Journal of Classical Analysis
Volume 25, Number 2 (2025), 117–136 doi:10.7153/jca-2025-25-08

HANKEL DETERMINANTS FOR LOGARITHMIC AND LOGARITHMIC INVERSE COEFFICIENTS FOR THE CLASS $\mathscr{U}(\lambda)$

RAJIB MANDAL AND SUDIP KUMAR GUIN*

Abstract. Let $\mathscr{U}(\lambda)$ be the class of analytic functions f in the open unit disc $\mathbb{D}$ with the normalization $f(0)=0$ and $f'(0)=1$ satisfying $\left|\left(\frac{z}{f(z)}\right)^2 f'(z)-1\right|<\lambda$ for $0<\lambda\leqslant 1$. In this paper, we obtain the sharp bounds of the second Hankel determinant of logarithmic and logarithmic inverse coefficients, and the sharp bounds of the first three logarithmic coefficients for $f\in\mathscr{U}(\lambda)$.

1. Introduction, definitions and results

Let $\mathscr{A}$ denote the class of analytic functions f in the unit disk $\mathbb{D}:=\{z\in\mathbb{C}:|z|<1\}$ such that $f(0)=0$ and $f'(0)=1$. Let $\mathscr{S}$ be the subclass of functions $f\in\mathscr{A}$ that are univalent in $\mathbb{D}$ and has the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \quad \text{for } z\in\mathbb{D}. \tag{1}$$

For $0<\lambda\leqslant 1$, consider the class

$$\mathscr{U}(\lambda)=\left\{f(z)\in\mathscr{A}:\left|\left(\frac{z}{f(z)}\right)^2 f'(z)-1\right|<\lambda \text{ in } \mathbb{D}\right\}.$$

In [1, 2, 26], it is known that every $f\in\mathscr{U}(\lambda)$ is univalent in $\mathbb{D}$, and hence $\mathscr{U}(\lambda)\subset\mathscr{S}$. Set $\mathscr{U}:=\mathscr{U}(1)$ and let $\mathscr{S}^*$ denote the class of starlike functions. In [31], an interesting fact is given that neither $\mathscr{U}$ is included in $\mathscr{S}^*$ nor includes $\mathscr{S}^*$. This rare property of $\mathscr{U}$ attracts huge attention in the past decades. In [24], authors obtained that the class $\mathscr{U}(\lambda)$ is preserved under rotation, conjugation, dilation and omitted-value transformations. For quite sometimes, the class $\mathscr{U}(\lambda)$ together with its various generalizations have been studied extensively. See for example, [12, 21].

The logarithmic coefficients γ_n associated with each $f\in\mathscr{S}$ are defined by

$$F_f(z):=\log\frac{f(z)}{z}=2\sum_{n=1}^{\infty}\gamma_nz^n \quad \text{for } z\in\mathbb{D}. \tag{2}$$

Mathematics subject classification (2020): 30C45, 30C50, 30C55.
Keywords and phrases: Univalent functions, class $\mathscr{U}(\lambda)$, Hankel determinant, logarithmic coefficients.

* Corresponding author.

Paper JCA-25-08

The logarithmic coefficients γ_n play a central role in the theory of univalent functions, for intriguing perspective, we refer to the reader [11, Chapter 5]. The problem of finding the sharp bounds of $|\gamma_n|$ for the class $\mathcal{S}$ and its various subclasses are studied recently by several authors (see [3, 4, 13, 17, 34, 36]) in different contexts. In [6, 22], coefficients criteria have been established for $\mathcal{U}$, and partial sums for functions from $\mathcal{U}$ are discussed in [23].

Differentiating (2) and using (1), we obtain

$$\begin{cases} \gamma_1 = \frac{1}{2}a_2, \\ \gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right), \\ \gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right), \\ \gamma_4 = \frac{1}{2}\left(a_5 - a_2a_4 + a_2^2a_3 - \frac{1}{2}a_3^2 - \frac{1}{4}a_2^4\right), \\ \gamma_5 = \frac{1}{2}\left(a_6 - a_2a_5 - a_3a_4 + a_2a_3^2 + a_2^2a_4 - a_2^3a_3 + \frac{1}{5}a_2^5\right). \end{cases} \tag{3}$$

If $f \in \mathcal{S}$, then by the Bieberbach's theorem, we have $|a_2| \leqslant 2$ and hence $|\gamma_1| \leqslant 1$. Using the Fekete-Szegö inequality [11, Theorem 3.8] for functions in $\mathcal{S}$, we have $|\gamma_2| = \frac{1}{2}\left|a_3 - \frac{1}{2}a_2^2\right| \leqslant \frac{1}{2} + e^{-2} = 0.635\ldots$. For $n \geqslant 3$, the problem seems much harder and no significant bound for $|\gamma_n|$ is obtained when $f \in \mathcal{S}$.

Let $f \in \mathcal{A}$ and $n, q \in \mathbb{N}$. Then the Hankel determinant $H_{q,n}(f)$ of Taylor's coefficients of f is defined by

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}.$$

The Hankel determinants play an important role, for instance, in the study of singularities [10, Chapter X] and in the study of power series with integral coefficients [8]. For more general results and applications of Hankel determinants, we refer [27, 28].

In 2022, Kowalczyk and Lecko [14] proposed the study of the Hankel determinant whose entries are logarithmic coefficients of $f \in \mathcal{S}$, which is given by

$$H_{q,n}\left(F_f/2\right) := \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q} & \cdots & \gamma_{n+2(q-1)} \end{vmatrix}.$$

Also, authors [14] obtained the sharp bound of second Hankel determinant $H_{2,1}\left(F_f/2\right)$ for starlike and convex functions. The problem of computing the sharp bounds of $H_{2,1}\left(F_f/2\right)$ has been considered by many authors (see [7, 5, 14, 15, 16, 20, 30, 32, 33]) for various subclasses of $\mathcal{S}$. Inspite this, the sharp bound of Hankel determinants of logarithmic coefficients remains relatively unknown, prompting the need for extensive studies for various function classes. Suppose that $f \in \mathcal{S}$ is given by (1). Then the second Hankel determinant of $F_f/2$ is given by

$$H_{2,1}\left(F_f/2\right) := \begin{vmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_3 \end{vmatrix} = \gamma_1\gamma_3 - \gamma_2^2 = \frac{1}{4}\left(a_2a_4 - a_3^2 + \frac{1}{12}a_2^4\right). \tag{4}$$

Further, $H_{2,1}\left(F_f/2\right)$ is invariant under rotation since for $f_\theta(z) = e^{-i\theta}f(e^{i\theta}z)$, we have

$$H_{2,1}\left(F_{f_\theta}/2\right) = \frac{e^{4i\theta}}{4}\left(a_2a_4 - a_3^2 + \frac{1}{12}a_2^4\right) = e^{4i\theta}H_{2,1}\left(F_f/2\right), \;\; f \in \mathcal{U}(\lambda), \theta \in \mathbb{R}.$$

Let g be the inverse function of $f \in \mathcal{S}$, which is defined by the Taylor series expansion

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n \tag{5}$$

in a neighbourhood of the origin, where we may choose $|w| < 1/4$, as we know from Köebe's one-fourth theorem that for each univalent function f defined in $\mathbb{D}$, it's inverse f^{-1} exists at least on a disc of radius $1/4$. Löwner [19] obtained the sharp bound $|A_n| \leqslant K_n$ for each $n \in \mathbb{N}$, where $K_n = (2n)!/(n!(n+1)!)$. Since $f(f^{-1})(w) = w$, from (1) and (5), we have

$$\begin{cases} A_2 = -a_2, \\ A_3 = -a_3 + 2a_2^2, \\ A_4 = -a_4 + 5a_2a_3 - 5a_2^3, \\ A_5 = -a_5 + 6a_4a_2 - 21a_3a_2^2 + 3a_3^2 + 14a_2^4. \end{cases} \tag{6}$$

