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FOUR DIMENSIONAL MATRIX SUMMABILITY OF DOUBLE SEQUENCE OF SETS WITH RESPECT TO MODULUS FUNCTION

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Abstract. In this article, we have extended the idea of strong Cesaro summability to the idea of strong T-summability of double sequence of closed sets with respect to modulus function, when T is non-negative regular four dimensional matrix summability method. We also show that strongly T-summable sequence of closed set is T-statistically convergent and that T-statistically convergence is equivalent to strong T-summability with respect to modulus function on bounded sequence of closed sets.

1. Introduction

Strong summability first appeared in the paper by Hardy and Littlewood ([15]) which improved Fejer's theorem on the Cesaro convergence of a Fourier series and the strong summability of Fourier series ([17]) that continues to be an active area of research.

The notion of convergence for double sequences was presented by A. Pringsheim in ([25]). The four dimensional matrix transformation $(Ax)_{m,n} = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{m,n,k,l}x_{k,l}$ (where $x = (x_{k,l})$) was widely studied by Hamilton and Robison in ([13]) and ([26]). In this paper, we consider the four dimensional matrices and the double set sequences with real-valued entries unless stated otherwise.

Throughout the paper $\mathbb{N}$ denotes the set of positive integers. Let Ω denote the set of all double sequences of complex numbers. By convergence of a double sequence we shall mean the convergence in Pringsheim sense, that is, a double sequence $x = (x_{ij})$ is said to be Pringsheim convergent (or briefly P-convergent) if for a given $\varepsilon > 0$, there exist $N \in \mathbb{N}$ such that $\mid x_{ij} - L \mid < \varepsilon$, whenever $i, j > N$. The number L is called *Pringsheim limit* and is denoted by $P - \lim x = L$

A double sequence $x = (x_{ij})$ is called bounded if there exists a positive number M such that $|x_{ij}| \leqslant M$ for all $(i, j) \in \mathbb{N}\times\mathbb{N}$.

Recall that a modulus function ([3], [29]) f is a function from $[0,\infty)$ to $[0,\infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$;
(ii) $f(x+y) \leqslant f(x) + f(y)$;
(iii) f is increasing;
(iv) f is continuous from the right at 0.

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A modulus function may be bounded or unbounded.

For example $f(x) = \dfrac{x}{x+1}$ is a bounded modulus function whereas $f(x) = x^p$ $(0 < p \leqslant 1)$ is unbounded.

Using modulus function, Maddox ([18]) define the following spaces:

$$w_o(f) = \left\{ x \in S : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(|x_k|) = 0 \right\} \tag{I}$$

$$w(f) = \left\{ x \in S : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(|x_k - L|) = 0, \quad \text{for some} \quad L \right\} \tag{II}$$

$$w_\infty(f) = \left\{ x \in S : \sup_n \frac{1}{n} \sum_{k=1}^{n} f(|x_k|) < \infty \right\} \tag{III}$$

where S is the space of all complex sequences.

Let us now remind A-statistical convergence and A-summability of double sequence.

Let $A = (a_{m,n,i,j})$ be a four-dimensional summability matrix. For a given double sequence $x = (x_{ij})$, the A-transform of x, denoted by $Ax = ((Ax)_{m,n})$, is given by

$$(Ax)_{m,n} = \sum\nolimits_{(i,j)\in\mathbb{N}\times\mathbb{N}} a_{m,n,i,j} x_{ij}$$

provided the double series converges in Pringsheim's sense for every $m,n \in \mathbb{N}$.

A two dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. The well-known characterization for two dimensional matrix transformations which are regular is known as Silverman-Toeplitz conditions ([14]). By considering an additional assumption of boundedness, Robinson ([26]) presented a four-dimensional analog of regularity. This was made because a double P-convergent sequence is not necessarily bounded. The characterization and the definition of regularity of four dimensional matrices is known as Robinson-Hamilton condition or RH-regularity ([13], [26]). The reader also may refer to the recent monographs ([1]) and ([2]), and references therein, devoted to summability theory and the spaces of double sequences generated by some four dimensional triangle matrices with a new approach.

