_1\left[\begin{matrix}a,b\\c-b\end{matrix}\middle| y\right].$$

Proof. Put

$$x_1+x_2-x_1x_2=1$$

in formula (12), which is equivalent to $(1-x_1)(1-x_2)=0$. Then use Gauss' summation formula. □

THEOREM 12. *A reduction formula for a quadratic first Appell function is given by*

$$F_1\left(a;b,b;c;x,\frac{x}{x-1}\right)={}_3F_2\left[\begin{matrix}a,b,c-a\\\frac{c}{2},\frac{c+1}{2}\end{matrix}\middle|\frac{x^2}{4(x-1)}\right].$$

Proof. Put

$$x_1+x_2-x_1x_2=0$$

in formula (12), which is equivalent to $x_2=\frac{x_1}{x_1-1}$. Then the second factor becomes 1. □

THEOREM 13. *Assume* $Re(c-a-b)>0$. *Then a reduction formula for a special third Appell function is given by*

$$F_3(a,a;b,b;c;1,y)=\Gamma\begin{bmatrix}c,c-a-b\\c-a,c-b\end{bmatrix}\ {}_3F_2\left[\begin{matrix}a,b,c-a-b\\c-a,c-b\end{matrix}\middle| y\right].$$

Proof. Consider the special case

$$x_1+x_2-x_1x_2=1$$

in formula (13), then use Gauß's summation formula. □

THEOREM 14. *Assume* $\max(|x|,|\frac{x}{x-1}|)<1$. *Then a reduction formula for a quadratic third Appell function is given by*

$$F_3\left(a,a;b,b;c;x,\frac{x}{x-1}\right)={}_3F_2\left[\begin{matrix}a,b,c-a-b\\\frac{c}{2},\frac{c+1}{2}\end{matrix}\middle|\frac{x^2}{4(x-1)}\right].$$

Proof. Put

$$x_1+x_2-x_1x_2=0$$

in formula (13) and solve for x_2. □

3. A q-analogue of a reduction formula for the third q-Appell function Φ_3

We are able to q-deform one of the reduction formulas. In order to do this, the two ${}_2\phi_1$ formulas must be summed by the Heine formula. First we state some definitions.

DEFINITION 6. [6] Let $\delta > 0$ be an arbitrary small number. We will always use the following branch of the logarithm: $-\pi+\delta < \mathrm{Im}(\log q) \leqslant \pi+\delta$. This defines a simply connected space in the complex plane. The power function is then defined by

$$q^a \equiv e^{a\log(q)}.$$

DEFINITION 7. [6] The q-shifted factorials are defined by

$$\langle a;q\rangle_n \equiv \prod_{m=0}^{n-1}(1-q^{a+m}),$$

$$\langle \widetilde{a};q\rangle_n \equiv \prod_{m=0}^{n-1}(1+q^{a+m}).$$

DEFINITION 8. We shall define a q-hypergeometric series by

$$\begin{aligned}&{}_{p+p'}\phi_{r+r'}\left[\begin{matrix}\hat{a}_1,\ldots,\hat{a}_p\\ \hat{b}_1,\ldots,\hat{b}_r\end{matrix}|q;z\right]\\ &\equiv \sum_{k=0}^{\infty}\frac{\langle \hat{a}_1;q\rangle_k\ldots\langle \hat{a}_p;q\rangle_k}{\langle 1,\hat{b}_1;q\rangle_k\ldots\langle \hat{b}_r;q\rangle_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+r+r'-p-p'} z^k,\end{aligned}$$

where

$$\widehat{a} \equiv a \vee \widetilde{a}.$$

It is assumed that the denominator contains no zero factors, i.e. $\widehat{b_k} \neq -l+\frac{2m\pi i}{\log q}$, $k=1,\ldots,r, l,m\in\mathbb{N}$. In a few cases the parameter $\widehat{a}$ will be the real plus infinity, $(0<|q|<1)$. They correspond to multiplication by 1.