Ponnusamy *et al.* [29] proposed the notion of logarithmic inverse coefficients. The logarithmic inverse coefficients $\Gamma_n, n \in \mathbb{N}$, of f are defined by the equation

$$F_{f^{-1}}(w) := \log\frac{f^{-1}(w)}{w} = 2\sum_{n=1}^{\infty}\Gamma_n w^n, \;\; |w| < \frac{1}{4}. \tag{7}$$

Differentiating (7) and using (6), we obtain

$$\begin{cases} \Gamma_1 = -\frac{1}{2}a_2, \\ \Gamma_2 = -\frac{1}{2}a_3 + \frac{3}{4}a_2^2, \\ \Gamma_3 = -\frac{1}{2}a_4 + 2a_2a_3 - \frac{5}{3}a_2^3, \\ \Gamma_4 = -\frac{1}{2}a_5 + \frac{5}{2}a_4a_2 - \frac{15}{2}a_3a_2^2 + \frac{5}{4}a_3^2 + \frac{35}{8}a_2^4. \end{cases} \tag{8}$$

Then the second Hankel determinant of $F_{f^{-1}}/2$ is given by

$$\begin{aligned} H_{2,1}\left(F_{f^{-1}}/2\right) &= \Gamma_1\Gamma_3 - \Gamma_2^2 \\ &= \frac{1}{4}\left(A_2A_4 - A_3^2 + \frac{1}{4}A_2^4\right) \\ &= \frac{1}{48}\left(13a_2^4 - 12a_2^2a_3 - 12a_3^2 + 12a_2a_4\right). \end{aligned} \tag{9}$$

Let $\mathscr{B}_0$ be the class of Schwarz functions, *i.e.*, analytic functions $\omega : \mathbb{D} \to \mathbb{D}$ such that $\omega(0) = 0$. The function $\omega \in \mathscr{B}_0$ can be written as a power series $\omega(z) = \sum_{n=1}^{\infty} c_n z^n$. Also, for any $\phi \in \mathbb{R}$, $\omega(z) \in \mathscr{B}_0$ if and only if $\omega(ze^{i\phi}) \in \mathscr{B}_0$.

The main aim of this paper is to establish the sharp bounds of the logarithmic coefficients $\gamma_1, \gamma_2, \gamma_3$, the second Hankel determinants $\left|H_{2,1}\left(F_f/2\right)\right|$ and $\left|H_{2,1}\left(F_{f^{-1}}/2\right)\right|$ for functions belonging to $\mathscr{U}(\lambda)$, where $0 < \lambda \leqslant 1$.

2. Main results

The following four results establish the sharp bounds of $\left|H_{2,1}\left(F_f/2\right)\right|$ and $\left|H_{2,1}\left(F_{f^{-1}}/2\right)\right|$ for functions $f \in \mathscr{U}(\lambda)$, for some $\lambda \in (0,1]$.

THEOREM 1. *Let $f \in \mathscr{U}(\lambda)$ be of the form (1). Then for $\left|(a_2^2 - a_3)/\lambda\right| = 1$ and $0 < \lambda \leqslant 0.225906\ldots$,*

$$\left|H_{2,1}\left(F_f/2\right)\right| \leqslant \frac{(1+\lambda)^4 - 12\lambda^2}{48}.$$

The inequality is sharp.

THEOREM 2. *Let $f \in \mathscr{U}(\lambda)$ be of the form (1). Then for $\left|2(a_2^2 - a_3)\right| = 1$ and $\lambda = \frac{1}{2}$,*

$$\left|H_{2,1}\left(F_f/2\right)\right| \leqslant \frac{\sqrt{11.012265\ldots}}{48}.$$

The inequality is sharp.

THEOREM 3. *Let $f \in \mathscr{U}(\lambda)$ be of the form (1). Then for $\left|a_2^2 - a_3\right| = 1$ and $\lambda = 1$,*

$$\left|H_{2,1}\left(F_f/2\right)\right| \leqslant \frac{1}{4}.$$

The inequality is sharp.

THEOREM 4. *Let* $f \in \mathscr{U}(\lambda)$ *be of the form* (1). *Then for* $\left|(a_2^2 - a_3)/\lambda\right| = 1$ *and* $0 < \lambda \leqslant 1$,

$$\left|H_{2,1}\left(F_{f^{-1}}/2\right)\right| \leqslant \frac{(1+\lambda)^4 + 12\lambda(1+\lambda)^2 - 12\lambda^2}{48}.$$

The inequality is sharp.

In the following result, we establish the sharp bounds of the logarithmic coefficients γ_1, γ_2 and γ_3 for functions $f \in \mathscr{U}(\lambda)$, $0 < \lambda \leqslant 1$.

THEOREM 5. *Let* $f \in \mathscr{U}(\lambda)$ *be of the form* (1). *Then for* $(a_2^2 - a_3)/\lambda = 1$ *and* $0 < \lambda \leqslant 1$,

$$|\gamma_1| \leqslant \frac{1+\lambda}{2}, \quad |\gamma_2| \leqslant \frac{1+\lambda^2}{4} \quad \text{and} \quad |\gamma_3| \leqslant \frac{1+\lambda^3}{6}.$$

Each inequality is sharp.

3. Lemmas

The following lemmas are necessary for this paper and will be used to prove the main results.

LEMMA 1. [9, 18, 25, 35] *For each function* $f \in \mathscr{U}(\lambda)$, $0 < \lambda \leqslant 1$, *there exists function* $\omega(z) = c_2 z + c_3 z^2 + c_4 z^3 + \cdots$, *analytic in* $\mathbb{D}$ *such that* $|\omega(z)| \leqslant |z| < 1$, *and* $|\omega'(z)| \leqslant 1$ *for all* $z \in \mathbb{D}$, *with*

$$\frac{z}{f(z)} = 1 - a_2 z + \lambda z \omega(z) = 1 - a_2 z + \lambda \omega_1(z), \tag{10}$$

where $\omega_1(z) = \sum_{n=2}^{\infty} c_n z^n \in \mathscr{B}_0$.

Additionally, $|z\omega'(z)| = |c_2 z + 2c_3 z^2 + 3c_4 z^3 + \cdots| < 1$ *for all* $z \in \mathbb{D}$ *gives*

$$\begin{cases} |c_2| \leqslant 1, \\ |c_3| \leqslant \frac{1}{2}(1 - |c_2|^2), \\ |c_4| \leqslant \frac{1}{3}\left(1 - |c_2|^2 - \frac{4|c_3|^2}{1+|c_2|}\right), \\ |c_5| \leqslant \frac{1}{4}\left(1 - |c_2|^2 - 4|c_3|^2\right). \end{cases}$$

LEMMA 2. [24, 35] *Let* $f \in \mathscr{U}(\lambda)$ *be such that* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, *where* $0 < \lambda \leqslant 1$. *Then* $|a_2| \leqslant 1 + \lambda$. *If* $|a_2| = 1 + \lambda$, *then f must be of the form*

$$f_\phi(z) = \frac{z}{1 - (1+\lambda)e^{i\phi} z + \lambda e^{2i\phi} z^2},$$

for some $\phi \in [0, 2\pi)$.

LEMMA 3. *Let $f(z)$ be of the form*

$$f(z) = \frac{z}{1 - a_2 z + \lambda z^2},$$

where $a_2 = le^{i\theta}$ with $|a_2| = l$ and $\theta \in \mathbb{R}$. Then for $1-\lambda < l \leqslant 1+\lambda$, $f(z) \in \mathscr{U}(\lambda)$ if and only if

$$\frac{l^2(1+\lambda^2) - (1-\lambda^2)^2}{2\lambda l^2} \leqslant \cos 2\theta \leqslant 1.$$

Also, for $l \leqslant 1-\lambda$, $f(z) \in \mathscr{U}(\lambda)$.

Proof. Following the proof of Obradović *et al.* [24, Theorem 4], let g be a function given by

$$\frac{z}{g(z)} = 1 - a_2 z + \lambda z^2 = 1 - (1 + \lambda e^{i\tau})z + \lambda z^2, \tag{11}$$

where $a_2 = 1 + \lambda e^{i\tau}$.

