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DOUBLE SHEHU TRANSFORM FOR TIME SCALES WITH APPLICATIONS

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Abstract. The classical Shehu transform is an the important integral transform used for solving differential and integral equations. The transform has already been extended for time scales. In this paper, we studied the double Shehu transform for time scales and solve partial and integro dynamic equations without converting them into an ordinary dynamic equation. The existence condition for the double Shehu transform is given. Further, some elementary properties, and the convolution theorem are discussed. Finally, applications are given for solving partial dynamic and integro-dynamic equations through examples.

1. Introduction

In real applications, we perpetually deal with various problems having both continuous and discrete aspects. To unify these cases S. Hilger initiated the concept of time scales [17], which is any closed subset of real numbers. Integral transforms have their own importance in various fields of science. Application of appropriate integral transform convert differential and integral equations to simple algebraic expressions that can be solved easily. Integral transforms on time scales have the same influence for solving various ordinary, partial-dynamic and integro-dynamic equations with given initial conditions. Motivating from this theory integral transforms such as Laplace, Fourier, Sumudu, and Shehu transforms have been generalized on time scales [2, 4, 5, 7, 8, 9, 10, 11, 13, 14, 16, 18, 19, 20, 21, 22, 23, 24, 25, 28, 29].

Hassen Eltayeb, Adem Kilicman, and Brian Fisher [12] introduced the double Laplace transform and concept of multiple convolution on time scales. The Shehu transform, which is a generalization of the Sumudu and Laplace transforms was introduced by Shehu Maitma and Weidong Zhao in [15]. In [25] we generalized the Shehu transform along with its fundamental properties on time scales and used this to solve dynamic equations. The double Shehu transform of a function $g(x,t)$ defined on a suitable domain was introduced by Suliman Alfaqeih and Emine Misirli in [3] and is defined as

$$H_{xt}^2(g(x,t)) = \int_0^\infty \int_0^\infty e^{-\left(\frac{px}{u}+\frac{qt}{v}\right)} g(x,t)\,dx\,dt. \tag{1}$$

In this paper, we have generalized the double Shehu transform on time scales and applied it to solve partial and integro-dynamic equations. Using this technique a

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partial dynamic equation can be solved without converting it into an ordinary dynamic equation, which reduces our calculations and is the main advantage of this technique. Thus the use of the double Shehu transform is very convenient.

2. Preliminaries

In this section, we recall some basic terminologies and results from [1, 4, 5, 6, 8, 12, 25, 26] which are as follows.

DEFINITION 1. A function $g:\mathbb{T}\to\mathbb{R}$ is called rd-continuous if it is continuous at right-dense points in $\mathbb{T}$ and its left sided limit exists at left-dense points in $\mathbb{T}$.

DEFINITION 2. A function $g:\mathbb{T}\to\mathbb{R}$ is called regressive provided

$$1+\mu(t)f(t)\neq 0 \text{ for all } \quad t\in\mathbb{T}.$$

We denote set of all regressive functions by $\mathscr{R}$.

Further, for $h>0$, the set of Hilger complex numbers is, $\mathbb{C}_h := \{z\in\mathbb{C} : z\neq \frac{-1}{h}\}$. And for $z\in\mathbb{C}$ the Hilger real part of z is $Re_h(z) := \frac{|zh+1|-1}{h}$.

DEFINITION 3. Let $g:\mathbb{T}\to\mathbb{C}$ is an rd-continuous function, then the generalized Shehu transform of $g(t)$ is defined as

$$G(u,v)=\mathscr{S}h\{g(t)\}(u,v)=\int_{t_0}^{\infty} e^{\sigma}_{\ominus\frac{u}{v}}(t,t_0)g(t)\,\Delta t \tag{2}$$

for all $\frac{u}{v}\in\mathscr{D}\{g\}$ where $\mathscr{D}\{g\}$ consists of all $\frac{u}{v}\in\mathscr{R}$ for which the improper integral exists.

DEFINITION 4. The function $g:\mathbb{T}\to\mathbb{C}$ is said to be of exponential type I, if there exists constants $\mathscr{M},c>0$ such that $|g(t)|\leqslant\mathscr{M}e^{ct}$. The function $g:\mathbb{T}\to\mathbb{C}$ is said to be of exponential type II, if there exists constants $\mathscr{M},c>0$ such that $|g(t)|\leqslant\mathscr{M}e_c(t,t_0)$.

Following concepts are important regarding convergence of the Shehu transform.

2.1. Domain of the Shehu transform

We will assume that $\mathbb{T}$ is a time scale with bounded graininess, that is, $0<\mu_{\min}\leqslant\mu(t)\leqslant\mu_{\max}<\infty$ for all $t\in\mathbb{T}$. We set $\mu_{\min}=\mu_*$ and $\mu_{\max}=\mu^*$. Let $\mathbb{H}$ denotes the Hilger circle given by,

$$\mathbb{H}=\mathbb{H}_t=\left\{z\in\mathbb{C}:0<\left|z+\frac{1}{\mu(t)}\right|<\frac{1}{\mu(t)}\right\}$$

$$\mathbb{H}_{\min}=\left\{z\in\mathbb{C}:0<\left|z+\frac{1}{\mu^*(t)}\right|<\frac{1}{\mu^*(t)}\right\}$$

$$\mathbb{H}_{\max}=\left\{z\in\mathbb{C}:0<\left|z+\frac{1}{\mu_*(t)}\right|<\frac{1}{\mu_*(t)}\right\}.$$

Clearly $\mathbb{H}_{\min} \subseteq \mathbb{H}_t \subseteq \mathbb{H}_{\max}$ In order to give a suitable domain for the transform, which is associated to the region of convergence of integral 2 for any $c > 0$, we define the set

$$\begin{aligned}\mathscr{D} &= \left\{\frac{u}{v} \in \mathbb{C} : Re_\mu\left(\frac{u}{v}\right) > Re_\mu(c) \ \ \forall t \in \mathbb{T}\right\}\\ &= \left\{\frac{u}{v} \in \mathbb{C} : \frac{u}{v} \in \overline{\mathbb{H}}^c_{\max} \text{ and } Re_{\mu_*}\left(\frac{u}{v}\right) > Re_{\mu_*}(c) \ \ \forall t \in \mathbb{T}\right\}\\ &= \left\{\frac{u}{v} \in \mathbb{C} : \ominus\frac{u}{v} \in \mathbb{H} \text{ and } Re_\mu\left(\frac{u}{v}\right) > Re_\mu(c) \ \ \forall t \in \mathbb{T}\right\},\end{aligned}$$

where $\overline{\mathbb{H}}^c_{\max}$ denotes the complement of closure of largest Hilger circle corresponding to μ. When $\mu_* = 0$ this set is a right half plane.

LEMMA 1. *If* $\ominus\frac{u}{v} \in \mathbb{H}$, *and* $Re_\mu\left(\frac{u}{v}\right) > Re_\mu(c)$ *for all* $t \in \mathbb{T}$ *then* $\left(\ominus\frac{u}{v} \oplus c\right) \in \mathbb{H}$.

Proof. Since $Re_\mu\left(\frac{u}{v}\right) > Re_\mu(c)$ implies $\left|1 + \frac{u}{v}\mu(t)\right| > \left|1 + c\mu(t)\right|$.

Now $\ominus\frac{u}{v} \oplus c = \left|\frac{-1+c\frac{v}{u}}{\frac{v}{u}+\mu(t)}\right|$ and $\left|\ominus\frac{u}{v} \oplus c + \frac{1}{\mu(t)}\right| = \left|\frac{c\mu(t)+1}{\left(1+\frac{u}{v}\mu(t)\right)\mu(t)}\right| < \frac{1}{\mu(t)}$.

Thus $\ominus\frac{u}{v} \oplus c \in \mathbb{H}$. □

LEMMA 2. *If* $\ominus\frac{u}{v} \in \mathbb{H}$, *then* $\left|1 + \mu(t)\frac{u}{v}\right| > 1$.

Proof. As $\ominus\frac{u}{v} \in \mathbb{H}$ then $\left|\ominus\frac{u}{v} + \frac{1}{\mu(t)}\right| < \frac{1}{\mu(t)}$ this implies $\left|\frac{-1}{\frac{v}{u}+\mu(t)} + \frac{1}{\mu(t)}\right| < \frac{1}{\mu(t)}$ this implies $\left|\frac{1}{\left(1+\mu(t)\frac{u}{v}\right)\mu(t)}\right| < \frac{1}{\mu(t)}$. Thus $\left|1 + \mu(t)\frac{u}{v}\right| > 1$. □

THEOREM 1. (Absolute convergence of the Shehu transform) *If* $g : \mathbb{T} \to \mathbb{C}$ *is an rd-continuous function of exponential type II with the exponential constant c, then the integral* $\int_{t_0}^{\infty} e^{\sigma}_{\ominus\frac{u}{v}}(t,t_0)g(t,t_0)\Delta t$ *converges absolutely for all* $\frac{u}{v} \in \mathscr{D}$.

