Journal of Classical Analysis
Volume 25, Number 2 (2025), 107–116 doi:10.7153/jca-2025-25-07

ON GENERALIZED β–ABSOLUTE CONVERGENCE OF SINGLE AND DOUBLE SERIES IN MULTIPLICATIVE SYSTEMS

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Abstract. In this paper, we obtain sufficient conditions for the generalized β-absolute convergence $(0 < \beta \leqslant 2)$ of single and double series in multiplicative systems of functions of generalized bounded fluctuation.

1. Introduction

Concerning the absolute convergence of Fourier series, the theorem of Bernstein [1, Vol. II, Theorem 2 of Bernstein, p. 154], the theorem of Szász [1, Vol. II, Szász Theorem, p. 155], and the theorem of Zygmund [1, Vol. II, p. 160] are classical. Gogoladze and Meskhia [3] generalized β-absolute convergence $(0 < \beta < 2)$ of the Fourier trigonometric series with gaps for some classes of functions, also, it was proved that these conditions are unimprovable in a certain sense. Similar problems are also studied in the papers [4] and [11]. In 2010, Móricz [8] obtained sufficient conditions for generalized β-absolute convergence of single Walsh-Fourier series. In 2011, Móricz and Veres [9] proved analogues of the results proved in [8] for the double Walsh-Fourier series. In 2012, Golubov and Volosivets [6] obtained several sufficient conditions for generalized β-absolute convergence of single and double series in multiplicative systems. Those conditions gave a multiplicative analogue of results due to Gogoladze and Meskhia [3]. They noticed that their results are analogous of the results obtained by Móricz in [8], and Móricz and Veres in [9]. Generalizing the results of Golubov and Volosivets [6], in 2017, Kuznetsova [7] obtained sufficient conditions for generalized β-absolute convergence of single series in multiplicative systems. Extending some of these results of Kuznetsova, Volosivets and Kuznetsova [13] obtained sufficient condition for generalized β-absolute convergence of double series in multiplicative systems. In this paper, generalizing the result of Kuznetsova [7, Theorem 1, p. 306] and Volosivets and Kuznetsova [13, Theorem 5, p. 227], we obtain sufficient conditions for generalized β-absolute convergence of single and double series in multiplicative systems. In particular case, our condition for single and double series are multiplicative analogue of the result obtained by Móricz [8, Theorem 3, p. 281] and Móricz and Veres [9, Theorem 3, p. 129], respectively.

Throughout the paper, C represents a constant vary time to time.

Mathematics subject classification (2020): 42C10.

Keywords and phrases: Multiplicative system, generalized β-absolute convergence, functions of p-Λ-bounded fluctuation, functions of p-(Λ^1,Λ^2)-bounded fluctuation.

Paper JCA-25-07

2. Results for functions of one variable

Let $\mathbf{P} = \{p_j\}_{j=1}^{\infty}$ be a sequence of natural numbers such that $2 \leqslant p_j \leqslant N$ for all $j \in \mathbb{N}$ and $\mathbb{Z}_j = \{0, 1, \ldots, p_j - 1\}$. Consider the sequence $\{m_j\}_{j=0}^{\infty}$ defined as follows: $m_0 = 1$, $m_n = m_{n-1}p_n$ for $n \in \mathbb{N}$. Then any number $x \in [0,1)$ can be represented as

$$x = \sum_{j=1}^{\infty} x_j m_j^{-1}, \quad x_j \in \mathbb{Z}_j, \tag{1}$$

and every $k \in \mathbb{Z}_+$ is uniquely represented as

$$k = \sum_{j=1}^{\infty} k_j m_{j-1}, \quad k_j \in \mathbb{Z}_j. \tag{2}$$

The expression (1) is also uniquely determined if, for $x = s/m_n$, $0 < s < m_n$, $s \in \mathbb{Z}$, a finite number of nonzero x_j is taken.

For numbers $x \in [0,1)$ and $k \in \mathbb{Z}_+$ with expansions (1) and (2), we set, by definition,

$$\chi_k(x) = \exp\left(2\pi i \left(\sum_{j=1}^{\infty} \frac{x_j k_j}{p_j}\right)\right).$$

The system $\{\chi_k(x)\}_{k=0}^{\infty}$ of functions is called a multiplicative system. It is known that it is orthonormal and complete in $L^1[0,1)$ (see [5, Chap. 1, Sec. 1.5]). It is easy to see that, for $0 \leqslant n < m_k$, the function $\chi_n(x)$ is constant on

$$I_j^k = \left[\frac{j-1}{m_k}, \frac{j}{m_k}\right), \quad 1 \leqslant j \leqslant m_k, \quad k \in \mathbb{Z}_+.$$