Recall that a four dimensional matrix $A = (a_{m,n,i,j})$ is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit. The Robinson-Hamilton conditions state that a four dimensional matrix $A = (a_{m,n,i,j})$ is RH-regular if and only if

$(RH1)$ $P\text{-}\lim_{m,n} a_{m,n,i,j} = 0$ for each $(i,j) \in \mathbb{N}\times\mathbb{N}$;

$(RH2)$ $P\text{-}\lim_{m,n} \sum_{(i,j)\in\mathbb{N}\times\mathbb{N}} a_{m,n,i,j} = 1$;

$(RH3)$ $P\text{-}\lim_{m,n} \sum_{i\in\mathbb{N}} |a_{m,n,i,j}| = 0$, for each j;

$(RH4)$ $P\text{-}\lim_{m,n} \sum_{j\in\mathbb{N}} |a_{m,n,i,j}| = 0$, for each i;

$(RH5)$ $\lim_{m,n} \sum_{(i,j)\in\mathbb{N}\times\mathbb{N}} |a_{m,n,i,j}|$ is P-convergent for every $(m,n) \in \mathbb{N}\times\mathbb{N}$,

$(RH6)$ there exist finite positive integers K_1, K_2 such that $\sum_{(i,j)>K_2} |a_{m,n,i,j}| < K_1$ holds for every $(m,n) \in \mathbb{N}\times\mathbb{N}$.

Now let $K \subseteq \mathbb{N}\times\mathbb{N}$ and $A=(a_{m,n,i,j})$ be a non-negative RH-regular summability matrix. Then the $A-density$ of K is given by

$$\delta_2^A\{K\} = P - \lim_{m,n} \sum\nolimits_{(i,j)\in D(\varepsilon)} a_{m,n,i,j}$$

where $D(\varepsilon)=\{(i,j)\in\mathbb{N}\times\mathbb{N} : |x_{ij}-L| \geqslant \varepsilon\}$ provided that the limit on the right-hand side exists in Pringsheim's sense.

A double sequence of real numbers $x=(x_{ij})$ is said to be A-statistically convergent to a number L if for every $\varepsilon>0$,

$$\delta_2^A\{D(\varepsilon)\}=0.$$

Note that a P-convergent double sequence is A-statistically convergent to the same limit but the converse is not always true. Also A-statistically convergent double sequence need not be bounded.

For example:

EXAMPLE 1. Let $x=(x_{ij})$ be defined by

$$\{x_{ij}\} = \begin{cases} 1, & \text{if } \ i,j \text{ are square integer,} \\ 0, & \text{otherwise.} \end{cases}$$

But if we take a double Cesaro matrix $C(1,1)$ is A, then $C(1,1)$-statistical convergence is same as statistical convergence for double sequence.

Let $E \subseteq \mathbb{N}\times\mathbb{N}$. The double natural density of E is given by

$$\delta_2(E) = P - \lim_{i,j} \frac{1}{ij}|E_{i,j}|,$$

provided the limit exists where $E_{i,j}=\{(i,j)\in E : i \leqslant m, j \leqslant n\}$ and $|.|$ denotes the cardinality of the enclosed set.

A double sequence $x=(x_{ij})$ is said to be statistically convergent to a number L if for every $\varepsilon>0$,

$$\delta_2(E(\varepsilon))=0,$$

in Pringsheim sense where $E(\varepsilon)=\{i\leqslant m, j\leqslant n : |x_{ij}-L|\geqslant\varepsilon\}$.

If we take the matrix A as the four dimensional identity matrix, then A-statistical convergence is reduced to Pringsheim convergence.

The notion of absolute matrix summability factors of infinite series and $(C,1)$, $(H,1)$-summability were introduced by Karakaş and Moricz in ([16]) and ([19], [20]) respectively. Demicri and Karakuş defined the idea of A-statistical summability for double sequences. Following Demicri and Karakuş ([6]), we know that a bounded double sequence which is A-statistically convergent to L is A-summable to L and hence it is statistically A-summable to L also but not conversely.