Proof. For $\frac{u}{v} \in \mathscr{D}$, $\ominus\frac{u}{v} \in \mathbb{H}$, then from Lemma 2, we have $\left|1 + \mu(t)\frac{u}{v}\right| > 1$.

$$\begin{aligned}\left|\int_{t_0}^{\infty} e^{\sigma}_{\ominus\frac{u}{v}}(t,t_0)f(t)\ \ \Delta(t)\right| &\leqslant \mathscr{M}\int_{t_0}^{\infty}\left|e_c(t,t_0)e^{\sigma}_{\ominus\frac{u}{v}}(t,t_0)\right|\ \ \Delta t\\ &\leqslant \mathscr{M}\int_{t_0}^{\infty}\left|\frac{e_{\ominus\frac{u}{v}}(t,t_0)}{1+\mu(t)\frac{u}{v}}\right|\ \ \left|e_c(t,t_0)\right|\ \Delta t\\ &\leqslant \mathscr{M}\int_{t_0}^{\infty}\left|e_{\ominus\frac{u}{v}\oplus c}(t,t_0)\right|\ \ \Delta t\\ &\leqslant \mathscr{M}\int_{t_0}^{\infty} exp\left(\int_{t_0}^{t}\frac{log\left|1+\mu(\tau)\left(\ominus\frac{u}{v}\oplus c\right)\right|}{\mu(\tau)}\ \ \Delta(\tau)\right)\Delta t\\ &\leqslant \mathscr{M}\int_{t_0}^{\infty} exp\left(\int_{t_0}^{t}\frac{log\left|\frac{1+\mu(\tau)c}{1+\mu(\tau)\frac{u}{v}}\right|}{\mu(\tau)}\Delta\tau\right)\Delta t\end{aligned}$$

$$\leqslant \mathcal{M}\int_{t_0}^{\infty} e^{-\alpha t}dt$$
$$= \frac{\mathcal{M}}{\alpha}$$

where $\alpha = \left|\frac{log\left|\frac{1+\mu_* c}{1+\mu^* z}\right|}{\mu^*}\right|$. $\square$

Following same estimates from above theorem we can show that if $g(t)$ is of exponential type II with constant c and $Re_\mu\left(\frac{u}{v}\right) > Re_\mu(c)$, then $\lim_{t\to\infty} e_{\ominus\frac{u}{v}}(t,t_0)g(t) = 0$.

THEOREM 2. (Shehu transform of derivative) *If $g : \mathbb{T} \to \mathbb{C}$ is rd-continuous function such that $g^\Delta : \mathbb{T} \to \mathbb{C}$ is rd-continuous and $\lim_{t\to\infty} e_{\ominus\frac{u}{v}}(t,t_0)g(t) = 0$, then*

$$\mathcal{S}h\{g^\Delta(t)\}(u,v) = \frac{u}{v}\mathcal{S}h\{g(t)\}(u,v) - g(t_0).$$

Proof. Applying Definition 3

$$\begin{aligned}
\mathcal{S}h\{g^\Delta(t)\}(u,v) &= \int_{t_0}^{\infty} e^{\sigma}_{\ominus\frac{u}{v}(t,t_0)} g^\Delta(t)\,\Delta t \\
&= \int_{t_0}^{\infty} \{(e_{\ominus\frac{u}{v}}(t,t_0)g(t))^\Delta - g(t)e^\Delta_{\ominus\frac{u}{v}}(t,t_0)\}\Delta t \\
&= -g(t_0) - \int_{t_0}^{\infty} g(t)\left(\ominus\frac{u}{v}\right) e_{\ominus\frac{u}{v}}(t,t_0)\,\Delta t \\
&= -g(t_0) + \frac{u}{v}\int_{t_0}^{\infty} g(t)\frac{e_{\ominus\frac{u}{v}}(t,t_0)}{1+\mu(t)\frac{u}{v}}\,\Delta t \\
&= -g(t_0) + \frac{u}{v}\int_{t_0}^{\infty} e^{\sigma}_{\ominus\frac{u}{v}}(t,t_0)g(t)\,\Delta t \\
&= \frac{u}{v}\mathcal{S}h\{g(t)\}(u,v) - g(t_0). \quad \square
\end{aligned}$$

More generally, if $g^{\Delta^{n-1}} : \mathbb{T} \to \mathbb{C}$ is an rd-continuous function such that $g^{\Delta^n}(t) : \mathbb{T} \to \mathbb{C}$ is also rd-continuous then,

$$\mathcal{S}h\{g^{\Delta^n}(t)\}(u,v) = \frac{u^n}{v^n}\mathcal{S}h\{g(t)\}(u,v) - \sum_{k=0}^{n-1}\left(\frac{u}{v}\right)^{n-(k+1)} g^{\Delta^k}(t_0).$$

THEOREM 3. *Assume that $h : \mathbb{T} \to \mathbb{C}$ is rd-continuous function.*
If $H(t) = \int_{t_0}^{t} h(\tau)\Delta\tau$ for all $t \in \mathbb{T}$, then $\mathcal{S}h\{H(t)\}(u,v) = \frac{v}{u}\mathcal{S}h\{h(t)\}(u,v)$.

Proof. We have $H^\Delta(t) = h(t)$ and $H(t_0) = 0$ then applying Theorem 2, we get

$$\mathcal{S}h\{H^\Delta(t)\}(u,v) = \frac{u}{v}\mathcal{S}h\{H(t)\}(u,v) - H(t_0)$$
$$\mathcal{S}h\{H(t)\}(u,v) = \frac{v}{u}\mathcal{S}h\{h(t)\}(u,v). \quad \square$$

THEOREM 4. *(1)* $\mathscr{S}h\{1\} = \frac{v}{u}$ *provided* $\lim_{t\to\infty} e_{\ominus\frac{u}{v}}(t_1,t_0) = 0$.

(2) If $\alpha \in \mathscr{R}$ *is a complex number then*

$$\mathscr{S}h\{e_\alpha(t,t_0)\}(u,v) = \frac{v}{u-\alpha v}$$

with $\alpha \neq \frac{u}{v}$ *and for* $\lim_{t\to\infty} e_{\alpha\ominus\frac{u}{v}}(t,t_0) = 0$.

(3) If $\alpha \in \mathscr{R}$ *is a complex number such that* $\mu\alpha^2 \in \mathscr{R}$*, then*

$$\mathscr{S}h\{\sin_\alpha(t,t_0)\}(u,v) = \frac{\alpha v^2}{u^2+\alpha^2 v^2},$$

and

$$\mathscr{S}h\{\cos_\alpha(t,t_0)\}(u,v) = \frac{uv}{u^2+\alpha^2 v^2}.$$

(4) If $\alpha \in \mathscr{R}$ *is a complex number such that* $-\mu\alpha^2 \in \mathscr{R}$ *then*

$$\mathscr{S}h\{\sinh_\alpha(t,t_0)\}(u,v) = \frac{\alpha^2 v}{u^2-\alpha^2 v^2},$$

and

$$\mathscr{S}h\{\cosh_\alpha(t,t_0)\}(u,v) = \frac{uv}{u^2-\alpha^2 v^2}.$$

(5) For a generalized polynomial defined in [4, 5] *as*

$$h_0(t,s) = 1$$
$$h_{k+1}(t,s) = \int_s^t h_k(\tau,s)\Delta\tau, \qquad k \in \mathbb{N}\cup\{0\}$$

then $\mathscr{S}h\{h_k(t_1,t_0)\}(u,v) = \frac{1}{\left(\frac{u}{v}\right)^{k+1}}$ *for* $\frac{u}{v} \in \mathscr{R}$, $u \neq 0$, $v \neq 0$ *with*

$$\lim_{t\to\infty} h_k(t,t_0) e_{\ominus\frac{u}{v}}(t,t_0) = 0.$$

Proof.

(1) Applying Definition 3

$$\begin{aligned}
\mathscr{S}h\{1\}(u,v) &= \int_{t_0}^{\infty} e^{\sigma}_{\ominus\frac{u}{v}}(t,t_0)\,\Delta t \\
&= \int_{t_0}^{\infty} \left(\mu(t) e^{\Delta}_{\ominus\frac{u}{v}} + e_{\ominus\frac{u}{v}}(t,t_0)\right)\Delta t \\
&= \int_{t_0}^{\infty} \left(1+\mu(t)\left(\ominus\frac{u}{v}\right)\right) e_{\ominus\frac{u}{v}}(t,t_0)\,\Delta t \\
&= \frac{-u}{v}\int_{t_0}^{\infty} \frac{-\frac{u}{v}}{1+\mu(t)\frac{u}{v}} e_{\ominus\frac{u}{v}}(t,t_0)\,\Delta t
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-v}{u}\int_{t_0}^{\infty} \ominus \frac{u}{v} e_{\ominus \frac{u}{v}}(t,t_0)\,\Delta t \\
&= \frac{-v}{u}\int_{t_0}^{\infty} e^{\Delta}_{\ominus \frac{u}{v}}(t,t_0)\,\Delta t \\
&= \frac{v}{u}.
\end{aligned}
$$

(2) Applying Definition 3

$$
\begin{aligned}
\mathscr{S}h\{e_{\alpha}(t,t_0)\}(u,v) &= \int_{t_0}^{\infty} e^{\sigma}_{\ominus \frac{u}{v}}(t,t_0) e_{\alpha}(t,t_0)\,\Delta t \\
&= \int_{t_0}^{\infty} \frac{e_{\alpha \ominus \frac{u}{v}}(t,t_0)}{1+\mu(t)\frac{u}{v}}\,\Delta t \\
&= \frac{v}{\alpha v - u}\int_{t_0}^{\infty} \left(\alpha \ominus \frac{u}{v}\right) e_{\alpha \ominus \frac{u}{v}}(t,t_0)\,\Delta t \\
&= \frac{-v}{u-\alpha v}\int_{t_0}^{\infty} e^{\Delta}_{\alpha \ominus \frac{u}{v}}(t,t_0)\,\Delta t \\
&= \frac{v}{u-\alpha v}.
\end{aligned}
$$

(3) We have the following relations,

$$
\sin_{\alpha}(t,t_0) = \frac{e_{i\alpha}(t,t_0) - e_{-i\alpha}(t,t_0)}{2i} \quad \text{and} \quad \cos_{\alpha}(t,t_0) = \frac{e_{i\alpha}(t,t_0) + e_{-i\alpha}(t,t_0)}{2}.
$$

Applying Definition 3, and using linearity of Shehu transform, we get

$$
\mathscr{S}h\{\sin_{\alpha}(t,t_0)\}(u,v) = \frac{\alpha v^2}{u^2+\alpha^2 v^2} \quad \text{and} \quad \mathscr{S}h\{\cos_{\alpha}(t,t_0)\}(u,v) = \frac{uv}{u^2+\alpha^2 v^2}.
$$

(4) We have following relations,

$$
\sinh_{\alpha}(t,t_0) = \frac{e_{\alpha}(t,t_0) - e_{-\alpha}(t,t_0)}{2} \quad \text{and} \quad \cosh_{\alpha}(t,t_0) = \frac{e_{\alpha}(t,t_0) + e_{-\alpha}(t,t_0)}{2}.
$$

Applying Definition 3, and using linearity of Shehu transform, we get

$$
\mathscr{S}h\{\sinh_{\alpha}(t,t_0)\}(u,v) = \frac{\alpha^2 v}{u^2-\alpha^2 v^2}, \quad \text{and} \quad \mathscr{S}h\{\cosh_{\alpha}(t,t_0)\}(u,v) = \frac{uv}{u^2-\alpha^2 v^2}.
$$

(5) From (1) the result holds for $k=0$. Now assuming that above formula holds for $k=p\in\mathbb{N}_0$. So we have $\mathscr{S}h\{h_p(t,t_0)\}(u,v) = \frac{1}{\left(\frac{u}{v}\right)^{p+1}}$.