Let $G(\mathbf{P})$ be the group consisting of sequences of the form $\widetilde{x} = (x_1, x_2, \ldots)$, $x_j \in \mathbb{Z}_+$, $0 \leqslant x_j < p_j$, with the operation $\widetilde{x} \oplus \widetilde{y} = \widetilde{z}$, where $z_j = x_j + y_j (\mathrm{mod}\ p_j)$, $j \in \mathbb{N}$. The inverse operation $\widetilde{x} \ominus \widetilde{y}$ is defined in a similar way. The mapping $\lambda_{\mathbf{P}}(\widetilde{x}) = \sum_{j=1}^{\infty} x_j m_j^{-1}$ of the group $G(\mathbf{P})$ to $[0,1)$ is not bijective, because each number of the form

$$x = \frac{k}{m_l}, \quad k, l \in \mathbb{N}, \quad k < m_l, \tag{3}$$

is the image of two different elements from $G(\mathbf{P})$. Let us define an inverse mapping $\lambda_{\mathbf{P}}^{-1}$. For $x \in [0,1)$ of the form (3), we set $x_j = [m_j x] (\mathrm{mod}\ p_j)$, $j \in \mathbb{N}$. Then $\lambda_{\mathbf{P}}^{-1}(x) = (x_1, \ldots, x_l, 0, \ldots)$. For any other $x \in [0,1)$, there exists a unique element $\widetilde{x} \in G(\mathbf{P})$ with the property $\lambda_{\mathbf{P}}(\widetilde{x}) = x$, and, in that case, we set $\lambda_{\mathbf{P}}^{-1}(x) = \widetilde{x}$. We define the generalized distance

$$\rho(x,y) = \lambda_{\mathbf{P}}(\lambda_{\mathbf{P}}^{-1}(x) \ominus \lambda_{\mathbf{P}}^{-1}(y))$$

and the addition

$$x \oplus y = \lambda_{\mathbf{P}}(\lambda_{\mathbf{P}}^{-1}(x) \oplus \lambda_{\mathbf{P}}^{-1}(y))$$

on $[0,1)$. Note that $x\oplus y$ is not defined if $\lambda_{\mathbf{P}}^{-1}(x)\oplus\lambda_{\mathbf{P}}^{-1}(y)=\tilde{z}$, where $z_j=p_j-1$ for $j\geqslant j_0$, i.e., $x\oplus y$ is defined for almost all $x\in[0,1)$ for a fixed $y\in[0,1)$. It is easy to see that $x\oplus/m_{k+1}$, $k\in\mathbb{Z}_+$, is always defined and that $\rho\left(x\oplus\frac{1}{m_{k+1}},x\right)<\frac{1}{m_k}$.

In the same way, we also define

$$x\ominus y=\lambda_{\mathbf{P}}(\lambda_{\mathbf{P}}^{-1}(x)\ominus\lambda_{\mathbf{P}}^{-1}(y))$$

with constrains similar to those in the case $x\oplus y$.

For $f\in L^1[0,1)$, the Fourier coefficients in the system $\{\chi_j(x)\}_{j=0}^{\infty}$ are given by the formula

$$\hat{f}(j)=\int_0^1 f(x)\,\overline{\chi_j(x)}\,dx,\quad j\in\mathbb{Z}_+.$$

We note that the space $L^p[0,1)$, $1\leqslant p<\infty$, is endowed with the standard norm

$$\|f\|_p=\left(\int_0^1|f(x)|^p\,dx\right)^{1/p}.$$

Let

$$\Delta_u f(x)=f(x\oplus u)-f(x).$$

It follows from the condition $f\in L^p[0,1)$ that $\|\Delta_u f(x)\|_p\to 0$ as $u\to 0$. Therefore, we can introduce the discrete modulus of continuity of a function $f\in L^p[0,1)$, $1\leqslant p<\infty$, as the sequence

$$\omega_k(f)_p=\sup\{\|\Delta_u f(x)\|_p : u\in I_1^k\},\ \ k\in\mathbb{Z}_+.$$

The space $C^*[0,1)$ with the corresponding norm

$$\|f\|_\infty=\sup_{x\in[0,1)}|f(x)|$$

is the closure of the set of polynomials in the system $\{\chi_k(x)\}_{k=0}^{\infty}$ in the norm $\|.\|_\infty$. For $f\in C^*[0,1)$, we put, by definition,

$$\omega_k(f)_\infty=\sup\left\{|f(x)-f(y)|:\ x,y\in[0,1),\ \rho(x,y)<\frac{1}{m_k}\right\},\quad k\in\mathbb{Z}_+.$$