Using modulus function and non-negative regular matrix $A=(a_{m,n,i,j})$ with $\sup_{m,n}\sum_{i,j=0,0} a_{m,n,i,j} < \infty$, Savas and Patterson ([29]) generalizes the strong A-summability and extend the sequence space (I), (II) and (III) as follows:

$$w_o(A,f) = \left\{ x\in\Omega : \lim_{m,n\to\infty} \sum_{i,j=0,0}^{\infty} a_{m,n,i,j} f(|x_{i,j}|) = 0 \right\},$$

$$w(A,f)=\Big\{x\in\Omega:\lim_{m,n\to\infty}\sum_{i,j=0,0}^{\infty}a_{m,n,i,j}f(|x_{i,j}-L|)=0,\quad\text{for some}\quad L\Big\},$$

$$w_\infty(A,f)=\Big\{x\in\Omega:\sup_{m,n}\sum_{i,j=0,0}^{\infty}a_{m,n,i,j}f(|x_{i,j}|)<\infty\Big\}.$$

We now introduce the definition of statistical A-summability ([6]):

Let $A=(a_{m,n,i,j})$ be a non-negative RH-regular summability matrix. A double sequence $x=(x_{ij})$ is said to be statistically A-summable to L if for every $\varepsilon>0$

$$\delta_2\{(m,n)\in\mathbb{N}\times\mathbb{N}:|(Ax)_{m,n}-L|\geqslant\varepsilon\}=0.$$

Note that a double sequence x is statistically A-summable to L if and only if $Ax=\{(Ax)_{mn}\}$ is statistically convergent to L.

Before we can start our main results, first we shall present the following definitions by using double sequence of sets.

2. Wijsman convergence and Cesaro summability

We recall some basic definitions and concepts ([5], [22], [23]).

Let (X,ρ) be a metric space. For any point $x\in X$ and for any non-empty subset U of X, we define the distance from x to U by

$$d(x,U)=\inf_{u\in U}\rho(x,u).$$

Let X be any non-empty set. The function $\varphi:\mathbb{N}\to P(X)$ is defined by $\varphi(n)=U_n(\in P(X))$ for each $n\in\mathbb{N}$, where $P(X)$ is a power set of X. The sequence $\{U_n\}=\{U_1,U_2,U_3,\dots\}$, which is the range's elements of φ, is said to be a set sequences or a sequence of sets.

There are different types of convergence notions for the sequences of sets. In ([4], [5], [9], [30], [31]) the concepts of different types invariant convergence and statistical convergence by double sequence of sets was discussed in the sense of Wijsman. We handled in this section is the concept of Wijsman convergence ([12]) using double sequence of sets.

Nuray et al. ([22], [23]) introduced the concepts of Wijsman convergence and Wijsman statistically convergence for a double sequence of sets, as follows:

DEFINITION 1. Let (X,ρ) be a metric space and U be any non-empty closed subset of X. A double sequence of closed sets $\{U_{ij}\}(\subseteq X)$ is said to be Wijsman convergent to U if for each $x\in X$,

$$\lim_{i,j\to\infty}\{d(x,U_{ij})-d(x,U)\}=0.$$

It is denoted by $[W_2]-\lim U_{ij}=U$.

EXAMPLE 2. Let $X = \mathbb{R}^2$ and a double sequence $\{U_{ij}\}$ be defined as

$$U_{ij} = \left\{ (x,y) \in \mathbb{R}^2 : x^2 + (y-1)^2 = \frac{1}{ij} \right\}.$$

The double sequence of sets is Wijsman convergent to $U = \{(0,1)\}$.

DEFINITION 2. Let (X,ρ) be a metric space and U be any non-empty closed subset of X. A double sequence of closed sets $\{U_{ij}\}(\subseteq X)$ is said to be Wijsman statistically convergent to U if for each $\varepsilon > 0$ and each $x \in X$,

$$\lim_{m,n\to\infty} |\{(i,j) : i \leqslant m, j \leqslant n : |d(x,U_{ij}) - d(x,U)| \geqslant \varepsilon\}| = 0.$$

It is denoted by $[WS] - \lim U_{ij} = U$.

DEFINITION 3. A double sequence of closed sets $\{U_{ij}\}(\subseteq X)$ is said to be bounded if for each $x \in X$,

$$\sup_{ij}\{d(x,U_{ij}\} < \infty.$$

The set of all bounded sequence of sets is denoted by L_∞.

EXAMPLE 3. Let $X = \mathbb{R}^2$ and a double sequence $\{U_{ij}\}$ be defined as

$$U_{ij} = \begin{cases} (x,y) \in \mathbb{R}^2 : (x-1)^2 + y^2 = \dfrac{1}{ij}, & \text{if } i,j \text{ are square integer,} \\ \{(0,0)\}, & \text{otherwise.} \end{cases}$$

Then the double sequence $\{U_{ij}\}$ is bounded and Wijsman statistically convergent to the set $U = \{(0,0)\}$ but it is not Wijsman convergent.