Now, by Theorem 3

$$
\begin{aligned}
\mathscr{S}h\{h_{p+1}(t,t_0)\}(u,v) &= \frac{v}{u}\mathscr{S}h\{h_p(t,t_0)\}(u,v) \\
&= \frac{v}{u}\frac{1}{\left(\frac{u}{v}\right)^{p+1}} \\
&= \frac{1}{\left(\frac{u}{v}\right)^{(p+1)+1}}. \quad \square
\end{aligned}
$$

It is important to note following properties of regressive and exponential functions given in [1, 5]. For $p_1, p_2 \in \mathscr{R}$, we have

(1) $p_1 \oplus p_2 = p_1 + p_2 + \mu p_1 p_2$,

(2) $p_1 \ominus p_2 = p_1 \oplus (\ominus p_2) = \frac{p_1 - p_2}{1+\mu p_2}$,

(3) $\ominus p_1 = \frac{-p}{1+\mu p}$,

(4) $e_{p_1}(\sigma(t), t_0) = e_{p_1}(t, t_0)(1 + \mu(t) p_1(t))$,

(5) $e_{\ominus p_1}(t, t_0) = \frac{1}{e_p(t, t_0)}$,

(6) $e_{p_1}(t, t_0).e_{p_2}(t, t_0) = e_{p_1 \oplus p_2}(t, t_0)$,

(7) $\frac{e_{p_1}(t, t_0)}{e_{p_2}(t, t_0)} = e_{p_1 \oplus p_2}(t, t_0)$,

(8) $e^{\sigma}_{\ominus p_1}(t, t_0) = \frac{e_{\ominus p_1}(t, t_0)}{1+\mu p_1}$.

In this paper we adopt the following notations

$$e_{a\oplus b}(t_1, t_2, t_0, t_0') = e_a(t_1, t_0) e_b(t_2, t_0') \quad \text{and} \quad e_{\ominus a \ominus b}(t_1, t_2, t_0, t_0') = e_{\ominus a}(t_1, t_0) e_{\ominus b}(t_2, t_0').$$

3. Double Shehu transform on time scales

THEOREM 5. *For given time scales $\mathbb{T}_1$ and $\mathbb{T}_2$ the generalized Shehu transform of rd-continuous function $g : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{C}$ with respect to variables t_1 and t_2 is given by*

$$\mathscr{S}h_{t_1}[g(t_1, t_2)](u_1, v_1; t_0) = \int_{t_0}^{\infty} e^{\sigma_1}_{\ominus \frac{u_1}{v_1}}(t_1, t_0) g(t_1, t_2)\, \Delta_1 t_1$$

and

$$\mathscr{S}h_{t_2}[g(t_1, t_2)](u_2, v_2; t_0') = \int_{t_0'}^{\infty} e^{\sigma_2}_{\ominus \frac{u_2}{v_2}}(t_2, t_0') g(t_1, t_2)\, \Delta_2 t_2$$

respectively.

Now we will extend classical double Shehu transform given by Equation 1 in time scale as.

DEFINITION 5. If $\mathbb{T}_1$ and $\mathbb{T}_2$ are time scales such that $sup\{\mathbb{T}_1, \mathbb{T}_2\} = \infty$ and $t_0 \in \mathbb{T}_1, t_0' \in \mathbb{T}_2$ are fixed. Let $g : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{C}$ is rd-continuous function then the generalized double Shehu transform of $g(t_1, t_2)$ is

$$\begin{aligned} G[(u_1, u_2), (v_1, v_2)] &= \mathscr{S}h_{t_1} \mathscr{S}h_{t_2}[g(t_1, t_2)] \\ &= \int_{t_0}^{\infty} \int_{t_0'}^{\infty} e^{\sigma_1 \sigma_2}_{\ominus \frac{u_1}{v_1} \ominus \frac{u_2}{v_2}}(t_1, t_2, t_0, t_0') g(t_1, t_2)\, \Delta_1 t_1 \Delta_2 t_2 \qquad (3) \end{aligned}$$

provided the integral exists with $1 + \mu_1 \frac{u_1}{v_1} \neq 0$ and $1 + \mu_2 \frac{u_2}{v_2} \neq 0$ for all $(t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$.

Analogous to [12, Lemma 7] we have,

LEMMA 3. *If $\frac{u_1}{v_1}, \frac{u_2}{v_2} \in \mathbb{C}$ are regressive, then*

$$e^{\sigma_1,\sigma_2}_{\ominus\frac{u_1}{v_1}\ominus\frac{u_2}{v_2}}(t_1,t_2,t_0,t_0') = \frac{e_{\ominus\frac{u_1}{v_1}}(t_1,t_0)e_{\ominus\frac{u_2}{v_2}}(t_2,t_0')}{\left[1+\mu_1\left(\frac{u_1}{v_1}\right)\right]\left[1+\mu_2\left(\frac{u_2}{v_2}\right)\right]}.$$

Proof.

$$\begin{aligned}
&e^{\sigma_1\sigma_2}_{\ominus\frac{u_1}{v_1}\ominus\frac{u_2}{v_2}}(t_1,t_2,t_0,t_0')\\
&= e^{\sigma_1}_{\ominus\frac{u_1}{v_1}}(t_1,t_0)\cdot e^{\sigma_2}_{\ominus\frac{u_2}{v_2}}(t_2,t_0')\\
&= \left[1+\mu_1\left(\ominus\frac{u_1}{v_1}\right)\right]e_{\ominus\frac{u_1}{v_1}}(t_1,t_0)\left[1+\mu_2\left(\ominus\frac{u_2}{v_2}\right)\right]e_{\ominus\frac{u_2}{v_2}}(t_2,t_0')\\
&= \left[1-\mu_1\frac{\left(\frac{u_1}{v_1}\right)}{1+\mu_1\left(\frac{u_1}{v_1}\right)}\right]e_{\ominus\frac{u_1}{v_1}}(t_1,t_0)\left[1-\mu_2\frac{\left(\frac{u_2}{v_2}\right)}{1+\mu_2\left(\frac{u_2}{v_2}\right)}\right]e_{\ominus\frac{u_2}{v_2}}(t_2,t_0')\\
&= \frac{e_{\ominus\frac{u_1}{v_1}}(t_1,t_0)e_{\ominus\frac{u_2}{v_2}}(t_2,t_0')}{\left[1+\mu_1\left(\frac{u_1}{v_1}\right)\right]\left[1+\mu_2\left(\frac{u_2}{v_2}\right)\right]}. \quad \square
\end{aligned}$$

THEOREM 6. (Existence Condition) *Let $g(t_1,t_2)$ be rd-continuous function on $\mathbb{T}_1\cap[t_0,k_1]\times\mathbb{T}_2\cap[t_0',k_2]$ and $|g(t_1,t_2)|\leqslant \mathscr{M} e_{c_1\oplus c_2}(t_1,t_2,t_0,t_0')$ then the double Shehu transform of $g(t_1,t_2)$ exists for all positively regressive $\ominus\frac{u_1}{v_1}$ and $\ominus\frac{u_2}{v_2}$ provided, $\lim_{t_1\to\infty} e_{c_1\ominus\frac{u_1}{v_1}}(t_1,t_0)\to 0$ and $\lim_{t_2\to\infty} e_{c_2\ominus\frac{u_2}{v_2}}(t_2,t_0')\to 0$.*

Proof.

$$\begin{aligned}
&\left|\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}[g(t_1,t_2)]\right|\\
&= \left|\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus\frac{u_1}{v_1}\ominus\frac{u_2}{v_2}}(t_1,t_2,t_0,t_0')g(t_1,t_2)\,\Delta_1t_1\Delta_2t_2\right|\\
&\leqslant \int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus\frac{u_1}{v_1}\ominus\frac{u_2}{v_2}}(t_1,t_2,t_0,t_0')\left|g(t_1,t_2)\right|\,\Delta_1t_1\Delta_2t_2\\
&\leqslant \mathscr{M}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus\frac{u_1}{v_1}\ominus\frac{u_2}{v_2}}(t_1,t_2,t_0,t_0')\,e_{c_1\oplus c_2}(t_1,t_2,t_0,t_0')\,\Delta_1t_1\Delta_2t_2\\
&= \mathscr{M}\int_{t_0}^{\infty}\int_{t_0'}^{\infty} \frac{e_{\ominus\frac{u_1}{v_1}\ominus\frac{u_2}{v_2}}(t_1,t_2,t_0,t_0')}{\left[1+\mu_1\left(\frac{u_1}{v_1}\right)\right]\left[1+\mu_2\left(\frac{u_2}{v_2}\right)\right]}\,e_{c_1\oplus c_2}(t_1,t_2,t_0,t_0')\,\Delta_1t_1\Delta_2t_2\\
&= \mathscr{M}\int_{t_0}^{\infty}\frac{e_{c_1\ominus\frac{u_1}{v_1}}(t_1,t_0)}{\left[1+\mu_1\left(\frac{u_1}{v_1}\right)\right]}\left[\int_{t_0'}^{\infty}\frac{e_{c_2\ominus\frac{u_2}{v_2}}(t_2,t_0')}{\left[1+\mu_2\left(\frac{u_2}{v_2}\right)\right]}\,\Delta_2t_2\right]\Delta_1t_1
\end{aligned}$$

$$= \mathscr{M}\frac{1}{\left(c_1 - \frac{u_1}{v_1}\right)}\int_{t_0}^{\infty}\left(c_1 \ominus \frac{u_1}{v_1}\right)e_{c_1\ominus\frac{u_1}{v_1}}(t_1,t_0)$$
$$\times\left[\frac{1}{\left(c_2 - \frac{u_2}{v_2}\right)}\int_{t_0'}^{\infty}\left(c_2 \ominus \frac{u_2}{v_2}\right)e_{c_2\ominus\frac{u_2}{v_2}}(t_2,t_0')\,\Delta_2 t_2\right]\Delta_1 t_1$$
$$= \frac{\mathscr{M}}{\left(c_1 - \frac{u_1}{v_1}\right)\left(c_2 - \frac{u_2}{v_2}\right)}\int_{t_0}^{\infty} e^{\Delta_1^1}_{c_1\ominus\frac{u_1}{v_1}}(t_1,t_0)\left[\int_{t_0'}^{\infty} e^{\Delta_2^1}_{c_2\ominus\frac{u_2}{v_2}}(t_2,t_0')\,\Delta_2 t_2\right]\Delta_1 t_1$$
$$= \frac{\mathscr{M}}{\left(\frac{u_1}{v_1} - c_1\right)\left(\frac{u_2}{v_2} - c_2\right)}$$
$$= \frac{\mathscr{M}v_1 v_2}{(u_1 - c_1 v_1)(u_2 - c_2 v_2)}. \quad \square$$

3.1. Double Shehu transform of some elementary functions

In this subsection using Definition 5 we find transform of some elementary functions.