Let $\alpha\geqslant 1$. We say that a sequence $\gamma=\{\gamma_i\}_{i=0}^{\infty}$ belongs to the class $A^*(\alpha)=A^*(\alpha,2)=A^*(\alpha,2,\mathbf{P})$, if $\gamma_i>0$ for all i and

$$\left(\sum_{i=m_k}^{m_{k+1}-1}\gamma_i^{\alpha}\right)^{1/\alpha}\leqslant Cm_k^{(1-\alpha)/\alpha}\sum_{i=m_{k-1}}^{m_k-1}\gamma_i=:Cm_k^{(1-\alpha)/\alpha}\Gamma_k,\quad k\in\mathbb{N}.$$

For $k=0$, we assume that a similar inequality is valid for $\Gamma_0=\gamma_0$. This definition is given in the paper of L. Gogoladze [2] and also in the paper of Gogoladze and Meskhia [3] for $m_k=2^k$.

A sequence $\gamma=\{\gamma_i\}_{i=0}^{\infty}$ belongs to the class $A^*(\infty,2)$, if $\gamma_i>0$ for all i and

$$\max_{m_k\leqslant i<m_{k+1}}\gamma_i\leqslant Cm_k^{-1}\Gamma_k,\ \ k\in\mathbb{N}.$$

We say that a non-decreasing sequences $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ of positive numbers belongs to the class $\mathbb{L}$ if $\lim_{n\to\infty} \Lambda_n = \infty$, where $\Lambda_n = \sum_{k=1}^{n} \lambda_k^{-1}$.

Let $\Lambda = \{\lambda_i\}_{i=1}^{\infty} \in \mathbb{L}$ and $p \geqslant 1$. Let $k \in \mathbb{Z}_+$ be fixed, and let $f(x)$ be bounded on $[0,1)$. If $i \in [1, m_k] \cap \mathbb{Z}$, then we denote

$$osc(f, I_i^k) = \sup\{|f(x) - f(y)| : x, y \in I_i^k\},$$

$$æ_k(f, p, \Lambda) = \sup \sum_{i=1}^{m_k} \frac{osc^p(f, I_{\alpha_i}^k)}{\lambda_i},$$

where the supremum in the formula for $æ_k$ is taken over all permutations $\{\alpha_i\}_{i=1}^{m_k}$ of the index set $\{1, 2, \dots, m_k\}$. If

$$V_\Lambda^p(f) = \sup\{æ_k(f, p, \Lambda) : k \in \mathbb{Z}_+\} < \infty,$$

then f belongs to the class $\Lambda Fl^{(p)}[0,1)$ of functions of p-Λ-bounded fluctuation.

Note that, for $p = 1$, the class $\Lambda Fl^{(p)}[0,1)$ reduce to the class $\Lambda Fl^{(1)}[0,1)$ of functions of Λ-bounded fluctuation and for $\Lambda = \{1\}$, the class $\Lambda Fl^{(p)}[0,1)$ reduce to the class $Fl^{(p)}[0,1)$ of functions of p-bounded fluctuation.

We prove the following results.

THEOREM 1. *Let* $f \in \Lambda Fl^{(p)}[0,1) \cap C^*[0,1)$ $(p \geqslant 1)$, *and let* $0 < \beta < 2$, $\gamma = \{\gamma_r\}_{r=1}^{\infty} \in A^*(2/(2-\beta), 2)$ *or* $\beta = 2$, $\gamma \in A^*(\infty, 2)$. *If the series*

$$\sum_{k=0}^{\infty} m_k^{-\beta/2} \Gamma_k \left(\frac{\omega_k^q(f)_\infty}{\Lambda_{m_k}} \right)^{\frac{\beta}{p+q}}$$

converges, then the series

$$\sum_{r=1}^{\infty} \gamma_r |\hat{f}(r)|^\beta \tag{4}$$

also converges, where $q \geqslant 1$.

Proof. For $k \in \mathbb{Z}_+$, consider $g(x) = f\left(x \oplus m_{k+1}^{-1}\right) - f(x)$. Then, for each $r \in \mathbb{Z}_+$, $\widehat{g}(r) = \left(\chi_r(m_{k+1}^{-1}) - 1\right) \widehat{f}(r)$. In view of Parseval's equality, we get

$$\sum_{r=0}^{\infty} \left| \widehat{f}(r) \left(\chi_r(m_{k+1}^{-1}) - 1\right) \right|^2 = \int_0^1 |g(x)|^2 dx = \int_0^1 |f_u(x)|^2 \, dx,$$

where $f_u(x) = f\left(x \oplus (up_{k+1} + 1) m_{k+1}^{-1}\right) - f\left(x \oplus u m_k^{-1}\right)$ for $0 \leqslant u < m_k$.