EXAMPLE 4. Let $X = \mathbb{R}^2$ and a double sequence $\{U_{ij}\}$ be defined as

$$U_{ij} = \begin{cases} (x,y) \in \mathbb{R}^2 : (x-i)^2 + (y-j)^2 = 1, & \text{if } i,j \text{ are square integer,} \\ \{(0,0)\}, & \text{otherwise.} \end{cases}$$

Then the double sequence $\{U_{ij}\}$ is Wijsman statistically convergent to the set $U = \{(0,0)\}$ but it is not Wijsman convergent.

If a double sequence of sets $\{U_{ij}\}$ is Wijsman statistically convergent to the set U, then $\{U_{ij}\}$ need not be Wijsman convergent. Also, it is not necessary be bounded.

EXAMPLE 5. Let $X = \mathbb{R}^2$ and a double sequence $\{U_{ij}\}$ be defined as

$$U_{ij} = \begin{cases} \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + (y-1)^2 = ij\}, & \text{if } i,j \text{ are square integer,} \\ \{(2,2)\}, & \text{otherwise.} \end{cases}$$

Then the double sequence $\{U_{ij}\}$ is Wijsman statistically convergent to the set $U = \{(2,2)\}$ but it is neither Wijsman convergent nor bounded.

DEFINITION 4. Let (X,ρ) be a metric space and U be any non-empty closed subset of X. A double sequence of closed sets $\{U_{ij}\}(\subseteq X)$ is said to be Wijsman Cesaro-summable to U if for each $x \in X$,

$$\lim_{p,q\to\infty} \frac{1}{pq} \sum_{i=1}^{p} \sum_{j=1}^{q} d(x,U_{ij}) = d(x,U).$$

DEFINITION 5. Let (X,ρ) be a metric space and U be any non-empty closed subset of X. A double sequence of closed sets $\{U_{ij}\}(\subseteq X)$ is said to be Wijsman strongly Cesaro-summable to U if for each $x \in X$,

$$\lim_{p,q\to\infty} \frac{1}{pq} \sum_{i=1}^{p} \sum_{j=1}^{q} |d(x,U_{ij}) - d(x,U)| = 0.$$

DEFINITION 6. Let (X,ρ) be a metric space and U be any non-empty closed subset of X. A double sequence of closed sets $\{U_{ij}\}(\subseteq X)$ is said to be Wijsman strongly α-Cesaro-summable to U if for each $x \in X$ and $0 < \alpha < \infty$,

$$\lim_{p,q\to\infty} \frac{1}{pq} \sum_{i=1}^{p} \sum_{j=1}^{q} |d(x,U_{ij}) - d(x,U)|^{\alpha} = 0.$$

EXAMPLE 6. Let $X = \mathbb{R}^2$ and a double sequence $\{U_{ij}\}$ be defined as

$$U_{ij} = \begin{cases} (x,y) \in \mathbb{R}^2 : x^2 + (y-1)^2 = \dfrac{1}{ij}, & \text{if } i,j \text{ are square integer} \\ \{(1,0)\}, & \text{otherwise.} \end{cases}$$

Then the double sequence $\{U_{ij}\}$ is Wijsman strongly Cesaro summable to the set $U = \{(1,0)\}$.

If $\{U_{ij}\}$ is convergent but unbounded then it is Wijsaman statistically convergent but it need not Wijsman Cesaro summable nor Wijsman strongly Cesaro summable.

EXAMPLE 7. Let $X = \mathbb{R}^2$ and a double sequence $\{U_{ij}\}$ be defined as

$$U_{ij} = \begin{cases} \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + (y-1)^2 = i\}, & \text{if } j = 1, \text{ for all } i, \\ \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + (y-1)^2 = j\}, & \text{if } i = 1, \text{ for all } j, \\ \{(0,0)\}, & \text{otherwise.} \end{cases}$$

Then the double sequence of sets $\{U_{ij}\}$ is Wijsman convergent to the set $U = \{(0,0)\}$, but the limit

$$\lim_{p,q\to\infty} \frac{1}{pq} \sum_{i=1}^{p} \sum_{j=1}^{q} d(x,U_{ij})$$

does not tend to a finite limit. Therefore $\{U_{ij}\}$ is not Wijsman Cesaro-summabe and hence is not Wijsman strongly Cesaro-summable although

$$\lim_{p,q\to\infty} \frac{1}{pq} |\{i \leqslant p, j \leqslant q : |d(x,U_{ij}) - d(x,\{0,0\})| \geqslant \varepsilon\}| = \lim_{p,q\longrightarrow\infty} \frac{p+q-1}{pq} = 0$$

so $\{U_{ij}\}$ is Wijsman statistically convergent to $\{(0,0)\}$.