(1) $\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}[1] = \frac{v_1 v_2}{u_1 u_2}$, provided $\lim_{t_1\to\infty} e_{\ominus\frac{u_1}{v_1}}(t_1,t_0) \to 0$ and $\lim_{t_2\to\infty} e_{\ominus\frac{u_2}{v_2}}(t_2,t_0') \to 0$.

Proof.

$$\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}[1]$$
$$= \int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus\frac{u_1}{v_1}\ominus\frac{u_2}{v_2}}(t_1,t_2,t_0,t_0')\,\Delta_1 t_1\,\Delta_2 t_2$$
$$= \int_{t_0}^{\infty}\int_{t_0'}^{\infty}\frac{e_{\ominus\frac{u_1}{v_1}}(t_1,t_0)e_{\ominus\frac{u_2}{v_2}}(t_2,t_0')}{\left[1+\mu_1\left(\frac{u_1}{v_1}\right)\right]\left[1+\mu_2\left(\frac{u_2}{v_2}\right)\right]}\,\Delta_1 t_1\,\Delta_2 t_2$$
$$= \frac{-1}{\left(\frac{u_1}{v_1}\right)}\int_{t_0}^{\infty}\frac{-\left(\frac{u_1}{v_1}\right)}{\left[1+\mu_1\left(\frac{u_1}{v_1}\right)\right]}e_{\ominus\frac{u_1}{v_1}}(t_1,t_0)$$
$$\times\left[\frac{1}{-\left(\frac{u_2}{v_2}\right)}\int_{t_0'}^{\infty}\frac{-\left(\frac{u_2}{v_2}\right)}{\left[1+\mu_2\left(\frac{u_2}{v_2}\right)\right]}e_{\ominus\frac{u_2}{v_2}}(t_2,t_0')\,\Delta_2 t_2\right]\Delta_1 t_1$$
$$= \frac{1}{\left(\frac{u_1}{v_1}\right)\left(\frac{u_2}{v_2}\right)}\int_{t_0}^{\infty}\frac{-\left(\frac{u_1}{v_1}\right)}{\left[1+\mu_1\left(\frac{u_1}{v_1}\right)\right]}e_{\ominus\frac{u_1}{v_1}}(t_1,t_0)$$
$$\times\left[\int_{t_0'}^{\infty}\frac{-\left(\frac{u_2}{v_2}\right)}{\left[1+\mu_2\left(\frac{u_2}{v_2}\right)\right]}e_{\ominus\frac{u_2}{v_2}}(t_2,t_0')\Delta_2 t_2\right]\Delta_1 t_1$$

$$= \frac{1}{\left(\frac{u_1}{v_1}\right)\left(\frac{u_2}{v_2}\right)} \int_{t_0}^{\infty} \ominus \frac{u_1}{v_1} e_{\ominus \frac{u_1}{v_1}}(t_1,t_0)\, \Delta_1 t_1 \int_{t'_0}^{\infty} \ominus \frac{u_2}{v_2} e_{\ominus \frac{u_2}{v_2}}(t_2,t'_0)\, \Delta_2 t_2$$
$$= \frac{1}{\left(\frac{u_1}{v_1}\right)\left(\frac{u_2}{v_2}\right)} \int_{t_0}^{\infty} e^{\Delta_1^1}_{\ominus \frac{u_1}{v_1}}(t_1,t_0)\, \Delta_1 t_1 \int_{t'_0}^{\infty} e^{\Delta_2^1}_{\ominus \frac{u_2}{v_2}}(t_2,t'_0)\, \Delta_2 t_2$$
$$= \frac{v_1 v_2}{u_1 u_2}. \quad \square$$

(2) $\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}[e_{\alpha\oplus\beta}(t_1,t_2,t_0,t'_0)] = \frac{v_1 v_2}{(u_1-\alpha v_1)(u_2-\beta v_2)}$, provided $\lim_{t_1\to\infty} e_{\alpha\ominus\frac{u_1}{v_1}}(t_1,t_0) \to 0$ and $\lim_{t_1\to\infty} e_{\beta\ominus\frac{u_2}{v_2}}(t_2,t'_0) \to 0$.

Proof.

$$\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}[e_{\alpha\oplus\beta}(t_1,t_2,t_0,t'_0)]$$
$$= \int_{t_0}^{\infty}\int_{t'_0}^{\infty} e^{\sigma_1\sigma_2}_{\ominus\frac{u_1}{v_1}\ominus\frac{u_2}{v_2}}(t_1,t_2,t_0,t'_0)\, e_{\alpha\oplus\beta}(t_1,t_2,t_0,t'_0)\, \Delta_1 t_1\, \Delta_2 t_2$$
$$= \int_{t_0}^{\infty}\int_{t'_0}^{\infty} \frac{e_{\ominus\frac{u_1}{v_1}\ominus\frac{u_2}{v_2}}(t_1,t_2,t_0,t'_0)}{\left[1+\mu_1\left(\frac{u_1}{v_1}\right)\right]\left[1+\mu_2\left(\frac{u_2}{v_2}\right)\right]}\, e_{\alpha\oplus\beta}(t_1,t_2,t_0,t'_0)\, \Delta_1 t_1\, \Delta_2 t_2$$
$$= \int_{t_0}^{\infty}\int_{t'_0}^{\infty} \frac{e_{\alpha\ominus\frac{u_1}{v_1}}(t_1,t_0)\, e_{\beta\ominus\frac{u_2}{v_2}}(t_2,t'_0)}{\left[1+\mu_1\left(\frac{u_1}{v_2}\right)\right]\left[1+\mu_2\left(\frac{u_2}{v_2}\right)\right]}\, \Delta_1 t_1\, \Delta_2 t_2$$
$$= \frac{1}{\left(\alpha-\frac{u_1}{v_1}\right)} \int_{t_0}^{\infty} \frac{\left(\alpha-\frac{u_1}{v_1}\right)}{\left[1+\mu_1\left(\frac{u_1}{v_1}\right)\right]}\, e_{\alpha\ominus\frac{u_1}{v_1}}(t_1,t_0)\, \Delta_1 t_1$$
$$\times \frac{1}{\left(\beta-\frac{u_2}{v_2}\right)} \int_{t'_0}^{\infty} \frac{\left(\beta-\frac{u_2}{v_2}\right)}{\left[1+\mu_1\left(\frac{u_2}{v_2}\right)\right]} e_{\beta\ominus\frac{u_2}{v_2}}(t_2,t'_0)\, \Delta_2 t_2$$
$$= \frac{1}{\left(\alpha-\frac{u_1}{v_1}\right)}\frac{1}{\left(\beta-\frac{u_2}{v_2}\right)} \int_{t_0}^{\infty} \left(\alpha\ominus\frac{u_1}{v_1}\right) e_{\alpha\ominus\frac{u_1}{v_1}}(t_2,t_0)\, \Delta_1 t_1$$
$$\times \int_{t'_0}^{\infty} \left(\beta\ominus\frac{u_2}{v_2}\right) e_{\beta\ominus\frac{u_2}{v_2}}(t_2,t'_0)\, \Delta_2 t_2$$
$$= \frac{1}{\left(\alpha-\frac{u_1}{v_2}\right)}\frac{1}{\left(\beta-\frac{u_2}{v_2}\right)} \int_{t_0}^{\infty} e^{\Delta_1^1}_{\alpha\ominus\frac{u_1}{v_1}}(t_1,t_0)\, \Delta_1 t_1 \int_{t'_0}^{\infty} e^{\Delta_2^1}_{\beta\ominus\frac{u_2}{v_2}}(t_2,t'_0)\, \Delta_2 t_2$$
$$= \frac{1}{\left(\frac{u_1}{v_1}-\alpha\right)\left(\frac{u_2}{v_2}-\beta\right)}$$
$$= \frac{v_1 v_2}{(u_1-\alpha v_1)(u_2-\beta v_2)}. \quad \square$$

Similarly we can prove the following result,

(3) $\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}[e_{i(\alpha\oplus\beta)}(t_1,t_2,t_0,t_0')] = \frac{v_1v_2}{(u_1-i\alpha v_1)(u_2-i\beta u_2)}$.

(4) $\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}[e_{-i(\alpha\oplus\beta)}(t_1,t_2,t_0,t_0')] = \frac{v_1v_2}{(u_1+i\alpha v_1)(u_2+i\beta u_2)}$.

Using (2), (3) and (4) we get,

(5) $\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}[\sin_{\alpha\oplus\beta}(t_1,t_2,t_0,t_0')] = \frac{v_1v_2(\beta u_1v_2+\alpha u_2v_1)}{(u_1^2+\alpha^2v_1^2)(u_2^2+\beta^2v_2^2)}$.

(6) $\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}[\cos_{\alpha\oplus\beta}(t_1,t_2,t_0,t_0')] = \frac{v_1v_2(u_1u_2-\alpha\beta v_1v_2)}{(u_1^2+\alpha^2v_1^2)(v_2^2+\beta^2v_2^2)}$.