Since, for $r \in [m_k, m_{k+1})$,

$$|\chi_r(m_{k+1}^{-1}) - 1| = \left| \exp\left(\frac{2\pi i j}{p_{k+1}}\right) - 1 \right| = 2\sin\left(\frac{\pi j}{p_{k+1}}\right) \geqslant 2\sin\left(\frac{\pi}{N}\right), \tag{5}$$

where N is the majorant of the generating sequence $\{p_j\}_{j=1}^{\infty}$ and $j=[r/m_k]\in[1,p_{k+1}-1]$, it follows that

$$B_k := \sum_{r=m_k}^{m_{k+1}-1} |\widehat{f}(r)|^2 \leqslant C\int_0^1 |f_u(x)|^2\,dx. \tag{6}$$

Applying Hölder's inequality on the right side of the above inequality, we obtain

$$B_k \leqslant C\left(\int_0^1 |f_u(x)|^{p+q}\,dx\right)^{\frac{2}{p+q}}.$$

Multiplying above inequality by λ_{u+1}^{-1} and, after that, summing the resulting inequality over $u=0,1,2,\ldots,m_k-1$, we can write

$$B_k \leqslant C\left(\frac{1}{(\Lambda_{m_k})^{\frac{2}{p+q}}}\left(\int_0^1\sum_{u=0}^{m_k-1}\frac{|f_u(x)|^{p+q}}{\lambda_{u+1}}dx\right)^{\frac{2}{p+q}}\right),$$

where $\Lambda_{m_k}=\sum_{u=0}^{m_k-1}\frac{1}{\lambda_{u+1}}$. Since $|f_u(x)|\leqslant\omega_k(f)_\infty$, we have

$$B_k \leqslant C\left(\left(\frac{\omega_k^q(f)_\infty}{\Lambda_{m_k}}\right)^{\frac{2}{p+q}}\left(\int_0^1\sum_{u=0}^{m_k-1}\frac{|f_u(x)|^{p}}{\lambda_{u+1}}dx\right)^{\frac{2}{p+q}}\right),$$

where

$$\sum_{u=0}^{m_k-1}\frac{|f_u(x)|^p}{\lambda_{u+1}} = O(1)\ \text{ as }\ f\in\Lambda Fl^{(p)}[0,1).$$

Hence,

$$B_k \leqslant C\left(\frac{\omega_k^q(f)_\infty}{\Lambda_{m_k}}\right)^{\frac{2}{p+q}}. \tag{7}$$

If $0<\beta<2$, then $1=\frac{\beta}{2}+\frac{2-\beta}{2}$ and, in view of (7) and Hölder's inequality, we have

$$\begin{aligned}\sum_{r=m_k}^{m_{k+1}-1}\gamma_r|\hat{f}(r)|^\beta &\leqslant \left(\sum_{r=m_k}^{m_{k+1}-1}\gamma^{2/(2-\beta)}\right)^{(2-\beta)/2} B_k^{\beta/2}\\ &\leqslant Cm_k^{-\beta/2}\Gamma_k\left(\frac{\omega_k^q(f)_\infty}{\Lambda_{m_k}}\right)^{\frac{\beta}{p+q}}.\end{aligned}$$

Summing over $k\in\mathbb{Z}_+$, we have

$$\sum_{r=1}^{\infty}\gamma_r|\hat{f}(r)|^\beta \leqslant C\sum_{k=0}^{\infty}m_k^{-\beta/2}\Gamma_k\left(\frac{\omega_k^q(f)_\infty}{\Lambda_{m_k}}\right)^{\frac{\beta}{p+q}}.$$

For $\beta=2$, the argument is similar. The theorem is proved. □

In the case when $p=q=1$ in Theorem 1, we have the following result proved by Kuznetsova [7, Theorem 1, p. 306].

COROLLARY 1. *Let* $f \in \Lambda Fl^{(1)}[0,1) \cap C^*[0,1)$. *If the series*

$$\sum_{k=0}^{\infty} m_k^{-\beta/2}\Gamma_k\left(\frac{\omega_k(f)_\infty}{\Lambda_{m_k}}\right)^{\frac{\beta}{2}}$$

is converges, then the series (4) *also converges, where* β *and* $\gamma = \{\gamma_r\}_{r=1}^{\infty}$ *are as in Theorem* 1.