DEFINITION 9. [6] The q-analogue of the third Appell function is defined by

$$\begin{aligned}&\Phi_3(a,a';b,b';c|q;x_1,x_2)\\ &\equiv \sum_{m_1,m_2=0}^{\infty}\frac{\langle a;q\rangle_{m_1}\langle a';q\rangle_{m_2}\langle b;q\rangle_{m_1}\langle b';q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}}x_1^{m_1}x_2^{m_2},\ \max(|x_1|,|x_2|)<1.\end{aligned}$$

DEFINITION 10. The Γ_q function is defined by

$$\Gamma_q(z) \equiv \begin{cases}\frac{\langle 1;q\rangle_\infty}{\langle z;q\rangle_\infty}(1-q)^{1-z} & \text{if } 0<|q|<1\\ \frac{\langle 1;q^{-1}\rangle_\infty}{\langle z;q^{-1}\rangle_\infty}(\mathrm{q}-1)^{1-z}q^{\binom{z}{2}}, & \text{if } |q|>1.\end{cases}$$

The simple poles of Γ_q are located at $z = -n \pm \frac{2k\pi i}{\log q}$, $n,k \in \mathbb{N}_0$.

DEFINITION 11. The generalized Γ_q function is defined as follows:

$$\Gamma_q \begin{bmatrix} a_1,\ldots,a_p \\ b_1,\ldots,b_r \end{bmatrix} \equiv \frac{\Gamma_q(a_1)\ldots\Gamma_q(a_p)}{\Gamma_q(b_1)\ldots\Gamma_q(b_r)}.$$

THEOREM 15. *Heines q-analogue of Gauß's summation formula,* [6, (7.57)], *states that*

$$_2\phi_1(a,b;c|q;q^{c-a-b}) = \Gamma_q \begin{bmatrix} c, c-a-b \\ c-a, c-b \end{bmatrix}, \quad |q^{c-a-b}| < 1. \tag{14}$$

We shall also need the following lemma.

LEMMA 1. [6, (10.217)], *an expansion formula for the third q-Appell function, and a q-analogue of (7) is given by*

$$\begin{aligned} \Phi_3(a,a';b,b';c|q;x_1,x_2) &= \sum_{r=0}^{\infty} \frac{(-1)^r \langle a,a',b,b';q\rangle_r}{\langle 1,c+r-1;q\rangle_r \langle c;q\rangle_{2r}} x_1^r x_2^r q^{rc+\frac{3}{2}r(r-1)} \\ &\times\ {}_2\phi_1(a+r,b+r;c+2r|q;x_1)\ {}_2\phi_1(a'+r,b'+r;c+2r|q;x_2). \end{aligned} \tag{15}$$

THEOREM 16. *Assume Re* $(c-a-b) > 0$, *Re* $(c-a'-b') > 0$. *Then a reduction formula with double q-power argument for the third q-Appell function is given by*

$$\begin{aligned} &\Phi_3(a,a';b,b';c|q;q^{c-a-b},q^{c-a'-b'}) \\ &= \Gamma_q \begin{bmatrix} c,c,c-a-b,c-a'-b' \\ c-a,c-b,c-a',c-b' \end{bmatrix} \\ &\times {}_7\phi_9 \left[\begin{matrix} a,a',b,b',c-1,\frac{c+1}{2},\widetilde{\frac{c+1}{2}} \\ c-a,c-b,c-a',c-b',\frac{c-1}{2},\widetilde{\frac{c-1}{2}},3\infty \end{matrix} \middle| q;q^c \right]. \end{aligned}$$