(7) $\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}[\sinh_{\alpha\oplus\beta}(t_1,t_2,t_0,t_0')] = \frac{1}{2}\left[\frac{v_1v_2}{(u_1-\alpha v_1)(u_2-\beta v_2)} - \frac{v_1v_2}{(u_1+\alpha v_1)(u_2+\beta v_2)}\right]$.

(8) $\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}[\cosh_{\alpha\oplus\beta}(t_1,t_2,t_0,t_0')] = \frac{1}{2}\left[\frac{v_1v_2}{(u_1-\alpha v_1)(u_2-\beta v_2)} + \frac{v_1v_2}{(u_1+\alpha v_1)(u_2+\beta v_2)}\right]$.

(9) The generalized polynomial is defined in [4, 5] as

$$h_0(t,s) = 1$$
$$h_{k+1}(t,s) = \int_s^t h_k(\tau,s)\Delta\tau, \qquad k\in\mathbb{N}\cup\{0\}.$$

Then for $h_n(t_1,t_0)\cdot h_m(t_2,t_0') : \mathbb{T}_1\times\mathbb{T}_2\to\mathbb{C}$

$$\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\left[h_n(t_1,t_0)\cdot h_m(t_2,t_0')\right] = \frac{1}{\left(\frac{u_1}{v_1}\right)^{n+1}}\frac{1}{\left(\frac{u_2}{v_2}\right)^{m+1}}.$$

Proof. We have $\mathscr{S}h_t\left[h_k(t,t_0)\right] = \frac{1}{\left(\frac{u_1}{v_1}\right)^{k+1}}$, then

$$\begin{aligned}
&\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\left[h_n(t_1,t_0)\cdot h_m(t_2,t_0')\right] \\
&= \int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus\frac{u_1}{v_1}\ominus\frac{u_2}{v_2}}(t_1,t_2,t_0,t_0')\, h_n(t_1,t_0)\cdot h_m(t_2,t_0')\,\Delta_1t_1\,\Delta_2t_2 \\
&= \int_{t_0}^{\infty}\int_{t_0'}^{\infty} \frac{e_{\ominus\frac{u_1}{v_1}\ominus\frac{u_2}{v_2}}(t_1,t_2,t_0,t_0')}{\left[1+\mu_1\left(\frac{u_1}{v_1}\right)\right]\left[1+\mu_2\left(\frac{u_2}{v_2}\right)\right]}\, h_n(t_1,t_0)\cdot h_m(t_2,t_0')\,\Delta_1t_1\,\Delta_2t_2 \\
&= \int_{t_0}^{\infty}\int_{t_0'}^{\infty} \frac{e_{\ominus\frac{u_1}{v_1}}(t_1,t_0)\, e_{\ominus\frac{u_2}{v_2}}(t_2,t_0')}{\left[1+\mu_1\left(\frac{u_1}{v_2}\right)\right]\left[1+\mu_2\left(\frac{u_2}{v_2}\right)\right]}\, h_n(t_1,t_0)\cdot h_m(t_2,t_0')\,\Delta_1t_1\,\Delta_2t_2 \\
&= \int_{t_0}^{\infty} \frac{e_{\ominus\frac{u_1}{v_1}}(t_1,t_0)}{\left[1+\mu_1\left(\frac{u_1}{v_1}\right)\right]}\, h_n(t_1,t_0)\,\Delta_1t_1 \int_{t_0'}^{\infty} \frac{e_{\ominus\frac{u_2}{v_2}}(t_1,t_0)}{\left[1+\mu_2\left(\frac{u_2}{v_2}\right)\right]}\, h_n(t_2,t_0')\,\Delta_2t_2
\end{aligned}$$

$$= \int_{t_0}^{\infty} e^{\sigma_1}_{\ominus \frac{u_1}{v_1}}(t_1,t_0)\, h_n(t_1,t_0)\, \Delta_1 t_1 \int_{t_0'}^{\infty} e^{\sigma_2}_{\ominus \frac{u_2}{v_2}}(t_2,t_0')\, h_m(t_2,t_0')\, \Delta_2 t_2$$
$$= \frac{1}{\left(\frac{u_1}{v_1}\right)^{n+1}} \frac{1}{\left(\frac{u_2}{v_2}\right)^{m+1}}. \quad \square$$

4. Some results

THEOREM 7. (Linearity Property) *If $g_1(t_1,t_2)$ and $g_2(t_1,t_2)$ are functions with double Shehu transforms $\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\left[g_1(t_1,t_2)\right]$ and $\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\left[g_2(t_1,t_2)\right]$ respectively. Then*

$$\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\left[a_1 g_1(t_1,t_2) + a_2 g_2(t_1,t_2)\right]$$
$$= a_1 \mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\left[g_1(t_1,t_2)\right] + a_2 \mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\left[g_2(t_2,t_2)\right].$$

Proof. Follows from Definition 5. $\square$

THEOREM 8. (Shifting Theorem) *For $\lambda_1 \in \mathbb{T}_1$ and $\lambda_2 \in \mathbb{T}_2$ with $\lambda_1, \lambda_2 > 0$, we have*

$$H_{\lambda_1,\lambda_2}(t_1,t_2) = \begin{cases} 0 & t_1 \in \mathbb{T}_1,\ t_2 \in \mathbb{T}_2\ \ \&\ \ t_1 < \lambda_1,\ t_2 < \lambda_2 \\ 1 & t_1 \in \mathbb{T}_1,\ t_2 \in \mathbb{T}_2\ \ \&\ \ t_1 \geqslant \lambda_1,\ t_2 \geqslant \lambda_2, \end{cases}$$

then

$$\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\Big[H_{\lambda_1,\lambda_2}(t_1,t_2)\ g(t_1,t_2)\Big] = e_{\ominus \frac{u_1}{v_1} \ominus \frac{u_2}{v_2}}(\lambda_1,\lambda_2,t_0,t_0')\ \mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\left[g(t_1,t_2)\right].$$

Proof. Applying Definition 5

$$\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\Big[H_{\lambda_1,\lambda_2}(t_1,t_2) g(t_1,t_2)\Big]$$
$$= \int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1\sigma_2}_{\ominus \frac{u_1}{v_1} \ominus \frac{u_2}{v_2}}(t_1,t_2,t_0,t_0')\, H_{\lambda_1,\lambda_2}(t_1,t_2) g(t_1,t_2)\, \Delta_1 t_1\, \Delta_2 t_2$$
$$= \int_{t_0}^{\infty}\int_{t_0'}^{\infty} \frac{e_{\ominus \frac{u_1}{v_1} \ominus \frac{u_2}{v_2}}(t_1,t_2,t_0,t_0')}{\left[1+\mu_1\left(\frac{u_1}{v_1}\right)\right]\left[1+\mu_2\left(\frac{u_2}{v_2}\right)\right]} H_{\lambda_1,\lambda_2}(t_1,t_2) g(t_1,t_2)\, \Delta_1 t_1\, \Delta_2 t_2$$
$$= \int_{\lambda_1}^{\infty}\int_{\lambda_2}^{\infty} \frac{e_{\ominus \frac{u_1}{v_1} \ominus \frac{u_2}{v_2}}(\lambda_1,\lambda_2,t_0,t_0') e_{\ominus \frac{u_1}{v_1} \ominus \frac{u_2}{v_2}}(t_1,t_2,\lambda_1,\lambda_2)}{\left[1+\mu_1\left(\frac{u_1}{v_1}\right)\right]\left[1+\mu_2\left(\frac{u_2}{v_2}\right)\right]} g(t_1,t_2)\, \Delta_1 t_1\, \Delta_2 t_2$$
$$= e_{\ominus \frac{u_1}{v_1} \ominus \frac{u_2}{v_2}}(\lambda_1,\lambda_2,t_0,t_0') \int_{\lambda_1}^{\infty}\int_{\lambda_2}^{\infty} \frac{e_{\ominus \frac{u_1}{v_1} \ominus \frac{u_2}{v_2}}(t_1,t_2,\lambda_1,\lambda_2)}{\left[1+\mu_1\left(\frac{u_1}{v_1}\right)\right]\left[1+\mu_2\left(\frac{u_2}{v_2}\right)\right]} g(t_1,t_2) \Delta_1 t_1 \Delta_2 t_2$$
$$= e_{\ominus \frac{u_1}{v_1} \ominus \frac{u_2}{v_2}}(\lambda_1,\lambda_2,t_0,t_0') \int_{\lambda_1}^{\infty}\int_{\lambda_2}^{\infty} e^{\sigma_1,\sigma_2}_{\ominus \frac{u_1}{v_1} \ominus \frac{u_2}{v_2}}(t_1,t_2,\lambda_1,\lambda_2) g(t_1,t_2)\, \Delta_1 t_1\, \Delta_2 t_2$$
$$= e_{\ominus \frac{u_1}{v_1} \ominus \frac{u_2}{v_2}}(\lambda_1,\lambda_2,t_0,t_0') \mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\left[g(t_1,t_2)\right]. \quad \square$$

THEOREM 9. (Transform of partial derivatives) *Let* $g(t_1,t_2):\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{C}$ *is an rd-continuous function such that*

$$g^{\Delta_1^1}(t_1,t_2)=\frac{\partial g(t_1,t_2)}{\Delta_1 t_1},\quad g^{\Delta_1^2}(t_1,t_2)=\frac{\partial^2 g(t_1,t_2)}{\Delta_1 t_1^2},\quad g^{\Delta_2^1}(t_1,t_2)=\frac{\partial g(t_1,t_2)}{\Delta_2 t_2}$$
$$g^{\Delta_2^2}(t_1,t_2)=\frac{\partial^2 g(t_1,t_2)}{\Delta_2 t_2^2},\quad g^{\Delta_1^1\Delta_2^1}(t_1,t_2)=\frac{\partial^2 g(t_1,t_2)}{\Delta t_1\Delta t_2}$$

are also rd-continuous, then

(i) $\mathscr{S}h_{t_2}\mathscr{S}h_{t_1}\left[g^{\Delta_1^1}(t_1,t_2)\right]=\frac{u_1}{v_1}\,\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\left[g(t_1,t_2)\right]-\mathscr{S}h_{t_2}\left[g(t_0,t_2)\right],$