In the case when $q = 2-p$ and $\Lambda = \{1\}$ in Theorem 1, we have the following result.

COROLLARY 2. *Let* $f \in Fl^{(p)}[0,1) \cap C^*[0,1)$. *If the series*

$$\sum_{k=0}^{\infty} m_k^{-\beta}\Gamma_k\left(\omega_k(f)_\infty\right)^{\frac{(2-p)\beta}{2}}$$

is converges, then the series (4) *also converges, where* β *and* $\gamma = \{\gamma_r\}_{r=1}^{\infty}$ *are as in Theorem* 1.

Corollary 2 is the multiplicative analogue of result proved by Móricz [8, Theorem 3, p. 281].

3. Results for functions of two variables

The system $\{\chi_i(x)\chi_j(y)\}_{i,j=0}^{\infty}$ is also orthonormal and complete in $L^1[0,1)^2$ and, therefore, for $f \in L^1[0,1)^2$, we can find the Fourier coefficients

$$\hat{f}(i,j) = \int_0^1\int_0^1 f(x,y)\,\overline{\chi_i(x)\chi_j(y)}\,dx\,dy, \quad i,j \in \mathbb{Z}_+.$$

We note that the space $L^p[0,1)^2$, $1 \leqslant p < \infty$, is endowed with the standard norm

$$\|f\|_p = \left(\int_0^1\int_0^1 |f(x,y)|^p\,dx\,dy\right)^{1/p}.$$

Let

$$\Delta_{uv}f(x,y) = f(x\oplus u, y\oplus v) - f(x\oplus u, y) - f(x, y\oplus v) + f(x,y).$$

It follows from the condition $f \in L^p[0,1)^2$ that $\|\Delta_{uv}f(x,y)\|_p \to 0$ as $u,v \to 0$. Therefore, we can introduce the discrete modulus of continuity of a function $f \in L^p[0,1)^2$, $1 \leqslant p < \infty$, as the double sequence

$$\omega_{kl}(f)_p = \sup\{\|\Delta_{uv}f(x,y)\|_p : u \in I_1^k, v \in I_1^l\}, \quad k,l \in \mathbb{Z}_+.$$

The space $C^*[0,1)^2$ with the corresponding norm

$$\|f\|_\infty = \sup_{(x,y)\in[0,1)^2} |f(x,y)|$$

is the closure of the set of polynomials in the system $\{\chi_k(x)\chi_j(y)\}_{k,j=0}^{\infty}$ in the norm $\|.\|_{\infty}$. For $f \in C^*[0,1)^2$, we put, by definition,

$$\omega_{kl}(f)_{\infty} = \sup\Big\{|f(x,y)-f(x,v)-f(u,y)+f(u,v)| : \\ x,y,u,v \in [0,1),\ \rho(x,u) < \frac{1}{m_k},\ \rho(y,v) < \frac{1}{m_l}\Big\}, \quad k,l \in \mathbb{Z}_+.$$

Móricz and Veres [10] defined a two-dimensional analog of the class $A^*(\alpha)$. Their variant is a particular case of the following definition for $m_n = 2^n$. Let $\{\gamma_{ij}\}_{i,j=1}^{\infty}$ be a double sequence of positive numbers and $\alpha \geqslant 1$. If, for all $k,l \in \mathbb{N}$,

$$\left(\sum_{i=m_k}^{m_{k+1}-1}\sum_{j=m_l}^{m_{l+1}-1}\gamma_{ij}^{\alpha}\right)^{1/\alpha} \leqslant C(m_k m_l)^{(1-\alpha)/\alpha}\sum_{i=m_{k-1}}^{m_k-1}\sum_{j=m_{l-1}}^{m_l-1}\gamma_{ij} =: C(m_k m_l)^{(1-\alpha)/\alpha}\Gamma_{kl},$$

then $\{\gamma_{ij}\}_{i,j=1}^{\infty}$ belongs to the class $A^*(\alpha) = A^*(\alpha,2) = A^*(\alpha,2,\mathbf{P})$.