3. Main results (matrix summability)

In ([10]), Ganguly and Dafadar investigated on the strong matrix summable double sequence space with respect to modulus and statistical convergence and in ([11]) they also investigated on strongly summable double sequence space defined by modulus functions and μ-statistical convergence and described some important results related to matrix summability. Also J. Connor and E. Savas ([3], [27]) explored some interesting results on strong A-summability with respect to a modulus and he extended the definition of Cesaro-summability ([7], [8]) to strong A-summability with respect to a modulus when A is a non-negative regular matrix summability method.

Using the concept of statistical A-summability S. Orhan et al. ([24]) obtain a Korovkin-type approximation theorem for double sequences of positive linear operators of two variables. Very recently F. Nuray ([21]) generalizes the concepts of summability of sequence to summability of sequence of sets and investigated some results concerning to matrix transformation of sequence of sets.

Inspiring from the above articles, our main aim in this paper is to give a definitions of four dimensional matrix transformation with help of double sequence of sets and then we study their summability.

DEFINITION 7. Let Γ and Θ be any two non-empty subset of the space of all sequence of sets. Let $T=(t_{mn})$ be a non-negative infinite matrix of real numbers.

We set $TU=(T_{mn}(U))$ if

$$T_{mn}(U)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}t_{mnij}d(x,U_{ij})$$

converges for all $m,n\in\mathbb{N}$ and for all $x\in X$.

If $\{U_{ij}\}\in\Gamma$ implies $TU=(T_{mn}(U))\in\Theta$, then we say that T defines a matrix transformation from Γ into Θ and we denote it by $T:\Gamma\longrightarrow\Theta$. The sequence TU is called the T-transform of U. By (Γ,Θ) we mean the class of matrices T such that $T:\Gamma\longrightarrow\Theta$.

A matrix method T is called regular if all convergent sequence of sets $U=(U_{ij})$ are T-summable and $\lim_{m,n}\ T_{mn}x=\ \lim U_{ij}$ and it is denoted by $T\in(c,c,P)$. The matrix T is called regular if and only if all the conditions RH1–RH6 holds.

If $\lim_{i,j}d(x,U_{ij})=d(x,V)$, $u_{ij}=d(x,U_{ij})$ and $v=d(x,V)$, then for the real double sequence (u_{ij}), we have $\lim_{i,j}u_{ij}=v$. Hence we can write $T\in(W,W,P)$ if and only if RH1–RH6 hold.

DEFINITION 8. Let $T=(t_{mnij})$ be a non-negative regular summability method and $\{U_{ij}\}$ be a double sequence of closed sets. Then $\{U_{ij}\}$ is said to be strongly T-summable to V if

$$\lim_{m,n\longrightarrow\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}t_{mnij}|d(x,U_{ij})-d(x,V)|=0,\quad \text{for every } x\in X.$$

The collection of all strongly T-summable sequences of closed sets is denoted by $W(T)$ where

$$W(T)=\Big\{\{U_{ij}\}:\lim_{m,n\longrightarrow\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}t_{mnij}|d(x,U_{ij})-d(x,V)|=0,\ \text{for every } x\in X\Big\}.$$

The double sequence of closed sets $\{U_{ij}\}$ is said to be strongly T-summable to V if $\{U_{ij}\} \in W(T)$. If we replace T by $(C,1)$, then we obtain strongly Cesaro summable sequences of closed sets.