(ii) $\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\left[g^{\Delta_2^1}(t_1,t_2)\right]=\frac{u_2}{v_2}\,\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\left[g(t_1,t_2)\right]-\mathscr{S}h_{t_1}\left[g(t_1,t_0')\right],$

(iii) $$\mathscr{S}h_{t_2}\mathscr{S}h_{t_1}\left[g^{\Delta_1^2}(t_1,t_2)\right]=\left(\tfrac{u_1}{v_1}\right)^2\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\left[g(t_1,t_2)\right]-\left(\tfrac{u_1}{v_1}\right)\mathscr{S}h_{t_2}\left[g(t_0,t_2)\right]-\mathscr{S}h_{t_2}\left[g^{\Delta_1^1}(t_0,t_2)\right],$$

(iv) $$\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\left[g^{\Delta_2^2}(t_1,t_2)\right]=\left(\tfrac{u_2}{v_2}\right)^2\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\left[g(t_1,t_2)\right]-\left(\tfrac{u_2}{v_2}\right)\mathscr{S}h_{t_1}\left[g(t_1,t_0')\right]-\mathscr{S}h_{t_1}\left[g^{\Delta_2^1}(t_1,t_0')\right],$$

(v) $$\mathscr{S}h_{t_2}\mathscr{S}h_{t_1}\left[g^{\Delta_1^1\Delta_2^1}(t_1,t_2)\right]=\left(\frac{u_1}{v_1}\frac{u_2}{v_2}\right)\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\left[g(t_1,t_2)\right]-\left(\frac{u_1}{v_1}\right)\mathscr{S}h_{t_1}\left[g(t_1,t_0')\right]-\left(\frac{u_2}{v_2}\right)\mathscr{S}h_{t_2}\left[g(t_0,t_2)\right]-g(t_0,t_0')$$

provided $\lim_{t_1\to\infty}e_{\ominus\frac{u_1}{v_1}}(t_1,t_0)\to 0$ *and* $\lim_{t_2\to\infty}e_{\ominus\frac{u_2}{v_2}}(t_2,t_0')\to 0.$

Proof.

(i) For $g^{\Delta_1^1}(t_1,t_2)$ taking single Shehu transform with respect to t_1 and using the product rule, we have

$$\begin{aligned}
\mathscr{S}h_{t_1}\left[g^{\Delta_1^1}(t_1,t_2)\right]&=\int_{t_0}^{\infty}e^{\sigma_1}_{\ominus\frac{u_1}{v_1}}(t_1,t_0)\,g^{\Delta_1^1}(t_1,t_2)\,\Delta_1 t_1\\
&=\int_{t_0}^{\infty}\left[\left(e_{\ominus\frac{u_1}{v_1}}(t_1,t_0)g(t_1,t_2)\right)^{\Delta_1^1}-g(t_1,t_2)e^{\Delta_1^1}_{\ominus\frac{u_1}{v_1}}(t_1,t_0)\right]\Delta_1 t_1\\
&=-g(t_0,t_2)-\int_{t_0}^{\infty}\ominus\frac{u_1}{v_1}\,e_{\ominus\frac{u_1}{v_1}}(t_1,t_0)\,g(t_1,t_2)\,\Delta_1 t_1\\
&=-g(t_0,t_2)+\frac{u_1}{v_1}\int_{t_0}^{\infty}e^{\sigma_1}_{\ominus\frac{u_1}{v_1}}(t_1,t_0)\,g(t_1,t_2)\,\Delta_1 t_1\\
&=-g(t_0,t_2)+\frac{u_1}{v_1}\,\mathscr{S}h_{t_1}\left[g(t_1,t_2)\right]\\
&=\frac{u_1}{v_1}\,\mathscr{S}h_{t_1}\left[g(t_1,t_2)\right]-g(t_0,t_2)\\
\mathscr{S}h_{t_1}\left[g^{\Delta_1^1}(t_1,t_2)\right]&=\frac{u_1}{v_1}\,\mathscr{S}h_{t_1}\left[g(t_1,t_2)\right]-g(t_0,t_2).
\end{aligned}$$

Now taking Shehu transform with respect to t_2 we get

$$
\begin{aligned}
\mathscr{S}h_{t_2}\mathscr{S}h_{t_1}\left[g^{\Delta_1^1}(t_1,t_2)\right] &= \int_{t_0'}^{\infty} e^{\sigma_2}_{\ominus\frac{u_2}{v_2}}(t_2,t_0')\,\frac{u_1}{v_1}\,\mathscr{S}h_{t_1}\left[g(t_1,t_2)\right]\Delta_2 t_2 \\
&\quad -\int_{t_0'}^{\infty} e^{\sigma_2}_{\ominus\frac{u_2}{v_2}}(t_2,t_0')\,g(t_0,t_2)\,\Delta_2 t_2 \\
&= \frac{u_1}{v_1}\int_{t_0'}^{\infty}\int_{t_0}^{\infty} e^{\sigma_1\sigma_2}_{\ominus\frac{u_1}{v_1}\ominus\frac{u_2}{v_2}}(t_1,t_2,t_0,t_0')\,g(t_1,t_2)\,\Delta_1 t_1\,\Delta_2 t_2 \\
&\quad -\int_{t_0'}^{\infty} e^{\sigma_2}_{\ominus\frac{u_2}{v_2}}(t_2,t_0')\,g(t_0,t_2)\,\Delta_2 t_2 \\
\mathscr{S}h_{t_2}\mathscr{S}h_{t_1}\left[g^{\Delta_1^1}(t_1,t_2)\right] &= \frac{u_1}{v_1}\,\mathscr{S}h_{t_2}\mathscr{S}h_{t_1}\left[g(t_1,t_2)\right]-\mathscr{S}h_{t_2}\left[g(t_0,t_2)\right].
\end{aligned}
$$

(ii), (iii), (iv) and (v) can be proved similarly. □

Above theorem can be generalized as follows, let $g(t_1,t_2):\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{C}$ is rd-continuous function such that $g^{\Delta_1^i}$ and $g^{\Delta_2^j}$ are also rd-continuous for $i=1,2,\ldots,n$ and $j=1,2,\ldots,m$ respectively. Then,

(i) for $g^{\Delta_1^n}(t_1,t_2)=\frac{\partial^n g(t_1,t_2)}{\Delta_1 t_1^n}$

$$
\begin{aligned}
\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\left[g^{\Delta_1^n}(t_1,t_2)\right] &= \left(\frac{u_1}{v_1}\right)^n \mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\left[g(t_1,t_2)\right]-\left(\frac{u_1}{v_1}\right)^{n-1}\mathscr{S}h_{t_2}\left[g(t_0,t_2)\right] \\
&\quad -\sum_{i=1}^{n-1}\left(\frac{u_1}{v_1}\right)^{n-1-i}\mathscr{S}h_{t_2}\left[g^{\Delta_1^i}(t_0,t_2)\right],
\end{aligned}
$$

(ii) for $g^{\Delta_2^m}(t_1,t_2)=\frac{\partial^m g(t_1,t_2)}{\Delta_2 t_2^m}$

$$
\begin{aligned}
\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\left[g^{\Delta_2^m}(t_1,t_2)\right] &= \left(\frac{u_2}{v_2}\right)^m \mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\left[g(t_1,t_2)\right]-\left(\frac{u_2}{v_2}\right)^{m-1}\mathscr{S}h_{t_1}\left[g(t_1,t_0')\right] \\
&\quad -\sum_{j=1}^{m-1}\left(\frac{u_2}{v_2}\right)^{m-1-j}\mathscr{S}h_{t_1}\left[g^{\Delta_2^j}(t_1,t_0')\right].
\end{aligned}
$$

The proof follows from induction.

THEOREM 10. (Transform of integrals) *If* $g(t_1,t_2):\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{C}$ *is regulated then*

$$
\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\left[\int_{t_0}^{t_1}\int_{t_0'}^{t_2} g(\gamma_1,\gamma_2)\,\Delta_1\gamma_1\,\Delta_2\gamma_2\right]=\frac{v_1v_2}{u_1u_2}\,\mathscr{S}h_1\mathscr{S}h_2[g(t_1,t_2)].
$$

Proof. Let

$$
G(t_1,t_2)=\int_{t_0}^{t_1}\int_{t_0'}^{t_2} g(\gamma_1,\gamma_2)\;\Delta_1\gamma_1\,\Delta_2\gamma_2.
$$

Hence we have, $G^{\Delta_1^1\Delta_2^1}(t_1,t_2)=g(t_1,t_2)$ with $G(t_0,t_0')=G(t_1,t_0')=G(t_0,t_2)=0$.

Applying double Shehu transform and using (v) from Theorem 9, we get

$$\mathscr{S}h_{t_1}\mathscr{S}h_{t_1}\left[G^{\Delta_1^1\Delta_2^1}(t_1,t_1)\right]=\left(\frac{u_1}{v_1}\frac{u_2}{v_2}\right)\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\Big[G(t_1,t_2)\Big]-\left(\frac{u_1}{v_1}\right)\mathscr{S}h_{t_1}\Big[G(t_1,t_0')\Big]$$
$$-\left(\frac{u_2}{v_2}\right)\mathscr{S}h_{t_2}\Big[G(t_0,t_2)\Big]-G(t_0,t_0').$$

Thus,

$$\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\left[\int_{t_0}^{t_1}\int_{t_0'}^{t_2}g(\gamma_1,\gamma_2)\,\Delta_1\gamma_1\,\Delta_2\gamma_2\right]=\frac{v_1v_2}{u_1u_2}\,\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\big[g(t_1,t_1)\big].\quad\Box$$

5. Convolution property

In this section we will prove the convolution theorem for double Shehu transform. Using [8, Definition 2.1] of delay of a function $x:\mathbb{T}\to\mathbb{R}$ by $\sigma(\tau)\in\mathbb{T}$, denoted by $x(t,\sigma(\tau))$ the double convolution is defined in [12] as follows.