It is clear that functions of the type g given by (11) belong to $\mathscr{U}(\lambda)$ if and only if

$$0 \neq 1 - (1 + \lambda e^{i\tau})z + \lambda z^2, \;\; z \in \mathbb{D}. \tag{12}$$

Let us suppose

$$1 + \lambda e^{i\tau} = |1 + \lambda e^{i\tau}| e^{i\theta}.$$

Then it follows that

$$\begin{aligned} &(1 + \lambda e^{i\tau})z - \lambda z^2 \\ &= |1 + \lambda e^{i\tau}| e^{i\theta} z - \lambda z^2 \\ &= e^{i2\theta}\left(|1 + \lambda e^{i\tau}| e^{-i\theta} z - \lambda e^{-i2\theta} z^2\right). \end{aligned}$$

Thus, (12) holds if and only if

$$e^{-i2\theta} \neq \left(|1 + \lambda e^{i\tau}| u - \lambda u^2\right), \;\; u \in \mathbb{D}.$$

Next we assume $l = |a_2| = |1 + \lambda e^{i\tau}| \in [1-\lambda, 1+\lambda]$, $u = e^{i\alpha}$ and $x + iy = le^{i\alpha} - \lambda e^{2i\alpha}$. Then we obtain

$$x - \lambda = \cos\alpha(l - 2\lambda\cos\alpha) \;\text{ and }\; y = \sin\alpha(l - 2\lambda\cos\alpha). \tag{13}$$

This is the parametric equation of a Limaçon (see Figure 1 for the graph of some Limaçons parameterized by (13) for different values of λ and l). The implicit equation of (13) is given by

$$(x^2 + y^2 - \lambda^2)^2 = l^2(x^2 + y^2 + \lambda^2 - 2\lambda x). \tag{14}$$

The intersection points (x,y) of the unit circle and (14) is

$$x = \frac{l^2(1+\lambda^2)-(1-\lambda^2)^2}{2\lambda l^2}.$$

Hence, for

$$\frac{l^2(1+\lambda^2)-(1-\lambda^2)^2}{2\lambda l^2} \leqslant \cos 2\theta \leqslant 1,$$

the functions g defined by (11) belong to $\mathscr{U}(\lambda)$.

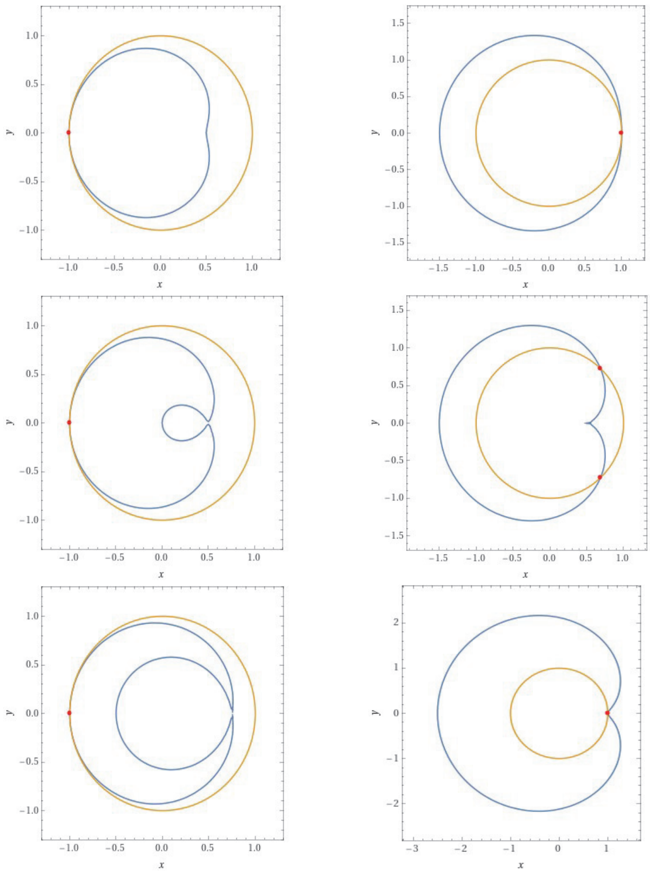

Figure 1: *The graph of some Limaçons parameterized by* (14) *for some different values of* λ *and* l *(* $(\lambda,l) = (0.25,0.75),(0.25,1.25),(0.5,0.5),(0.5,1),(0.75,0.25),(0.75,1.75)$ *respectively) together with unit circle.*

Observe that for $l = 1+\lambda$, $\theta = 0$ is the only one that produces a function belonging to $\mathscr{U}(\lambda)$ in (11), whereas for $l = 1-\lambda$, all functions defined by (11) are in the class $\mathscr{U}(\lambda)$.

Also, the condition $1-a_2z+\lambda z^2 \neq 0$ is satisfied if $|a_2| \leqslant 1-\lambda$, as

$$|1-a_2z+\lambda z^2| \geqslant 1-|\lambda z^2|-|a_2z| > 0 \text{ for } |a_2| \leqslant 1-\lambda.$$

This completes the proof. □

REMARK 1. From (10), we have for $\omega_1(z) = \sum_{n=2}^{\infty} c_n z^n$,

$$f(z) = \frac{z}{1-a_2z+\lambda\omega_1(z)}. \tag{15}$$

Comparing the coefficients on both sides of (15), we get

$$\begin{cases} a_3 = a_2^2 - c_2\lambda, \\ a_4 = a_2^3 - 2\lambda a_2c_2 - c_3\lambda, \\ a_5 = a_2^4 - 3\lambda a_2^2c_2 + \lambda^2c_2^2 - 2\lambda a_2c_3 - \lambda c_4, \\ a_6 = a_2^5 - 4\lambda a_2^3c_2 - 3\lambda a_2^2c_3 + 3\lambda^2a_2c_2^2 + 2\lambda^2c_2c_3 - 2\lambda a_2c_4 - \lambda c_5. \end{cases} \tag{16}$$

4. Proof of the main results

Proof of Theorem 1. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be a function in $\mathscr{U}(\lambda)$. Then using Lemma 1, we have

$$\frac{z}{f(z)} = 1-a_2z+\lambda\omega_1(z),$$

where $\omega_1(z) = \sum_{n=2}^{\infty} c_n z^n$.

Now substituting the expressions for a_3 and a_4 from (16) in (4), we get

$$\begin{aligned} 48H_{2,1}\left(F_f/2\right) &= 12a_2a_4 - 12a_3^2 + a_2^4 \\ &= 12a_2\left(a_2^3 - 2\lambda a_2c_2 - c_3\lambda\right) - 12\left(a_2^2 - c_2\lambda\right)^2 + a_2^4 \\ &= a_2^4 - 12a_2\lambda c_3 - 12c_2^2\lambda^2. \end{aligned}$$

It follows from Lemma 1 that

$$\begin{aligned} 48\left|H_{2,1}\left(F_f/2\right)\right| &= \left|a_2^4 - 12a_2\lambda c_3 - 12c_2^2\lambda^2\right| \\ &\leqslant \left|a_2^4 - 12c_2^2\lambda^2\right| + 12\lambda|a_2||c_3| \\ &\leqslant \left|a_2^4 - 12c_2^2\lambda^2\right| + 6\lambda|a_2|(1-|c_2|^2). \end{aligned}$$

In view of the first equation of (16), we have $|c_2| = |(a_2^2 - a_3)/\lambda| = 1$, and since $\mathscr{U}(\lambda)$ and $H_{2,1}\left(F_f/2\right)$ are invariant under rotations, we may assume $c_2 = 1$. Hence

$$48\left|H_{2,1}\left(F_f/2\right)\right| \leqslant |a_2^4 - 12\lambda^2|. \tag{17}$$