DEFINITION 9. Let $T = (t_{mnij})$ be a non-negative regular summability method and $\{U_{ij}\}$ be a double sequence of closed sets. Then $\{U_{ij}\}$ is said to be strongly T-statistically convergent to V if for every $\varepsilon > 0$

$$\chi_{S(U_{ij},V,;\varepsilon)} \subset W_{\phi}(T)$$

where

$$W_{\phi}(T) = \Big\{\{U_{ij}\} : \lim_{m,n\longrightarrow\infty} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} t_{mnij} d(x,U_{ij}) = 0\Big\}$$

and

$$(U_{ij},V,;\varepsilon) = \big\{i,j \in \mathbb{N} : |d(x,U_{ij}) - d(x,V)| \geqslant \varepsilon\big\}.$$

THEOREM 1. *Let $T = (t_{mnij})$ be a non-negative regular summability method and $\{U_{ij}\}$ be a bounded double sequence of closed sets. Then $\{U_{ij}\}$ is T-statistically convergent to V if and only if it is strongly T-summable to V.*

Proof. First we take $\{U_{ij}\}$ is bounded and T-statistically convergent to V. Then for every $\varepsilon > 0$, $\chi_{S(U_{ij},V,;\varepsilon)} \subset W_{\phi}(T)$ holds.

Now for every $x \in X$, we have

$$\begin{aligned}
&\sum_i \sum_j t_{mnij} |d(x,U_{ij}) - d(x,V)| \\
\leqslant &\sum\sum\nolimits_{i,j \in S(U_{ij},V;\varepsilon)} t_{mnij} |d(x,U_{ij}) - d(x,V)| \\
&+ \sum\sum\nolimits_{i,j \notin S(U_{ij},V;\varepsilon)} t_{mnij} |d(x,U_{ij}) - d(x,V)| \\
\leqslant &\sup_{i,j} |d(x,U_{ij}) - d(x,V)| \sum\sum\nolimits_{i,j \in S(U_{ij},V;\varepsilon)} t_{mnij} + \varepsilon . \sum\sum\nolimits_{i,j \notin S(U_{ij},V;\varepsilon)} t_{mnij}.
\end{aligned}$$

Since $\{U_{ij}\}$ is T-statistically convergent to V and T is regular, therefore taking limit as $m,n \longrightarrow \infty$ on both sides, we have for arbitrary $\varepsilon > 0$,

$$\lim_{m,n\longrightarrow\infty} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} t_{mnij} |d(x,U_{ij}) - d(x,V)| = 0.$$

Hence, $U_{ij} \in W(T)$.

Conversely, suppose that $U_{ij} \in W(T)$.

Then

$$\lim_{m,n\longrightarrow\infty} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} t_{mnij} |d(x,U_{ij}) - d(x,V)| \longrightarrow 0 \quad \text{as } m,n \to \infty. \tag{IV}$$

We have,

$$\sum\sum\nolimits_{i,j \in S(U_{ij},V;\varepsilon)} t_{mnij} |d(x,U_{ij}) - d(x,V)| \leqslant \sum\sum\nolimits_{i,j} t_{mnij} |d(x,U_{ij}) - d(x,V)|.$$

Taking limit as $m,n \longrightarrow \infty$ on both sides and using (IV) for every $x \in X$, we find that, $\{U_{ij}\}$ is T-statistically convergent to V. $\square$

By using modulus function we now introduce and investigate some properties of two sequence spaces, which are the generalization of $W_\phi(T)$ and $W(T)$.

DEFINITION 10. Let f be a modulus function and $T = (t_{mnij})$ be a non-negative regular summability method. Then a double sequence of closed sets $\{U_{ij}\}$ is said to be strongly T-summable to V with respect to f if

$$\lim_{m,n\longrightarrow\infty} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} t_{mnij} f(|d(x,U_{ij}) - d(x,V)|) = 0, \quad \text{for every } x \in X.$$

The collection of all strongly T-summable sequence of closed sets with respect to f is denoted by $W(T,f)$ where

$$W(T,f) = \Big\{\{U_{ij}\} : \lim_{m,n\longrightarrow\infty} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} t_{mnij} f(|d(x,U_{ij}) - d(x,V)|) = 0, \text{ for every } x \in X\Big\}.$$

The double sequence of closed sets $\{U_{ij}\}$ is said to be strongly T-summable to V with respect to f if $\{A_{ij}\} \in W(T,f)$.