Proof. Put $x_1 = q^{c-a-b}$, $x_2 = q^{c-a'-b'}$ in formula (15) to obtain for the left hand side

$$\begin{aligned} &\overset{\text{by (14), (15)}}{=} \sum_{r=0}^{\infty} \frac{(-1)^r \langle a,a',b,b',c-1;q\rangle_r}{\langle 1;q\rangle_r \langle c,c-1;q\rangle_{2r}} \\ &\times q^{rc+\frac{3}{2}r(r-1)} \Gamma_q \begin{bmatrix} c+2r,c+2r,c-a-b,c-a'-b' \\ c-a+r,c-b+r,c-a'+r,c-b'+r \end{bmatrix} \\ &\overset{\text{by [6, (1.46), (6.33), (6.34)]}}{=} \Gamma_q \begin{bmatrix} c,c,c-a-b,c-a'-b' \\ c-a,c-b,c-a',c-b' \end{bmatrix} \\ &\times \sum_{r=0}^{\infty} \frac{(-1)^r \langle a,a',b,b',c-1,\frac{c}{2},\frac{c+1}{2},\widetilde{\frac{c+1}{2}};q\rangle_r}{\langle 1,c-a,c-b,c-a',c-b',\frac{c-1}{2},\frac{c}{2},\widetilde{\frac{c-1}{2}};q\rangle_r} q^{rc+\frac{3}{2}r(r-1)}, \end{aligned}$$

which is equal to the desired formula. □

Acknowledgement. Thanks to the referee for his/her constructive criticism.

Conclusion. We have found several special reduction formulas which are seldom seen in the literature. Some with short and some with long proofs. Of course, the formulas with proofs which use Gauss' summation formula and/or quadratic transformations require certain restrictions of the variables. We have pointed this out, when possible. As a by-product, we have found proper restrictions of the parameters for the third Appell function to converge for double unit argument.

Discussion. There are also confluent formulas and reduction formulas for Horn and triple functions etc. These can be derived from our new formulas.

It was not possible to find q-analogues of the formulas, where the argument was set to zero. The reason was that this would require q-analogues of formulas (12) and (13) with the second q-addition.

Conflict of interest. The manuscript contains no conflicts of interest. There is no Data Availability Statement, since this is pure mathematics.

REFERENCES

[1] P. APPELL, *Sur des séries hypergéométriques de deux variables...*, C. R. Acad. Sci. **90** (1880), 296–298, 731–734.

[2] P. APPELL, J. KAMPÉ DE FÉRIET, *Fonctions hypergéométriques et hypersphériques*, Paris 1926.

[3] W. N. BAILEY, *Generalized hypergeometric series*, Cambridge 1935, reprinted by Stechert-Hafner, New York (1964).

[4] J. L. BURCHNALL, T. W. CHAUNDY, *Expansions of Appell's double hypergeometric functions*, Quart. J. Math., Oxford, **11**, (1940), 249–270.

[5] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER, F. G. TRICOMI, *Higher transcendental functions. vols. I,*, Based, in part, on notes left by Harry Bateman, McGraw-Hill Book Company, Inc., New York-Toronto-London (1953).

[6] T. ERNST, *A comprehensive treatment of q-calculus*, Birkhäuser 2012.

[7] H. EXTON, *Multiple hypergeometric functions and applications*, Mathematics and its Applications. Ellis Horwood Ltd., Chichester; Halsted Press [John Wiley & Sons, Inc.], New York-London-Sydney, 1976.

[8] E. D. RAINVILLE, *Special functions*, Reprint of 1960 first edition, Chelsea Publishing Co., Bronx, N.Y., 1971.

[9] H. M. SRIVASTAVA, B. R. K. KASHYAP, *Special functions in queuing theory. And related stochastic processes*, New York etc.: Academic Press, A Subsidiary of Harcourt Brace Jovanovich, Publishers, XII, 308 p. (1982).

[10] H. M. SRIVASTAVA, P. W. KARLSSON, *Multiple Gaussian hypergeometric series*, Ellis Horwood, New York, 1985.

[11] L. TOSCANO, *Formule di trasformazione e sivuluppi sulle funzioni ipergeometriche a due variabili*, (Italian), Ann. Mat. Pura Appl., IV. Ser. 33 (1952), 119–134.

(Received October 21, 2024)

Thomas Ernst
Department of Mathematics
Uppsala University
P.O. Box 480, SE-751 06 Uppsala, Sweden
e-mail: thomas@math.uu.se

Per W. Karlsson
Department of Mathematics
Uppsala University
P.O. Box 480, SE-751 06 Uppsala, Sweden