A double sequence $\{\gamma_{ij}\}_{i,j=1}^{\infty}$ belongs to the class $A^*(\infty,2)$, if $\gamma_{ij} > 0$ for all i,j and

$$\max_{m_k \leqslant i < m_{k+1},\ m_l \leqslant j < m_{l+1}} \gamma_{ij} \leqslant C(m_k m_l)^{-1}\Gamma_{kl}, \quad k,l \in \mathbb{N}.$$

Let $\Lambda^1 = \{\lambda_i^1\}_{i=1}^{\infty}$, $\Lambda^2 = \{\lambda_j^2\}_{j=1}^{\infty} \in \mathbb{L}$ and $p \geqslant 1$. Let $k,l \in \mathbb{Z}_+$ be fixed, and let $f(x,y)$ be bounded on $[0,1)^2$. If $i \in [1,m_k]\cap\mathbb{Z}$ and $j \in [1,m_l]\cap\mathbb{Z}$, then we denote

$$osc(f,I_{ij}^{kl}) = \sup\{|f(x,y)-f(x,v)-f(u,y)+f(u,v)| : x,u \in I_i^k, y,v \in I_j^l\},$$

$$æ_{kl}(f,p,\Lambda^1,\Lambda^2) = \sup\sum_{i=1}^{m_k}\sum_{j=1}^{m_l}\frac{osc^p(f,I_{\alpha_i,\beta_j}^{kl})}{\lambda_i^1\lambda_j^2},$$

where the supremum in the formula for $æ_{kl}$ is taken over all permutations $\{\alpha_i\}_{i=1}^{m_k}$ and $\{\beta_j\}_{j=1}^{m_l}$ of the index sets $\{1,2,\dots,m_k\}$ and $\{1,2,\dots,m_l\}$. If

$$V_{\Lambda^1,\Lambda^2}^p(f) = \sup\{æ_{kl}(f,p,\Lambda^1,\Lambda^2) : k,l \in \mathbb{Z}_+\} < \infty,$$

then f belongs to the class $(\Lambda^1,\Lambda^2)Fl^{(p)}[0,1)^2$ of functions of p-(Λ^1,Λ^2)-bounded fluctuation.

Note that, for $p=1$, the class $(\Lambda^1,\Lambda^2)Fl^{(p)}[0,1)^2$ reduce to the class $(\Lambda^1,\Lambda^2)Fl^{(1)}[0,1)^2$ of functions of (Λ^1,Λ^2)-bounded fluctuation and for $\Lambda^1=\Lambda^2=\{1\}$, the class $(\Lambda^1,\Lambda^2)Fl^{(p)}[0,1)^2$ reduce to the class $Fl^{(p)}[0,1)^2$ of functions of p-bounded fluctuation.

We prove the following results.

THEOREM 2. *Let* $f \in (\Lambda^1,\Lambda^2)Fl^{(p)}[0,1)^2 \cap C^*[0,1)^2$ $(p \geqslant 1)$, *and let* $0 < \beta < 2$, $\gamma = \{\gamma_{rs}\}_{r,s=1}^{\infty} \in A^*(2/(2-\beta),2)$ *or* $\beta = 2$, $\gamma \in A^*(\infty,2)$. *If the series*

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(m_k m_l)^{-\beta/2}\Gamma_{kl}\left(\frac{\omega_{kl}^q(f)_{\infty}}{\Lambda_{m_k}^1\Lambda_{m_l}^2}\right)^{\frac{\beta}{p+q}}$$

converges, then the series

$$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\gamma_{rs}|\hat{f}(r,s)|^{\beta} \tag{8}$$

also converges, where $q \geqslant 1$.

Proof. For $k,l \in \mathbb{Z}_+$, consider

$$g(x,y) = f\left(x \oplus m_{k+1}^{-1}, y \oplus m_{l+1}^{-1}\right) - f\left(x \oplus m_{k+1}^{-1}, y\right) - f\left(x, y \oplus m_{l+1}^{-1}\right) + f(x,y).$$

Then, for each $r,s \in \mathbb{Z}_+$, $\widehat{g}(r,s) = \left(\chi_r(m_{k+1}^{-1}) - 1\right)\left(\chi_s(m_{l+1}^{-1}) - 1\right)\widehat{f}(r,s)$.

In view of Parseval's equality, we get

$$\begin{aligned}
&\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\left|\widehat{f}(r,s)\left(\chi_r(m_{k+1}^{-1}) - 1\right)\left(\chi_s(m_{l+1}^{-1}) - 1\right)\right|^2 \\
&= \int_0^1\int_0^1 |g(x,y)|^2 dx\, dy \\
&= \int_0^1\int_0^1 |f_{uv}(x,y)|^2\, dx\, dy,
\end{aligned}$$

where

$$\begin{aligned}
f_{uv}(x,y) = &\ f\left(x \oplus u m_k^{-1}, y \oplus v m_l^{-1}\right) - f\left(x \oplus (u p_{k+1}+1) m_{k+1}^{-1}, y \oplus v m_l^{-1}\right) \\
&- f\left(x \oplus u m_k^{-1}, y \oplus (v p_{l+1}+1) m_{l+1}^{-1}\right) \\
&+ f\left(x \oplus (u p_{k+1}+1) m_{k+1}^{-1}, y \oplus (v p_{l+1}+1) m_{l+1}^{-1}\right)
\end{aligned}$$

for $0 \leqslant u < m_k$ and $0 \leqslant v < m_l$.