THEOREM 2. *Let f be a modulus function and $T = (t_{mnij})$ be a non-negative regular summability method. If a double sequence of closed sets $\{U_{ij}\}$ is strongly T-summable to V, then it is strongly T-summable to V with respect to the modulus function f.*

Proof. Since $\{U_{ij}\}$ is strongly T-summable to V, we have

$$s_{mn} = \sum_i \sum_j t_{mnij} |d(x,U_{ij}) - d(x,V)| \longrightarrow 0 \quad \text{as } m,n \longrightarrow \infty.$$

Let $\varepsilon > 0$ be given and choose δ with $0 < \delta < 1$ such that $f(u) < \varepsilon$ for $0 \leqslant u \leqslant \delta$.

Set $\lambda_{ij} = |d(x,U_{ij}) - d(x,V)|$ and consider

$$\begin{aligned} \sum_i \sum_j t_{mnij} f(\lambda_{ij}) &= \sum\sum_{\{i,j:\lambda_{ij}>\delta\}} t_{mnij} f(\lambda_{ij}) + \sum\sum_{\{i,j:\lambda_{ij}\leqslant\delta\}} t_{mnij} f(\lambda_{ij}) \\ &\leqslant \sum\sum_{\{i,j:\lambda_{ij}>\delta\}} t_{mnij} f(\lambda_{ij}) + \varepsilon . \sum\sum_{\{i,j:\lambda_{ij}\leqslant\delta\}} t_{mnij}. \end{aligned} \tag{V}$$

Now for $\lambda_{ij} > \delta$, note that

$$\lambda_{ij} < \frac{\lambda_{ij}}{\delta} < 1 + \left[\frac{\lambda_{ij}}{\delta}\right]$$

where $[x]$ is the integer part of x.

By using modulus function, we have for $\lambda_{ij} > \delta$,

$$f(\lambda_{ij}) \leqslant 1 + \frac{\lambda_{ij}}{\delta} f(1) \leqslant 2f(1)\left[\frac{\lambda_{ij}}{\delta}\right].$$

Applying T-summability on the left and right sides of the above inequality we have,

$$\sum\sum_{\{i,j:\lambda_{ij}>\delta\}} t_{mnij} f(\lambda_{ij}) \leqslant 2\delta^{-1} f(1) \sum\sum_{\{i,j:\lambda_{ij}>\delta\}} t_{mnij}. \qquad (VI)$$

By using (VI), (V) is reduced to

$$\sum_i\sum_j t_{mnij} f(\lambda_{ij}) \leqslant \varepsilon . \sum\sum_{\{i,j:\lambda_{ij}\leqslant\delta\}} t_{mnij} + 2\delta^{-1} f(1) \sum\sum_{\{i,j:\lambda_{ij}>\delta\}} t_{mnij}.$$

Since T is regular and $\varepsilon > 0$ is arbitrary, we can write $\{U_{ij}\} \in W(T,f)$. □

LEMMA 1. ([27]) *Let f be a modulus function and let $\alpha > 0$ be a given constant. Then there exists a constant $c > 0$ such that $f(x) > cx$ $(0 < x < \alpha)$.*

THEOREM 3. *Let f be a modulus function and $T = (t_{mnij})$ be a non-negative regular summability method. Then a bounded double sequence of closed sets $\{U_{ij}\}$ is strongly T-summable to V with respect to the modulus function f if and only if $\{U_{ij}\}$ is strongly T-summable to V.*

The proof of the above theorem follows from Theorem 2 and Lemma 1.

COROLLARY 1. *Let $\{U_{ij}\}$ be a double sequence of closed sets such that $d(x,U_{ij}) = O(\sqrt{ij})$ and let*

$$\lim_{ij} \frac{1}{\sqrt{ij}} |\{kl \leqslant ij : |d(x,U_{kl}) - d(x,V)| \geqslant \varepsilon\}| = 0, \quad \textit{for all } \varepsilon > 0.$$

Then $\{U_{ij}\}$ is strongly summable to V.

Proof. Note that $|d(x,U_{kl}) - d(x,V)| \leqslant B\sqrt{ij}$ for all i,j.
Then we have,

$$\lim_{ij} \frac{1}{\sqrt{ij}} \sum_{k=1}^{i}\sum_{l=1}^{j} |d(x,U_{ij}) - d(x,V)|$$
$$\leqslant \varepsilon + B\frac{1}{\sqrt{ij}} |\{kl \leqslant ij : |d(x,U_{kl}) - d(x,V)| \geqslant \varepsilon\}|,$$

holds for all $\varepsilon > 0$.
Hence for arbitrary $\varepsilon > 0$, $\{U_{ij}\}$ is strongly summable to V. □

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