Let us assume $a_2 = le^{i\theta}$, where $|a_2| = l$ and $\theta \in \mathbb{R}$. Then

$$\begin{aligned} 48\left|H_{2,1}\left(F_f/2\right)\right| &\leqslant |l^4 e^{4i\theta} - 12\lambda^2| \\ &= \sqrt{l^8 - 24 l^4 \lambda^2 \cos 4\theta + 144\lambda^4} \\ &= \sqrt{l^8 + 24 l^4 \lambda^2 + 144\lambda^4 - 48 l^4 \lambda^2 \cos^2 2\theta} \\ &= \sqrt{g(l, \cos 2\theta)}, \end{aligned}$$

where

$$g(x,y) = x^8 + 24\lambda^2 x^4 - 48\lambda^2 x^4 y^2 + 144\lambda^4.$$

In view of Lemma 3, now we will find out the maximum value of $g(x,y)$ in the region

$$\Omega = \left\{ (x,y) : 1-\lambda < x \leqslant 1+\lambda, \frac{x^2(1+\lambda^2) - (1-\lambda^2)^2}{2\lambda x^2} \leqslant y \leqslant 1 \right\}.$$

Note that $\frac{\partial}{\partial x}\{g(x,y)\} = 0$ and $\frac{\partial}{\partial y}\{g(x,y)\} = 0$ together imply that it has no solution in the interior of Ω. Hence $g(x,y)$ attains its maximum on the boundary of Ω. Thus for $-1 \leqslant y \leqslant 1$, on the boundary of Ω, we have

$$g(1-\lambda, y) = (1-\lambda)^8 + 24\lambda^2(1-\lambda)^4 - 48(1-\lambda)^4\lambda^2 y^2 + 144\lambda^4.$$

Note that for $\lambda = 1$, $g(1-\lambda, y) = 144\lambda^4 = 144$. Also, for $0 < \lambda < 1$, $g'(1-\lambda, y) = -96(1-\lambda)^4\lambda^2 y = 0$ implies $y = 0$. Now $g''(1-\lambda, y) = -96(1-\lambda)^4\lambda^2 < 0$ at $y = 0$, which shows $g(1-\lambda, y)$ attains its maximum at $y = 0$. Thus,

$$\begin{aligned} g(1-\lambda, y) &\leqslant (1-\lambda)^8 + 24\lambda^2(1-\lambda)^4 + 144\lambda^4 \\ &= ((1-\lambda)^4 + 12\lambda^2)^2. \end{aligned} \tag{18}$$

Now for $1-\lambda < x \leqslant 1+\lambda$, on the boundary of Ω, we have

$$g(x,1) = x^8 - 24\lambda^2 x^4 + 144\lambda^4 = (x^4 - 12\lambda^2)^2 = h_\lambda^2(x), \text{ say,}$$

where $h_\lambda(x) = x^4 - 12\lambda^2$. Then $h'_\lambda(x) = 4x^3 \geqslant 0$ for $1-\lambda < x \leqslant 1+\lambda$ implies $h_\lambda(x)$ is an increasing function for $1-\lambda < x \leqslant 1+\lambda$. Hence for $1-\lambda < x \leqslant 1+\lambda$,

$$\begin{aligned} &\max h_\lambda(x) = (1+\lambda)^4 - 12\lambda^2 \\ \text{and } &\min h_\lambda(x) = (1-\lambda)^4 - 12\lambda^2. \end{aligned}$$

Note that

$$\begin{aligned} &((1+\lambda)^4 - 12\lambda^2)^2 \geqslant ((1-\lambda)^4 - 12\lambda^2)^2, \text{ whenever } 0 < \lambda \leqslant \sqrt{2}-1 \\ \text{and } &((1+\lambda)^4 - 12\lambda^2)^2 < ((1-\lambda)^4 - 12\lambda^2)^2, \text{ whenever } \sqrt{2}-1 < \lambda \leqslant 1. \end{aligned}$$

Hence

$$\max g(x,1) = \begin{cases} ((1+\lambda)^4 - 12\lambda^2)^2 & \text{for } 0 < \lambda \leqslant \sqrt{2}-1, \\ ((1-\lambda)^4 - 12\lambda^2)^2 & \text{for } \sqrt{2}-1 < \lambda \leqslant 1. \end{cases} \tag{19}$$

Next for $1-\lambda < x \leqslant 1+\lambda$, on the boundary of Ω, we have

$$\begin{aligned} & g\left(x, \frac{x^2(1+\lambda^2)-(1-\lambda^2)^2}{2\lambda x^2}\right) \\ & = x^8+24\lambda^2x^4-12\left\{x^2(1+\lambda^2)-(1-\lambda^2)^2\right\}^2+144\lambda^4 \\ & = t_\lambda(x), \text{ say.} \end{aligned} \tag{20}$$

Then

$$\frac{\partial t_\lambda(x)}{\partial x} = 8x(6-6\lambda^2+6\lambda^6-6x^2+x^6-6\lambda^4(1+x^2)) = q_\lambda(x), \text{ say.}$$

We wish to find out the range of λ for which $q_\lambda(x) \geqslant 0$. Observe that $q'_\lambda(x) = 0$ implies $x^2 = \sqrt{2+2\lambda^4}$. Clearly $q''_\lambda(x) > 0$ at $x^2 = \sqrt{2+2\lambda^4}$. Hence the minimum value of $q_\lambda(x)$, which attains at $x^2 = \sqrt{2+2\lambda^4}$ is

$$6-6\lambda^2+6\lambda^6-4\sqrt{2}\sqrt{1+\lambda^4}-2\lambda^4(3+2\sqrt{2}\sqrt{1+\lambda^4}),$$

which will be $\geqslant 0$ if $0<\lambda \leqslant 0.225906\ldots$. Thus,

$$\frac{\partial t_\lambda(x)}{\partial x} \geqslant 0$$

for $1-\lambda < x \leqslant 1+\lambda$ and $0<\lambda \leqslant 0.225906\ldots$. Hence, $t_\lambda(x)$ being an increasing function for $1-\lambda < x \leqslant 1+\lambda$ and $0<\lambda \leqslant 0.225906\ldots$, attains its maximum at $x = 1+\lambda$. Thus for $0<\lambda \leqslant 0.225906\ldots$, we have

$$\begin{aligned} & g\left(x, \frac{x^2(1+\lambda^2)-(1-\lambda^2)^2}{2\lambda x^2}\right) \\ & \leqslant (1+\lambda)^8+24\lambda^2(1+\lambda)^4-12\left\{(1+\lambda)^2(1+\lambda^2)-(1-\lambda^2)^2\right\}^2+144\lambda^4 \\ & = \left\{(1+\lambda)^4-12\lambda^2\right\}^2. \end{aligned} \tag{21}$$

Therefore for $0<\lambda \leqslant 0.225906\ldots$ and $1-\lambda < |a_2| \leqslant 1+\lambda$, combining (18), (19) and (21), we get

$$g(x,y) \leqslant \left\{(1+\lambda)^4-12\lambda^2\right\}^2,$$

which implies

$$\left|H_{2,1}\left(F_f/2\right)\right| \leqslant \frac{(1+\lambda)^4-12\lambda^2}{48}. \tag{22}$$

From (17), now for $|a_2| \leqslant 1-\lambda$, we have

$$\begin{aligned} 48\left|H_{2,1}\left(F_f/2\right)\right| & \leqslant |a_2^4-12\lambda^2| \\ & \leqslant |a_2|^4+12\lambda^2 \\ & \leqslant (1-\lambda)^4+12\lambda^2 \end{aligned} \tag{23}$$

for $0<\lambda\leqslant 1$.