Since, for $r \in [m_k, m_{k+1})$, in view of (5), $|\chi_r(m_{k+1}^{-1}) - 1| \geqslant 2\sin\left(\frac{\pi}{N}\right)$, and similar inequality is valid for $|\chi_s(m_{l+1}^{-1}) - 1|$, it follows that

$$B_{kl} := \sum_{r=m_k}^{m_{k+1}-1}\sum_{s=m_l}^{m_{l+1}-1} |\widehat{f}(r,s)|^2 \leqslant C\int_0^1\int_0^1 |f_{uv}(x,y)|^2\, dx\, dy.$$

Applying Hölder's inequality on the right side of the above inequality, we obtain

$$B_{kl} \leqslant C\left(\int_0^1\int_0^1 |f_{uv}(x,y)|^{p+q}\, dx\, dy\right)^{\frac{2}{p+q}}.$$

Multiplying above inequality by $(\lambda_{u+1}^1\lambda_{v+1}^2)^{-1}$ and, after that, summing the resulting inequality over $u = 0,1,2,\ldots,m_k - 1$ and $v = 0,1,2,\ldots,m_l - 1$, we can write

$$B_{kl} \leqslant C\left(\frac{1}{(\Lambda_{m_k}^1\Lambda_{m_l}^2)^{\frac{2}{p+q}}}\left(\int_0^1\int_0^1\sum_{u=0}^{m_k-1}\sum_{v=0}^{m_l-1}\frac{|f_{uv}(x,y)|^{p+q}}{\lambda_{u+1}^1\lambda_{v+1}^2}dx\, dy\right)^{\frac{2}{p+q}}\right),$$

where $\Lambda^1_{m_k} = \sum_{u=0}^{m_k-1} \frac{1}{\lambda_{u+1}}$ and $\Lambda^2_{m_l} = \sum_{v=0}^{m_l-1} \frac{1}{\lambda_{v+1}}$.

Since $|f_{uv}(x,y)| \leqslant \omega_{kl}(f)_\infty$, we have

$$B_{kl} \leqslant C\left(\left(\frac{\omega^q_{kl}(f)_\infty}{\Lambda^1_{m_k}\Lambda^2_{m_l}}\right)^{\frac{2}{p+q}}\left(\int_0^1\int_0^1 \sum_{u=0}^{m_k-1}\sum_{v=0}^{m_l-1}\frac{|f_{uv}(x,y)|^p}{\lambda^1_{u+1}\lambda^2_{v+1}}dx\,dy\right)^{\frac{2}{p+q}}\right),$$

where

$$\sum_{u=0}^{m_k-1}\sum_{v=0}^{m_l-1}\frac{|f_{uv}(x,y)|^p}{\lambda^1_{u+1}\lambda^2_{v+1}} = O(1) \text{ as } f \in (\Lambda^1,\Lambda^2)Fl^{(p)}[0,1)^2.$$

Hence,

$$B_{kl} \leqslant C\left(\frac{\omega^q_{kl}(f)_\infty}{\Lambda^1_{m_k}\Lambda^2_{m_l}}\right)^{\frac{2}{p+q}}. \tag{9}$$

If $0<\beta<2$, then $1 = \frac{\beta}{2} + \frac{2-\beta}{2}$ and, in view of (9) and Hölder's inequality, we have

$$\begin{aligned}\sum_{r=m_k}^{m_{k+1}-1}\sum_{s=m_l}^{m_{l+1}-1}\gamma_{rs}|\hat f(r,s)|^\beta &\leqslant \left(\sum_{r=m_k}^{m_{k+1}-1}\sum_{s=m_l}^{m_{l+1}-1}\gamma^{2/(2-\beta)}\right)^{(2-\beta)/2} B^{\beta/2}_{kl}\\ &\leqslant C(m_km_l)^{-\beta/2}\Gamma_{kl}\left(\frac{\omega^q_{kl}(f)_\infty}{\Lambda^1_{m_k}\Lambda^2_{m_l}}\right)^{\frac{\beta}{p+q}}.\end{aligned}$$

Summing over $k,l \in \mathbb{Z}_+$, we have

$$\sum_{r=1}^\infty\sum_{s=1}^\infty \gamma_{rs}|\hat f(r,s)|^\beta \leqslant C\sum_{k=0}^\infty\sum_{l=0}^\infty (m_km_l)^{-\beta/2}\Gamma_{kl}\left(\frac{\omega^q_{kl}(f)_\infty}{\Lambda^1_{m_k}\Lambda^2_{m_l}}\right)^{\frac{\beta}{p+q}}.$$

For $\beta = 2$, the argument is similar. The theorem is proved. □

In the case when $p=q=1$ in Theorem 2, we have the following result proved by Volosivets and Kuznetsova [13, Theorem 5, p. 227].