DEFINITION 6. Let $g:\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{C}$ is rd-continuous and $h:\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{C}$ is piecewise rd-continuous function of exponential type II, then the double convolution of g and h denoted by $g**h$ is given by,

$$(g**h)(t_1,t_2)=\int_{t_0}^{t_1}\int_{t_0'}^{t_2}g(s_1,s_2)h(t_1,t_2,\sigma_1(s_1),\sigma_2(s_2))\Delta_1s_1\Delta_2s_2,$$

where $h(t_1,t_2,\sigma_1(s_1),\sigma_2(s_2))$ is delay of $h(t_1,t_2)$ by $\sigma_1(s_1)\in\mathbb{T}_1$ and $\sigma_2(s_2)\in\mathbb{T}_2$.

THEOREM 11. (Convolution theorem for double Shehu transform) *Let $g_1(t_1,t_2):\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{C}$ and $g_2(t_1,t_2):\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{C}$ are rd-continuous functions of exponential type II having double Shehu transforms $\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\big[g_1(t_1,t_2)\big]$ and $\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\big[g_2(t_1,t_2)\big]$ respectively, then*

$$\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\big[\big(g_1**g_2\big)(t_1,t_2)\big]=\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\big[g_1(t_1,t_2)\big]\cdot\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\big[g_2(t_1,t_2)\big].$$

Proof. Applying Definition 5 to convolution of g_1 and g_2, we get

$$\begin{aligned}
&\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\big[\big(g_1**g_2\big)(t_1,t_2)\big]\\
&=\int_{t_0}^{\infty}\int_{t_0'}^{\infty}e^{\sigma_1\sigma_2}_{\ominus\frac{u_1}{v_1}\ominus\frac{u_2}{v_2}}(t_1,t_2,t_0,t_0')\big[\big(g_1**g_2\big)(t_1,t_2)\big]\,\Delta_1t_1\,\Delta_2t_2\\
&=\int_{t_0}^{\infty}\int_{t_0'}^{\infty}e^{\sigma_1\sigma_2}_{\ominus\frac{u_1}{v_1}\ominus\frac{u_2}{v_2}}(t_1,t_2,t_0,t_0')\bigg[\int_{t_0}^{t_1}\int_{t_0'}^{t_2}g_1(t_1,t_2,\sigma_1(s_1),\sigma_2(s_2))\\
&\qquad\qquad\times g_2(s_1,s_2)\,\Delta_1s_1\,\Delta_2s_2\bigg]\,\Delta_1t_1\,\Delta_2t_2\\
&=\int_{t_0}^{\infty}\int_{t_0'}^{\infty}g_2(s_1,s_2)\bigg[\int_{\sigma_1(s_1)}^{\infty}\int_{\sigma_2(s_2)}^{\infty}g_1(t_1,t_2,\sigma_1(s_1),\sigma_2(s_2))\\
&\qquad\qquad\times e^{\sigma_1\sigma_2}_{\ominus\frac{u_1}{v_1}\ominus\frac{u_2}{v_2}}(t_1,t_2,t_0,t_0')\,\Delta_1t_1\,\Delta_2t_2\bigg]\,\Delta_1s_1\,\Delta_2s_2
\end{aligned}$$

$$= \int_{t_0}^{\infty}\int_{t_0'}^{\infty} g_2(s_1,s_2)\Big[\int_{t_0}^{\infty}\int_{t_0'}^{\infty} g_1(t_1,t_2,\sigma_1(s_1),\sigma_2(s_2))\ H_{\sigma_1(s_1),\sigma_2(s_2)}(t_1,t_2)$$
$$\times e^{\sigma_1\sigma_2}_{\ominus\frac{u_1}{v_1}\ominus\frac{u_2}{v_2}}(t_1,t_2,t_0,t_0')\ \Delta_1 t_1\ \Delta_2 t_2\Big]\ \Delta_1 s_1\ \Delta_2 s_2$$
$$= \int_{t_0}^{\infty}\int_{t_0'}^{\infty} g_2(s_1,s_2)\ \mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\Big[H_{\sigma_1(s_1),\sigma_2(s_2)}(t_1,t_2)\ g_1(t_1,t_2,\sigma_1(s_1),\sigma_2(s_2))\Big]\ \Delta_1 s_1\ \Delta_2 s_2$$
$$= \int_{t_0}^{\infty}\int_{t_0'}^{\infty} g_2(s_1,s_2)\ e_{\ominus\frac{u_1}{v_1}\ominus\frac{u_2}{v_2}}(\sigma_1(s_1),\sigma_2(s_2),t_0,t_0')\ \mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\ \big[g_1(t_1,t_2)\big]\ \Delta_1 s_1\ \Delta_2 s_2$$
$$= \mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\big[g_1(t_1,t_2)\big]\ \int_{t_0}^{\infty}\int_{t_0'}^{\infty} e^{\sigma_1,\sigma_2}_{\ominus\frac{u_1}{v_1}\ominus\frac{u_2}{v_2}}(s_1,s_2,t_0,t_0')\ g_2(s_1,s_2)\ \Delta_1 s_1\ \Delta_2 s_2$$
$$= \mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\big[g_1(t_1,t_2)\big]\ \mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\big[g_2(t_1,t_2)\big]. \quad \square$$

6. Applications

Now we are ready to give applications of double Shehu transform for solving some integro and partial dynamic equations through following examples. Here we have used the result, for $f:\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{C}$, $\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}[f(t_1,t_2)] = \mathscr{S}h_{t_2}\mathscr{S}h_{t_1}[f(t_1,t_2)]$ which follows from Definition 5 and proof of Lemma 3.

EXAMPLE 1. Solve the following integro-dynamic equation for $\mathbb{T}_1\times\mathbb{T}_2$ such that $0\in\mathbb{T}_1$ and $0\in\mathbb{T}_2$.

$$\begin{aligned}&\frac{\partial h(t_1,t_2)}{\Delta_1 t_1}+\frac{\partial h(t_1,t_2)}{\Delta_2 t_2}+1-e_1(t_1,0)-e_1(t_2,0)-e_{1\oplus 1}(t_1,t_2,0,0)\\&=\int_0^{t_1}\int_0^{t_2} h(s_1,s_2)\ \Delta_1 s_1\ \Delta_2 s_2,\end{aligned}\tag{4}$$

with initial conditions $h(t_1,0)=e_1(t_1,0),\ \ h(0,t_2)=e_1(t_2,0)$.

SOL. In order to solve above integro-dynamic equation we need to find $h(t_1,t_2)$. Taking double Shehu transform on both sides of given equation.

$$\begin{aligned}&\left(\frac{u_1}{v_1}\right)\mathscr{S}h_{t_2}\mathscr{S}h_{t_1}\big[h(t_1,t_2)\big]-\mathscr{S}h_{t_2}\big[h(0,t_2)\big]\ +\left(\frac{u_2}{v_2}\right)\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\big[h(t_1,t_2)\big]\\&\quad-\mathscr{S}h_{t_1}\big[h(t_1,0)\big]+\left(\frac{u_1}{v_1}\right)\left(\frac{u_2}{v_2}\right)-\frac{v_1v_2}{(u_1-v_1)u_2}-\frac{v_1v_2}{(u_2-v_2)u_1}-\frac{v_1v_2}{(u_1-v_1)(u_2-v_2)}\\&=\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\big[(h**1)(t_1,t_2)\big].\end{aligned}$$

Taking single Shehu transform of the initial conditions

$$\mathscr{S}h_{t_1}\big[h(t_1,0)\big]=\frac{v_1}{u_1-v_1},\quad \mathscr{S}h_{t_2}\big[h(0,t_2)\big]=\frac{v_2}{u_2-v_2}.$$

Substituting this into above equation we get

$$\begin{aligned}&\left(\frac{u_1}{v_1}\right)\mathscr{S}h_{t_2}\mathscr{S}h_{t_1}\big[h(t_1,t_2)\big] - \frac{v_2}{(u_2-v_2)} + \left(\frac{u_2}{v_2}\right)\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\big[h(t_1,t_2)\big]\\ &\quad - \frac{v_1}{(u_1-v_1)} - \left(\frac{v_1v_2}{u_1u_2}\right)\\ &\quad - \frac{v_1v_2}{(u_1-v_1)u_2} - \frac{v_1v_2}{(u_2-v_2)u_1} - \frac{v_1v_2}{(u_2-v_2)(u_1-v_1)}\\ &= \left(\frac{v_1v_2}{u_1u_2}\right)\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\big[h(t_1,t_2)\big].\end{aligned}$$

On simplifying we get

$$\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\big[h(t_1,t_2)\big] = \frac{v_1v_2}{(u_1-v_1)(u_2-v_2)}.$$

Taking inverse Shehu transform we get

$$\begin{aligned}h(t_1,t_2) &= e_{1\oplus 1}(t_1,t_2,0,0)\\ &= e_1(t_1,0)e_1(t_2,0)\end{aligned} \tag{5}$$

is requied solution.

REMARK. When $\mathbb{T}_1\times\mathbb{T}_2 = \mathbb{R}\times\mathbb{R}$ then Equation 4 becomes

$$\frac{\partial}{\partial t_1}h(t_1,t_2) + \frac{\partial}{\partial t_2}h(t_1,t_2) + 1 - e^{t_1} - e^{t_2} - e^{t_1+t_2} = \int_0^{t_1}\int_0^{t_2} h(s_1,s_2)ds_1ds_2$$

with initial conditions $h(t_1,0) = e^{t_1},\ h(t_2,0) = e^{t_2}$. And then from Equation 5 its solution is $h(t_1,t_2) = e^{t_1+t_2}$.