Hence (22) and (23) imply that for $0<\lambda\leqslant 0.225906\ldots$,

$$\left|H_{2,1}\left(F_f/2\right)\right| \leqslant \tfrac{(1+\lambda)^4-12\lambda^2}{48}.$$

To prove the sharpness, we consider the functions $f(z)=z+\sum_{n=2}^{\infty}a_nz^n=z/(1-(1+\lambda)z+\lambda z^2)$ for $0<\lambda\leqslant 0.225906\ldots$. It is easy to see that $f(z)\in\mathscr{U}(\lambda)$ and in view of $a_2=1+\lambda, a_3=1+\lambda+\lambda^2, a_4=1+\lambda+\lambda^2+\lambda^3$, we get

$$\left|H_{2,1}\left(F_f/2\right)\right| = \frac{(1+\lambda)^4-12\lambda^2}{48}.$$

This completes the proof. □

Proof of Theorem 2. We consider $\lambda=1/2$ and $1/2<|a_2|\leqslant 3/2$. Then as in the proof of Theorem 1, we obtain from (18) on the boundary of Ω that

$$g\left(\frac{1}{2},y\right)\leqslant\left(\frac{49}{16}\right)^2=(3.0625)^2 \text{ for } -1\leqslant y\leqslant 1.$$

Now on the boundary of Ω, (19) implies

$$g(x,1)\leqslant\left(\frac{47}{16}\right)^2=(2.9375)^2 \text{ for } 1/2<x\leqslant 3/2.$$

Next on the boundary of Ω, for $1/2<x\leqslant 3/2$, (20) implies

$$\begin{aligned} & g\left(x,\frac{x^2(1+\lambda^2)-(1-\lambda^2)^2}{2\lambda x^2}\right)\\ &= x^8+24\lambda^2x^4-12\left\{x^2(1+\lambda^2)-(1-\lambda^2)^2\right\}^2+144\lambda^4\\ &= x^8-\frac{51}{4}x^4+\frac{135}{8}x^2+\frac{333}{64}\\ &= h(x),\ \text{say.}\end{aligned}$$

Then for $1/2<x\leqslant 3/2$, $h'(x)=8x^7-51x^3+\frac{135}{4}x=0$ implies $x=0.84877\ldots$, $1.44443\ldots$. Now as $h''(0.84877\ldots)<0$ and $h''(1.44443\ldots)>0$, $h(x)$ is an increasing function for $1/2<x\leqslant 0.84877\ldots$ and $1.44443\ldots<x\leqslant 3/2$, and is an decreasing function for $0.84877\ldots<x\leqslant 1.44443\ldots$. Clearly, $h(x)$ attains its maximum at $x=0.84877\ldots$. Thus,

$$g\left(x,\frac{x^2(1+\lambda^2)-(1-\lambda^2)^2}{2\lambda x^2}\right)\leqslant 11.012265\ldots.$$

Hence combining the above cases, for $1/2<|a_2|\leqslant 3/2$, we have

$$\left|H_{2,1}\left(F_f/2\right)\right|\leqslant\tfrac{\sqrt{11.012265\ldots}}{48}.$$

Now for $|a_2| \leqslant 1/2$, using (23), we get

$$48\left|H_{2,1}\left(F_f/2\right)\right| \leqslant \frac{49}{16} = 3.0625.$$

Therefore, combining all the cases, we obtain

$$\left|H_{2,1}\left(F_f/2\right)\right| \leqslant \frac{\sqrt{11.012265\ldots}}{48} \text{ for } \lambda = \frac{1}{2}.$$

To prove the sharpness, we consider the functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n = z/(1 - a_2 z + \frac{1}{2}z^2)$, where $a_2 = le^{i\theta}$ with $l \approx 0.84877$ and $\cos 2\theta \approx 0.46919$. Then in view of Lemma 3, $f(z) \in \mathscr{U}(\lambda)$. Also,

$$\left|H_{2,1}\left(F_f/2\right)\right| \approx \frac{\sqrt{11.012265\ldots}}{48}.$$

This completes the proof. $\square$

Proof of Theorem 3. We consider $\lambda = 1$ and $0 < |a_2| \leqslant 2$. Then as in the proof of Theorem 1, we obtain from (18) on the boundary of Ω that

$$g(0,y) \leqslant 144 \text{ for } -1 \leqslant y \leqslant 1.$$

Now on the boundary of Ω, (19) implies

$$g(x,1) \leqslant 144 \text{ for } 0 < x \leqslant 2.$$

Next on the boundary of Ω, for $0 < x \leqslant 2$, (20) implies

$$\begin{aligned} & g\left(x, \frac{x^2(1+\lambda^2)-(1-\lambda^2)^2}{2\lambda x^2}\right) \\ &= x^8 + 24\lambda^2 x^4 + 144\lambda^4 - 12\left\{x^2(1+\lambda^2)-(1-\lambda^2)^2\right\}^2 \\ &= (x^4-12)^2 \leqslant 144. \end{aligned}$$

Hence combining the above cases, for $0 < |a_2| \leqslant 2$, we have

$$\left|H_{2,1}\left(F_f/2\right)\right| \leqslant \frac{1}{4}.$$

Now for $|a_2| = 0$, using (23), we get

$$48\left|H_{2,1}\left(F_f/2\right)\right| \leqslant 12.$$

Thus combining all the cases, for $\lambda = 1$, we obtain

$$\left|H_{2,1}\left(F_f/2\right)\right| \leqslant \frac{1}{4}.$$

To prove the sharpness, we consider the functions $f(z) = z/(1+z^2)$. It is easy to see that $f(z) \in \mathcal{U}(\lambda)$ and

$$\left|H_{2,1}\left(F_f/2\right)\right| = \frac{1}{4}.$$

This completes the proof. $\square$

Proof of Theorem 4. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be a function in $\mathcal{U}(\lambda)$ given by (10). Then using (16) in (9), we have

$$\begin{aligned}
&48H_{2,1}\left(F_{f^{-1}}/2\right)\\
&= 13a_2^4 - 12a_2^2a_3 - 12a_3^2 + 12a_2a_4\\
&= 13a_2^4 - 12a_2^2(a_2^2 - c_2\lambda) - 12(a_2^2 - c_2\lambda)^2 + 12a_2(a_2^3 - c_3\lambda - 2\lambda a_2c_2)\\
&= a_2^4 + 12\lambda a_2^2c_2 - 12\lambda a_2c_3 - 12\lambda^2c_2^2.
\end{aligned}$$

Then it follows from Lemma 1 that,

$$\begin{aligned}
48\left|H_{2,1}\left(F_{f^{-1}}/2\right)\right| &\leqslant \left|a_2^4 + 12\lambda a_2^2c_2 - 12\lambda^2c_2^2\right| + 12\lambda|a_2||c_3|\\
&\leqslant \left|a_2^4 + 12\lambda a_2^2c_2 - 12\lambda^2c_2^2\right| + 6\lambda|a_2|(1-|c_2|^2).
\end{aligned}$$

In view of the first equation of (16), we have $|c_2| = |(a_2^2 - a_3)/\lambda| = 1$, and since $\mathcal{U}(\lambda)$ and $H_{2,1}\left(F_{f^{-1}}/2\right)$ are invariant under rotations, we may assume $c_2 = 1$. Hence

$$48\left|H_{2,1}\left(F_{f^{-1}}/2\right)\right| \leqslant \left|a_2^4 + 12\lambda a_2^2 - 12\lambda^2\right|. \tag{24}$$

Let us assume $a_2 = le^{i\theta}$, where $|a_2| = l$ and $\theta \in \mathbb{R}$. Then

$$\begin{aligned}
&48\left|H_{2,1}\left(F_{f^{-1}}/2\right)\right|\\
&\leqslant |l^4e^{4i\theta} + 12\lambda l^2e^{2i\theta} - 12\lambda^2|\\
&= \sqrt{l^8 + 24\lambda l^6\cos 2\theta - 24\lambda^2l^4\cos 4\theta - 288\lambda^3l^2\cos 2\theta + 144\lambda^2l^4 + 144\lambda^4}\\
&= \sqrt{g(l, \cos 2\theta)},
\end{aligned}$$

where

$$g(x,y) = x^8 + 24\lambda x^6y - 48\lambda^2x^4y^2 - 288\lambda^3x^2y + 168\lambda^2x^4 + 144\lambda^4.$$