COROLLARY 3. *Let* $f \in (\Lambda^1,\Lambda^2)Fl^{(1)}[0,1)^2 \cap C^*[0,1)^2$. *If the series*

$$\sum_{k=0}^\infty\sum_{l=0}^\infty (m_km_l)^{-\beta/2}\Gamma_{kl}\left(\frac{\omega_{kl}(f)_\infty}{\Lambda^1_{m_k}\Lambda^2_{m_l}}\right)^{\frac{\beta}{2}}$$

is converges, then the series (8) *also converges, where* β *and* $\gamma = \{\gamma_{rs}\}_{r,s=1}^\infty$ *are as in Theorem* 2.

In the case when $q = 2-p$ and $\Lambda^1 = \Lambda^2 = \{1\}$ in Theorem 2, we have the following result.

COROLLARY 4. *Let* $f \in Fl^{(p)}[0,1)^2 \cap C^*[0,1)^2$. *If the series*

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(m_k m_l)^{-\beta}\,\Gamma_{kl}\,(\omega_{kl}(f)_\infty)^{\frac{(2-p)\beta}{2}}$$

is converges, then the series (8) *also converges, where* β *and* $\gamma = \{\gamma_{rs}\}_{r,s=1}^{\infty}$ *are as in Theorem* 2.

Corollary 4 is the multiplicative analogue of result proved by Móricz and Veres [9, Theorem 3, p. 129].

REFERENCES

[1] N. K. BARY, *A Treatise on Trigonometric Series*, Vols. I and II, Pergamon, New York (1964).

[2] L. GOGOLADZE, *Uniform strong summational multiple trigonometric Fourier series*, Reports of the extended session at the seminar of the I.Vekua Institute of Applied Mathematics, **1**, 2 (1985), 48–51.

[3] L. GOGOLADZE AND R. MESKHIA, *On the absolute convergence of trigonometric Fourier series*, Proc. Razmadze. Math. Inst., **141**, (2006), 29–40.

[4] B. I. GOLUBOV, *On Fourier series of continuous functions with respect to a Haar system*, Izv. Akad. Nauk SSSR Ser. Mat., **28**, (6) (1964), 1271–1296.

[5] B. GOLUBOV, A. EFIMOV AND V. SKVORTSOV, *Walsh Series and Transforms: Theory and Applications*, Springer Science+Business Media Dordrecht, 1991.

[6] B. I. GOLUBOV AND S. S. VOLOSIVETS, *Generalized absolute convergence of single and double Fourier series in multiplicative systems*, Anal. Math., **38**, 2 (2012), 105–122.

[7] M. A. KUZNETSOVA, *Generalized absolute convergence of series with respect to multiplicative systems of functions of generalized bounded variation*, Izv. Saratov Univ. Math. Mech. Inform., **17**, 3 (2017), 304–312.

[8] F. MÓRICZ, *Absolute convergence of Walsh-Fourier series and related results*, Anal. Math., **36**, 4 (2010), 275–286.

[9] F. MÓRICZ AND A. VERES, *Absolute convergence of double Walsh-Fourier series and related results*, Acta Math. Hungar., **131**, 1–2 (2011), 122–137.

[10] F. MÒRICZ AND A. VERES, *Absolute convergence of multiple Fourier series revisited*, Anal. Math., **34**, (2008), 145–162.

[11] V. TSAGAREISHVILI AND G. TUTBERIDZE, *Absolute convergence factors of Lipschitz class functions for general Fourier series*, Georgian Math. J., **29**, 2 (2022), 309–315.

[12] P. L. UL'YANOV, *Series with respect to a Haar system with monotone coefficients*, Izv. Akad. Nauk. SSSR Ser. Mat., **28**, (1964), 925–950.

[13] S. S. VOLOSIVETS AND M. A. KUZNETSOVA, *Generalized absolute convergence of single and double series in multiplicative systems*, Math. Notes, **107**, 2 (2020), 217–230.

(Received May 12, 2024)

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