EXAMPLE 2. Consider the following partial dynamic equation for $\mathbb{T}_1\times\mathbb{T}_2$ such that $0\in\mathbb{T}_1,\ 0\in\mathbb{T}_2$ with for all $t_1\in\mathbb{T}_1,\ \mu(t_1)\neq 1$.

$$\frac{\partial^3 f(t_1,t_2)}{\Delta_1 t_1^3} + \frac{\partial f(t_1,t_2)}{\Delta_1 t_1} + \frac{\partial f(t_1,t_2)}{\Delta_2 t_2} = 0, \tag{6}$$

with initial and boundary conditions

$$\begin{aligned}f(t_1,0) &= e_{-1}(t_1,0), \quad f(0,t_2) = e_2(t_2,0),\\ \frac{\partial f(0,t_2)}{\Delta_1 t_1} &= -e_2(t_2,0), \quad \frac{\partial^2 f(0,t_2)}{\Delta_1 t_1^2} = e_2(t_2,0).\end{aligned}$$

In order to solve above partial dynamic equation we need to find $f(t_1,t_2)$. Applying double Shehu transform to given equation.

$$\begin{aligned}&\left(\frac{u_1}{v_1}\right)^3\mathscr{S}h_{t_2}\mathscr{S}h_{t_1}\Big[f(t_1,t_2)\Big] - \left(\frac{u_1}{v_1}\right)^2\mathscr{S}h_{t_2}\Big[f(0,t_2)\Big] - \left(\frac{u_1}{v_1}\right)\mathscr{S}h_{t_2}\left[\frac{\partial f(0,t_2)}{\Delta_1 t_1}\right]\\ &- \mathscr{S}h_{t_2}\left[\frac{\partial^2 f(0,t_2)}{\Delta_1 t_1^2}\right] + \left(\frac{u_1}{v_1}\right)\mathscr{S}h_{t_2}\mathscr{S}h_{t_1}\Big[f(t_1,t_2)\Big] - \mathscr{S}h_{t_2}\Big[f(0,t_2)\Big]\\ &+ \left(\frac{u_2}{v_2}\right)\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\Big[f(t_1,t_2)\Big] - \mathscr{S}h_{t_1}\Big[f(t_1,0)\Big] = 0.\end{aligned}$$

Taking single Shehu transform of the initial and boundary conditions.

$$\mathscr{S}h_{t_1}\left[f(t_1,0)\right] = \frac{v_1}{u_1+v_1}, \quad \mathscr{S}h_{t_2}\left[f(0,t_2)\right] = \frac{v_2}{u_2-2v_2},$$

$$\mathscr{S}h_{t_2}\left[\frac{\partial f(0,t_2)}{\Delta_1 t_1}\right] = \frac{-v_2}{u_2-2v_2}, \quad \mathscr{S}h_{t_2}\left[\frac{\partial^2 f(0,t_2)}{\Delta_1 t_1^2}\right] = \frac{v_2}{u_2-2v_2}.$$

Substituting into above equation

$$\begin{aligned}\left[\left(\frac{u_1}{v_1}\right)^3 + \left(\frac{u_1}{v_1}\right) + \left(\frac{u_2}{v_2}\right)\right]&\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\left[f(t_1,t_2)\right] \\ = \left(\frac{u_1}{v_1}\right)^2&\left(\frac{v_2}{u_2-2v_2}\right) - \left(\frac{u_1}{v_1}\right)\left(\frac{v_2}{u_2-2v_2}\right) \\ &+ \left(\frac{v_2}{u_2-2v_2}\right) + \left(\frac{v_2}{u_2-2v_2}\right) + \left(\frac{v_1}{u_1+v_1}\right).\end{aligned}$$

Simplifying we get

$$\mathscr{S}h_{t_1}\mathscr{S}h_{t_2}\left[f(t_1,t_2)\right] = \left[\left(\frac{v_1}{u_1+v_1}\right)\left(\frac{v_2}{u_2-2v_2}\right)\right].$$

Taking inverse Shehu transform we get

$$\begin{aligned} f(t_1,t_2) &= e_{-1\oplus 2}(t_1,t_2,0,0) \\ &= e_{-1}(t_1,0)e_2(t_2,0) \end{aligned} \tag{7}$$

is required solution.

REMARK. When $\mathbb{T}_1 \times \mathbb{T}_2 = \mathbb{R} \times \mathbb{R}$ then Equation 6 becomes

$$\frac{\partial^3 f(t_1,t_2)}{\partial t_1^3} + \frac{\partial f(t_1,t_2)}{\partial t_1} + \frac{\partial f(t_1,t_2)}{\partial t_2} = 0$$

with initial and boundary conditions

$$f(t_1,0) = e^{-t_1}, \quad f(0,t_2) = e^{2t_2}, \quad \frac{\partial f(0,t_2)}{\partial t_1} = -e^{2t_2}, \quad \frac{\partial^2 f(0,t_2)}{\partial t_1^2} = e^{2t_2}.$$

And then from Equation 7 its solution is $f(t_1,t_2) = e^{-t_1+2t_2}$.

7. Conclusion

We generalized the double Shehu transform on time scales. Existence conditions are given coupled with remarkable properties such as shifting theorem, transform of integral, and transform of partial derivatives. The convolution theorem is proved. Eventually an integro-dynamic and a partial dynamic equations are solved to exhibit the significance of the proposed double transform.

Researchers can develop novel and enhanced methods to solve systems of partial dynamic equations with given initial conditions for the existence of solutions for partial dynamic equations involving both delta and nabla derivatives.

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REFERENCES

[1] R. Agarwal, M. Bohner, D. O'Regan and A. Peterson, *Dynamic equations on time scales: a survey*, J. Comput. Appl. Math., **141**, (2002), 1–26.
[2] H. Agwa, A. Kilicman and F. A. Ali, *A new integral transform on time scales and its applications*, Advances in Difference Equations, **2012**, 60, (2012), 1–14.
[3] S. Alfaqeih, E. Misirli, *On double Shehu transform and its properties with applications*, Int. J. Anal. Appl., **18**, 3, (2020), 381–395.
[4] M. Bohnar, A. Peterson, *Laplace transform and Z-transform : unification and extension*, Methods Appl. Anal., **9**, 1, (2002), 155–158.
[5] M. Bohner, A. Peterson, *Dynamic equations on time scales an introduction with applications*, Birkhauser, Boston, (2002).
[6] M. Bohner, G. Sh. Guseinov, *The convolution on time scales*, Abstr. Appl. Anal., **2007**, (2007), 1–24.
[7] T. Cuchta, S. Georgiev, *Analysis of the bilateral Laplace transform on time scales with applications*, Int. J. Dyn. Syst. Differ. Equ., **11**, 3–4, (2021), 255–274.
[8] J. M. Davis, I. A. Gravagne, B. J. Jackson, R. J. Marks II, and A. A. Romas, *The Laplace transform on time scales revisited*, J. Math. Anal. Appl., **332**, (2007), 1291–1307.
[9] J. M. Davis, I. A. Gravagne, and R. J. Marks II, *Bilateral Laplace transforms on time scales: convergence, convolution, and the characterization of stationary stochastic time series*, Circuits, Systems and Signal Process., **29**, 6, (2010), 1141–1165.
[10] J. M. Davis, I. A. Gravagne, and R. J. Marks II, *Time scale discrete Fourier transforms*, Proceedings of the Southeastern Symposium on System Theory, **42**, (2010), 102–110.
[11] H. Eltayeb, A. Kilicman, *On double Sumudu transform and double Laplace transform*, Malays. J. Math. Sci., **4**, 1, (2010), 17–30.
[12] H. Eltayeb, A. Kilicman, and B. Fisher, *On multiple convolutions and time scales*, Math. Prob. Engrg., **2013**, (2013), 1–7.
[13] S. Hilger, *Special functions, Laplace and Fourier transform on measure chains*, Dynam. Systems. Appl., **8**, (1999), 471–488.
[14] R. J. Marks II, I. A. Gravagne, and J. M. Davis, *A generalized Fourier transform and convolution on time scales*, J. Math. Anal. Appl., **340**, 2, (2008), 901–919.
[15] S. Maitama, W. Zhao, *New integral transform : Shehu transform a generalization of Sumudu and Laplace transform for solving differential equations*, Inter. J. Anal. Appl., **17**, 2, (2019), 167–190.
[16] J. M. Tchuenche, N. S. Mbare, *An application of the double Sumudu transform*, Appl. Math. Sci., **1**, 1, (2007), 31–39.
[17] S. Hilger, *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*, Ph.D thesis, Universität Würzburg, (1988).
[18] T. G. Thange, A. R. Gade, *Fractional Shehu transform and its applications*, South East Asian J. Math. Math. Sci., **17**, 2, (2021), 1–14.

[19] T. G. THANGE, A. R. GADE, *Laplace-Carson transform of fractional order*, Malaya Journal of Matematik, **8**, 4, (2020), 2253–2258.

[20] T. G. THANGE, S. S. GANGANE, *On Henstock-Kruzweil Sumudu transform*, Stochastic Model. Appl., **25**, 2, (2021), 41–47.

[21] T. G. THANGE, A. M. ALURE, *Generalized Shehu transform*, J. Math. Comp. Sci., **12**, 27, (2022), 12–27.

[22] T. G. THANGE, *On extended fractional Fourier transform*, International Journal of Mathematical Sciences and Engineering Applications, **11**, 1, (2017), 143–149.

[23] T. G. THANGE, *On multiple Laguerre transform in two variables*, International Journal of Mathematical Sciences and Engineering Applications, **11**, 1, (2017), 133–144.

[24] T. G. THANGE, *New integral transform using Curzon's integral*, International Electronic Journal of Pure and Applied Mathematics, **1**, 1, (2010), 81–84.

[25] T. G. THANGE, S. CHHATRABAND, *Generalization of Shehu transform on time scales with applications*, Paper presentated in International Conference On Analysis And Applied Mathematics, Tamil Nadu, India, (2022)

[26] T. G. THANGE, S. M. CHHATRABAND, *Laplace-Sumudu integral transform on time scales*, South East Asian J. Math. Math. Sci., **19**, 1, (2023), 91–108.

[27] T. G. THANGE, S. M. CHHATRABAND, *A New α-Laplace transform on time scales*, Jnanabha, **53**, 2, (2024), 151–160.

[28] T. G. THANGE, S. M. CHHATRABAND, *On Nabla Shahe transform and its applications*, Journal of Fractional Calculus and Applications, **15**, 1, (2024), 1–13.

[29] T. G. THANGE, S. M. CHHATRABAND, *New general integral transform on time scales*, J. Math. Model., **12**, 4, (2024), 655–669.

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