In view of Lemma 3, now we will find out the maximum value of $g(x,y)$ in the region

$$\Omega = \left\{(x,y) : 1-\lambda < x \leqslant 1+\lambda, \frac{x^2(1+\lambda^2)-(1-\lambda^2)^2}{2\lambda x^2} \leqslant y \leqslant 1\right\}.$$

Note that $\frac{\partial}{\partial x}\{g(x,y)\}=0$ and $\frac{\partial}{\partial y}\{g(x,y)\}=0$ together imply it has no solution in the interior of Ω. Thus for $-1\leqslant y\leqslant 1$, on the boundary of Ω, we have

$$\begin{aligned} g(1-\lambda,y) &= (1-\lambda)^8+24\lambda(1-\lambda)^6y-48\lambda^2(1-\lambda)^4y^2\\ &\quad -288\lambda^3(1-\lambda)^2y+168\lambda^2(1-\lambda)^4+144\lambda^4\\ &= h(y),\ \text{say.} \end{aligned}$$

Then

$$h'(y)=24\lambda(1-\lambda)^6-96\lambda^2(1-\lambda)^4y-288\lambda^3(1-\lambda)^2\geqslant 0$$

whenever $0<\lambda\leqslant 4-\sqrt{15}$. Also, for $4-\sqrt{15}<\lambda\leqslant 2-\sqrt{3}$, $h(y)$ attains its maximum at $y=\frac{1-4\lambda-6\lambda^2-4\lambda^3+c^4}{4\lambda(1-\lambda)^2}$. Furthermore, $h'(y)\leqslant 0$ whenever $2-\sqrt{3}<\lambda\leqslant 1$. Hence

$$g(1-\lambda,y)\leqslant\begin{cases}(1+8\lambda-30\lambda^2+8\lambda^3+\lambda^4)^2 & \text{for } 0<\lambda\leqslant 4-\sqrt{15},\\ 4(1-4\lambda+18\lambda^2-4\lambda^3+\lambda^4)^2 & \text{for } 4-\sqrt{15}<\lambda\leqslant 2-\sqrt{3},\\ (1-16\lambda+18\lambda^2-16\lambda^3+\lambda^4)^2 & \text{for } 2-\sqrt{3}<\lambda\leqslant 1.\end{cases}\tag{25}$$

Now for $1-\lambda<x\leqslant 1+\lambda$, on the boundary of Ω, we have

$$\begin{aligned} g(x,1) &= x^8+24\lambda x^6-48\lambda^2x^4-288\lambda^3x^2+168\lambda^2x^4+144\lambda^4\\ &= (x^4+12\lambda x^2-12\lambda^2)^2, \end{aligned}$$

which attains its maximum at $x=1+\lambda$. Hence

$$g(x,1)\leqslant((1+\lambda)^4+12\lambda(1+\lambda)^2-12\lambda^2)^2.\tag{26}$$

Next for $1-\lambda<x\leqslant 1+\lambda$, on the boundary of Ω, we have

$$\begin{aligned} & g\left(x,\frac{x^2(1+\lambda^2)-(1-\lambda^2)^2}{2\lambda x^2}\right)\\ &= x^8+12x^4\left\{x^2(1+\lambda^2)-(1-\lambda^2)^2\right\}-12\left\{x^2(1+\lambda^2)-(1-\lambda^2)^2\right\}^2\\ &\quad -144\lambda^2\left\{x^2(1+\lambda^2)-(1-\lambda^2)^2\right\}+168\lambda^2x^4+144\lambda^4, \end{aligned}$$

which attains its maximum at $x=1+\lambda$. Hence

$$\begin{aligned} & g\left(x,\frac{x^2(1+\lambda^2)-(1-\lambda^2)^2}{2\lambda x^2}\right)\\ &\leqslant (\lambda^4+16\lambda^3+18\lambda^2+16\lambda+1)^2\\ &= \left\{(1+\lambda)^4+12\lambda(1+\lambda)^2-12\lambda^2\right\}^2. \end{aligned}\tag{27}$$

Therefore for $1-\lambda<|a_2|\leqslant 1+\lambda$ and $0<\lambda\leqslant 1$, combining (25), (26) and (27), we get

$$g(x,y)\leqslant\left\{(1+\lambda)^4+12\lambda(1+\lambda)^2-12\lambda^2\right\}^2,$$

which shows that

$$\left|H_{2,1}\left(F_{f^{-1}}/2\right)\right| \leqslant \frac{(1+\lambda)^4+12\lambda(1+\lambda)^2-12\lambda^2}{48}. \tag{28}$$

From (24), now for $|a_2| \leqslant 1-\lambda$, we have

$$\begin{aligned} 48\left|H_{2,1}\left(F_{f^{-1}}/2\right)\right| &\leqslant \left|a_2^4+12\lambda a_2^2-12\lambda^2\right| \\ &\leqslant |a_2|^4+12\lambda|a_2|^2+12\lambda^2 \\ &\leqslant (1-\lambda)^4+12\lambda(1-\lambda)^2+12\lambda^2 \text{ for } 0<\lambda\leqslant 1. \end{aligned} \tag{29}$$

Hence (28) and (29) imply that

$$\left|H_{2,1}\left(F_{f^{-1}}/2\right)\right| \leqslant \frac{(1+\lambda)^4+12\lambda(1+\lambda)^2-12\lambda^2}{48} \text{ for } 0<\lambda\leqslant 1.$$

In order to prove the sharpness, we consider the function $f(z)=z/(1-(1+\lambda)z+\lambda z^2)$ for $0<\lambda\leqslant 1$. Then $f(z)\in\mathscr{U}(\lambda)$ and it is easy to see that

$$\left|H_{2,1}\left(F_{f^{-1}}/2\right)\right| = \frac{(1+\lambda)^4+12\lambda(1+\lambda)^2-12\lambda^2}{48}.$$

This completes the proof. □

Proof of Theorem 5. Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ be a function in $\mathscr{U}(\lambda)$. We consider the following cases to determine the sharp bounds for γ_1,γ_2 and γ_3 respectively.

Case 1. Applying Lemma 2 to the first equation of (3), we get

$$|\gamma_1| = \frac{1}{2}|a_2| \leqslant \frac{1+\lambda}{2}.$$

In order to prove the sharpness of the bound, we consider the function $f(z)=z/(1-(1+\lambda)z+\lambda z^2)$ for $0<\lambda\leqslant 1$. Then $f(z)\in\mathscr{U}(\lambda)$ and it is easy to check that $|\gamma_1|=(1+\lambda)/2$.

Case 2. The second equation of (3) and the first equation of (16) yield

$$|\gamma_2| = \frac{1}{2}\left|a_3-\frac{1}{2}a_2^2\right| = \frac{1}{2}\left|a_2^2-c_2\lambda-\frac{1}{2}a_2^2\right| = \frac{1}{2}\left|\frac{1}{2}a_2^2-\lambda c_2\right|.$$

In view of the first equation of (16), we have $c_2=(a_2^2-a_3)/\lambda=1$. Hence

$$|\gamma_2| \leqslant \frac{1}{2}\left|\frac{1}{2}a_2^2-\lambda\right|.$$

Let us assume $a_2 = le^{i\theta}$, where $|a_2| = l$ and $\theta \in \mathbb{R}$. Then, we have

$$\begin{aligned}|\gamma_2| &\leqslant \frac{1}{2}\left|\frac{1}{2}l^2e^{2i\theta} - \lambda\right| \\ &= \frac{1}{2}\left|\frac{1}{2}l^2\cos 2\theta + i\frac{1}{2}l^2\sin 2\theta - \lambda\right| \\ &= \frac{1}{2}\sqrt{\frac{1}{4}l^4 - \lambda l^2\cos 2\theta + \lambda^2} \\ &= \frac{1}{2}\sqrt{g(l,\cos 2\theta)},\end{aligned}$$

where

$$g(x,y) = \frac{1}{4}x^4 - \lambda x^2 y + \lambda^2.$$

In view of Lemma 3, now we will find out the maximum value of $g(x,y)$ in the region

$$\Omega = \left\{(x,y) : 1-\lambda < x \leqslant 1+\lambda, \frac{x^2(1+\lambda^2)-(1-\lambda^2)^2}{2\lambda x^2} \leqslant y \leqslant 1\right\}.$$

Note that $\frac{\partial}{\partial x}\{g(x,y)\} = 0$ and $\frac{\partial}{\partial y}\{g(x,y)\} = 0$ together imply it has no solution in the interior of Ω. Thus for $-1 \leqslant y \leqslant 1$, on the boundary of Ω, we have

$$g(1-\lambda,y) = \frac{1}{4}(1-\lambda)^4 - \lambda(1-\lambda)^2 y + \lambda^2,$$

which attains its maximum at $y = -1$. Hence

$$g(1-\lambda,y) \leqslant \frac{1}{4}(1-\lambda)^4 + \lambda(1-\lambda)^2 + \lambda^2 = \frac{1}{4}\left(1+\lambda^2\right)^2 \quad \text{for} \quad -1 \leqslant y \leqslant 1.$$

Now for $1-\lambda < x \leqslant 1+\lambda$, on the boundary of Ω, we have

$$g(x,1) = \frac{1}{4}x^4 - \lambda x^2 + \lambda^2,$$

which attains its maximum at $x = 1+\lambda$. Hence for $1-\lambda < x \leqslant 1+\lambda$, we have

$$g(x,1) \leqslant \frac{1}{4}(1+\lambda)^4 - \lambda(1+\lambda)^2 + \lambda^2 = \frac{1}{4}\left(1+\lambda^2\right)^2.$$

Next for $1-\lambda < x \leqslant 1+\lambda$, on the boundary of Ω, we have

$$g\left(x, \frac{x^2(1+\lambda^2)-(1-\lambda^2)^2}{2\lambda x^2}\right) = \frac{1}{4}x^4 - \frac{1}{2}\left\{x^2(1+\lambda^2)-(1-\lambda^2)^2\right\} + \lambda^2,$$

which attains its maximum at $x = 1+\lambda$. Hence

$$g\left(x, \frac{x^2(1+\lambda^2)-(1-\lambda^2)^2}{2\lambda x^2}\right) \leqslant \frac{1}{4}(1+\lambda^2)^2 \quad \text{for} \quad 1-\lambda < x \leqslant 1+\lambda.$$

Also, for $|a_2| \leqslant 1-\lambda$, we obtain

$$|\gamma_2| \leqslant \frac{|a_2|^2+2\lambda}{4} \leqslant \frac{1}{4}(1+\lambda^2).$$

Thus combining all the cases, we obtain

$$|\gamma_2| \leqslant \frac{1+\lambda^2}{4}.$$

In order to prove the sharpness of the bound, we consider the function

$$\begin{aligned} f(z) &= \frac{z}{1-(1+\lambda)z+\lambda z^2} \\ &= z+(1+\lambda)z^2+\left(1+\lambda+\lambda^2\right)z^3+\left(1+\lambda+\lambda^2+\lambda^3\right)z^4+\cdots \end{aligned}$$

for $0<\lambda\leqslant 1$. Then $f(z)\in\mathscr{U}(\lambda)$ and it is easy to check that $|\gamma_2| = (1+\lambda^2)/4$.

Case 3. The third equation of (3), (16) and Lemma 1 yield

$$\begin{aligned} |\gamma_3| &= \frac{1}{2}\left|a_4-a_2a_3+\frac{1}{3}a_2^3\right| \\ &= \frac{1}{2}\left|a_2^3-2\lambda a_2c_2-c_3\lambda-a_2(a_2^2-c_2\lambda)+\frac{1}{3}a_2^3\right| \\ &= \frac{1}{2}\left|\frac{1}{3}a_2^3-\lambda a_2c_2-c_3\lambda\right| \\ &\leqslant \frac{1}{2}\left|\frac{1}{3}a_2^3-\lambda a_2c_2\right|+\frac{1}{2}\lambda|c_3| \\ &\leqslant \frac{1}{2}\left|\frac{1}{3}a_2^3-\lambda a_2c_2\right|+\frac{1}{4}\lambda(1-|c_2|^2). \end{aligned}$$

In view of the first equation of (16), we have $c_2=(a_2^2-a_3)/\lambda=1$. Hence

$$|\gamma_3| \leqslant \frac{1}{2}\left|\frac{1}{3}a_2^3-\lambda a_2\right|.$$

Let us assume $a_2 = le^{i\theta}$, where $|a_2|=l$ and $\theta\in\mathbb{R}$. Then

$$\begin{aligned} |\gamma_3| &\leqslant \frac{1}{2}\left|\frac{1}{3}l^3e^{3i\theta}-\lambda le^{i\theta}\right| \\ &= \frac{1}{2}\left|\frac{1}{3}l^3(\cos 3\theta+i\sin 3\theta)-\lambda l(\cos\theta+i\sin\theta)\right| \\ &= \frac{1}{2}\sqrt{\frac{1}{9}l^6+\lambda^2l^2-\frac{2}{3}\lambda l^4\cos 2\theta} \\ &= \frac{1}{2}\sqrt{g(x,y)}, \end{aligned}$$

where

$$g(x,y)=\frac{1}{9}x^6+\lambda^2x^2-\frac{2}{3}\lambda x^4y,$$

with $x=l$ and $y=\cos 2\theta$. In view of Lemma 3, now we will find out the maximum value of $g(x,y)$ in the region

$$\Omega=\left\{(x,y):1-\lambda<x\leqslant 1+\lambda,\frac{x^2(1+\lambda^2)-(1-\lambda^2)^2}{2\lambda x^2}\leqslant y\leqslant 1\right\}.$$

Note that $\frac{\partial}{\partial x}\{g(x,y)\}=0$ and $\frac{\partial}{\partial y}\{g(x,y)\}=0$ together imply it has no solution in the interior of Ω. Thus for $-1\leqslant y\leqslant 1$, on the boundary of Ω, we have

$$g(1-\lambda,y)=\frac{1}{9}(1-\lambda)^6+\lambda^2(1-\lambda)^2-\frac{2}{3}\lambda(1-\lambda)^4y,$$

which attains its maximum at $y=-1$. Hence for $-1\leqslant y\leqslant 1$, we obtain

$$\begin{aligned}g(1-\lambda,y)&\leqslant\frac{1}{9}(1-\lambda)^6+\lambda^2(1-\lambda)^2+\frac{2}{3}\lambda(1-\lambda)^4\\&\leqslant\frac{1}{9}(1-\lambda)^2(1+\lambda+\lambda^2)^2\\&=\frac{1}{9}(1-\lambda^3)^2.\end{aligned}$$

Now for $1-\lambda<x\leqslant 1+\lambda$, on the boundary of Ω, we have

$$g(x,1)=\frac{1}{9}x^6+\lambda^2x^2-\frac{2}{3}\lambda x^4,$$

which attains its maximum at $x=1+\lambda,1$ for $0<\lambda<1,\lambda=1$ respectively. Hence for $1-\lambda<x\leqslant 1+\lambda$, we obtain

$$g(x,1)\leqslant\frac{1}{9}(\lambda^3+1)^2.$$

Next for $1-\lambda<x\leqslant 1+\lambda$, on the boundary of Ω, we have

$$g\left(x,\frac{x^2(1+\lambda^2)-(1-\lambda^2)^2}{2\lambda x^2}\right)=\frac{1}{9}x^6+\lambda^2x^2-\frac{1}{3}x^2\left\{x^2(1+\lambda^2)-(1-\lambda^2)^2\right\},$$

which attains its maximum at $x=\sqrt{1-\lambda+\lambda^2}$. Hence for $1-\lambda<x\leqslant 1+\lambda$, we obtain

$$g\left(x,\frac{x^2(1+\lambda^2)-(1-\lambda^2)^2}{2\lambda x^2}\right)\leqslant\frac{1}{9}(\lambda^3+1)^2.$$

Also, for $|a_2|\leqslant 1-\lambda$, we have

$$|\gamma_3|\leqslant\frac{|a_2|^3+3\lambda|a_2|}{6}\leqslant\frac{1}{6}(1-\lambda^3).$$

Thus combining all the cases, we obtain

$$|\gamma_3| \leqslant \frac{1+\lambda^3}{6}.$$

In order to prove the sharpness of the bound, we consider the function $f(z) = z/(1-(1+\lambda)z+\lambda z^2)$ for $0 < \lambda \leqslant 1$. Then $f(z) \in \mathscr{U}(\lambda)$ and it is easy to check that $|\gamma_3| = (1+\lambda^3)/6$. $\square$

Acknowledgements. We would like to thank the referee and the editing team for their suggestions.

Funding. The second Author is supported by University Grants Commission (IN) fellowship (No. NBCFDC/CSIR-UGC-DECEMBER-2023).

Conflict of Interest. Authors declare that we have no conflict of interest.

REFERENCES

[1] L. A. AKSENT'EV, *Sufficient conditions for univalence of regular functions*, Izv. Vyssh. Uchebn. Zaved. Matematika., **3**, (1958), 3–7.
[2] L. A. AKSENT'EV AND F. G. AVKHADIEV, *A certain class of univalent functions*, Izv. Vyssh. Uchebn. Zaved. Matematika., **10**, (1970), 12–20.
[3] M. F. ALI AND V. ALLU, *On logarithmic coefficients of some close-to-convex functions*, Proc. Amer. Math. Soc., **146**, 3 (2018), 1131–1142.
[4] M. F. ALI, V. ALLU, D. K. THOMAS AND A. LECKO, *On the third logarithmic coefficients of close-to- convex functions*, Curr. Res. Math. Comput. Sci. II, (2018), 271–278.
[5] V. ALLU, V. ARORA AND A. SHAJI, *On the Second Hankel Determinant of Logarithmic Coefficients for Certain Univalent Functions*, Mediterr. J. Math., **20**, 2 (2023), 81.
[6] R. M. ALI, M. OBRADOVIĆ AND S. PONNUSAMY, *Necessary and sufficient conditions for univalent functions*, Complex Var. Elliptic Equ., **58**, 5 (2013), 611–620.
[7] R. BISWAS, *Second Hankel determinant of logarithmic coefficients for $\mathscr{G}(\alpha)$ and $\mathscr{P}(M)$*, J. Anal., **32**, (2024), 3019–3037.
[8] D. G. CANTOR, *Power series with integral coefficients*, Bull. Amer. Math. Soc., **69**, (1963), 362–366.
[9] F. CARLSON, *Sur les coefficients d'une fonction bornée dans le cercle unité*, Ark. Mat. Astr. Fys., **27**, 1 (1940), 8.
[10] P. DIENES, *The Taylor series*, Springer-Verlag, New York, 1957.
[11] P. L. DUREN, *Univalent Functions. Springer-Verlag*, New York Inc, 1983.
[12] R. FOURNIER AND S. PONNUSAMY, *A class of locally univalent functions defined by a differential inequality*, Complex Var. Elliptic Equ., **52**, 1 (2007), 1–8.
[13] D. GIRELA, *Logarithmic coefficients of univalent functions*, Ann. Acad. Sci. Fenn., **25**, (2000), 337–350.
[14] B. KOWALCZYK AND A. LECKO, *Second Hankel determinant of logarithmic coefficients of convex and starlike functions*, Bull. Aust. Math. Soc., **105**, 3 (2022), 458–467.
[15] B. KOWALCZYK AND A. LECKO, *Second Hankel determinant of logarithmic coefficients of convex and starlike functions of order alpha*, Bull. Malays. Math. Sci. Soc., **45**, 2 (2022), 727–740.
[16] B. KOWALCZYK AND A. LECKO, *The second Hankel determinant of the logarithmic coefficients of strongly starlike and strongly convex functions*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat., **117**, 2 (2023), 91.
[17] U. P. KUMAR AND V. ALLU, *Logarithmic coefficients for certain subclasses of close-to-convex functions*, Monatsh. Math., **187**, (2018), 543–563.

[18] L. LI, S. PONNUSAMY AND K. J. WIRTHS, *Relations of the class $\mathcal{U}(\lambda)$ to other families of functions*, Bull. Malays. Math. Sci. Soc., **45**, 3 (2022), 955–972.

[19] K. LÖWNER, *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises*, I. Mathematische Annalen, **89**, 1 (1923), 103–121.

[20] M. MUNDALIA AND S. S. KUMAR, *Coefficient problems for certain close-to-convex functions*, Bull. Iran. Math. Soc., **49**, (2023), 5.

[21] M. OBRADOVIĆ AND S. PONNUSAMY, *New criteria and distortion theorems for univalent functions*, Complex Variables Theory Appl., **44**, (2016), 173–191.

[22] M. OBRADOVIĆ AND S. PONNUSAMY, *Criteria for univalent functions in the unit disk*, Arch. Math. (Basel), **100** (2), (2013), 149–157.

[23] M. OBRADOVIĆ AND S. PONNUSAMY, *Partial sums and radius problem for certain class of conformal mappings*, Sib. Math. J. **52**, 2 (2011), 291–302.

[24] M. OBRADOVIĆ, S. PONNUSAMY AND K. J. WIRTHS, *Geometric studies on the class $\mathcal{U}(\lambda)$*, Bull. Malays. Math. Sci. Soc., **39**, (2016), 1259–1284.

[25] M. OBRADOVIĆ AND N. TUNESKI, *Some properties of the class $\mathcal{U}$*, Ann. Univ. Mariae Curie-Sklodowska Sect. A **73**, (2019), 49–56.

[26] S. OZAKI AND M. NUNOKAWA, *The Schwarzian derivative and univalent functions*, Proc. Amer. Math. Soc., (1972), 392–394.

[27] C. POMMERENKE, *On the coefficients and Hankel determinants of univalent functions*, J. Lond. Math. Soc., **1**, 1 (1966), 111–122.

[28] C. POMMERENKE, *On the Hankel determinants of Univalent functions*, Mathematika, **14**, 1 (1967), 108–112.

[29] S. PONNUSAMY, N. L. SHARMA AND K. J. WIRTHS, *Logarithmic Coefficients of the Inverse of Univalent Functions*, Results Math., **73**, (2018), 1–20.

[30] S. PONNUSAMY AND T. SUGAWA, *Sharp inequalities for logarithmic coefficients and their applications*, Bull. Sci. Math., **166**, (2021), 102931.

[31] S. PONNUSAMY AND K. J. WIRTHS, *Coefficient problems on the class $\mathcal{U}(\lambda)$*, Probl. Anal. Issues Anal., **7**, 25 (2018), 87–103.

[32] M. RAZA, A. RIAZ AND D. K. THOMAS, *The third Hanekl determinant for inverse coefficients of convex functions*, Bull. Aust. Math. Soc., **109**, 1 (2024), 94–100.

[33] Y. J. SIM, A. LECKO AND D. K. THOMAS, *The second Hankel determinant for strongly convex and Ozaki close to convex functions*, Ann. Mat. Pura Appl., **200**, (2021), 2515–2533.

[34] D. K. THOMAS, *On logarithmic coefficients of close to convex functions*, Proc. Amer. Math. Soc., **144**, 4 (2016), 1681–1687.

[35] A. VASUDEVARAO AND H. YANAGIHARA, *On the growth of analytic functions in the class $\mathcal{U}(\lambda)$*, Comput. Methods Funct. Theory., **13**, (2013), 613–634.

[36] P. ZAPRAWA, *Initial logarithmic coefficients for functions starlike with respect to symmetric points*, Bol. Soc. Mat. Mex., **27**, 3 (2021), 62.

(Received September 5, 2024)

Rajib Mandal
Department of Mathematics
Raiganj University
Raiganj, West Bengal-733134, India
e-mail: rajibmathresearch@gmail.com

Sudip Kumar Guin
Department of Mathematics
Raiganj University
Raiganj, West Bengal-733134, India
e-mail: sudipguin20@gmail.com